Chiral limit of the two-dimensional fermionic determinant in a general magnetic field

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Abstract

We consider the effective action for massive two-dimensional QED in flat Euclidean space-time in the background of a general square-integrable magnetic field with finite range. It is shown that its small mass limit is controlled by the chiral anomaly. New results for the low-energy scattering of electrons in 2+1 dimensions in static, inhomogenous magnetic fields are also presented.

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I. INTRODUCTION

Fermionic determinants lie at the heart of gauge field theories with fermions. They are obtained by integrating over the fermionic degrees of freedom in the presence of a background potential A_μ, producing the one-loop effective action $S_{\text{eff}} = -\ln \det$, where the fermionic determinant, det, is formally the ratio $\det(\slashed{P} - \slashed{A} + m)/\det(\slashed{P} + m)$ of determinants of Dirac operators. The coupling constant e has been absorbed into A_μ. This action is exact and appears in the calculation of every physical process. Therefore, any truly nonperturbative calculation must deal with S_{eff} in its full generality. The main problem with calculating S_{eff} is that it must be known for generic potentials, typically tempered distributions, if it is to be part of an effective measure for A_μ. A summary of what is known about S_{eff} in quantum electrodynamics in 1+1, 2+1 and 3+1 dimensions for general fields is given in Sec.I of Ref.1. Recall that S_{eff} in QED only depends on the field strength tensor $F_{\mu\nu}$. It is seen that there are upper and lower bounds on S_{eff}, with some bounds holding only for restricted fields, such as unidirectional ones. After fifty years or so there are still no equalities in QED for general fields, except for massless QED in 1+1 dimensions - the Schwinger model [2].

In this paper an equality is obtained for the chiral limit of S_{eff} in two-dimensional, Wick-rotated Euclidean QED for a general field, hereafter referred to as a static magnetic field $B(\mathbf{r})$. Of course B is not completely unrestricted. We described elsewhere [1,3] precisely, how rough potentials and fields are to be smoothed as part of the regularization process required to make the functional integration over A_μ well-defined. It is sufficient to assume in this paper that A_μ is differentiable and that B is square integrable with finite range R. Then B is guaranteed to have finite flux, Φ, since $||B|| \geq |\Phi|/\sqrt{\pi}\, R$, where $||B||^2 = \int d^2r\, B(\mathbf{r})^2$ and

$\Phi = \int d^2 r\, B(\mathbf{r})$. The author knows of no definition of a determinant that can handle infinite flux fields; there is simply too much degeneracy [4], resulting in volume-like divergences (which are ignored) as in the constant field case. Furthermore, finite flux and range are consistent with the need to introduce a volume cutoff to define QED_2 before taking the thermodynamic limit.

With the foregoing restrctions on B our result is

$$\lim_{m^2 \to 0} m^2 \frac{\partial}{\partial m^2} \ln \det = \frac{|\Phi|}{4\pi}, \tag{1.1}$$

where m is the fermion mass. Together with the exact scaling relation

$$\ln \det(\lambda^2 B(\lambda \mathbf{r}), m^2) = \ln \det(B(\mathbf{r}), m^2/\lambda^2), \tag{1.2}$$

(1.1) implies the strong field limit

$$\ln \det(\lambda^2 B(\lambda \mathbf{r}), m^2) \underset{\lambda \gg 1}{\sim} -\frac{|\Phi|}{2\pi} \ln \lambda + R(\lambda), \tag{1.3}$$

where $\lim_{\lambda \to \infty}(R(\lambda)/\ln \lambda) = 0$. Note that the chiral limit in (1.1) implies that QED_2's fermionic determinant behaves like $(|\Phi|/4\pi) \ln m^2$ as $m \to 0$, which does not coincide with that of the Schwinger model.

For nonwinding background fields with $\Phi = 0$ one can prove continuity at $m = 0$. As a result, massive QED_2's fermionic determinant does coincide with that of the Schwinger model at $m = 0$:

$$\lim_{m \to 0} \ln \det = \frac{1}{4\pi^2} \int d^2 r\, d^2 r'\, B(\mathbf{r}) B(\mathbf{r}') \ln |\mathbf{r} - \mathbf{r}'|. \tag{1.4}$$

This follows from results of Seiler [5] and Simon [6] as will be shown in a future paper.

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It is reasonable to ask what is the relevance of QED_2's fermionic determinant, and its mass dependence in particular, to physics? The answer is that the integral of this determinant over the fermion mass fully determines QED_4's fermionic determinant for the same magnetic field $B(\mathbf{r})$ [7]. This determinant is still unknown except for a constant field [8,9] and a $sech^2(x/R)$ varying unidirectional field [10].

From the input parameters to the QED_2 determinant one can form the dimensionless ratios $e\|B\|/mc^2$ and $\hbar/mcR$, where e has been temporarily restored. This paper deals with the nonperturbative, small mass region $e\|B\|/mc^2$, $\hbar/mcR \gg 1$. The large mass region can be dealt with by a derivative expansion of $\ln \det$ [11]. What remains in order to estimate QED_4's fermionic determinant for general unidirectional fields are optimal upper and lower bounds on QED_2's determinant for intermediate values of the mass.

The derivation of (1.1) is really just a problem in quantum mechanics dealing with a particle confined to a planar surface with an inhomogenous magnetic field normal to it. The proportionality of the limit in (1.1) to the two-dimensional chiral anomaly, $\Phi/2\pi$, as well as its sign and its connection with paramagnetism, are discussed in Ref.12. Equation (1.1) was established in Ref.12 in finite volume for a unidirectional field $B(\mathbf{r}) \geq 0$. There is a missing volume factor in Equation (2.7) that was corrected in Ref.13. These restrictions are dropped in this paper.

Finally, the chiral limit of QED_2's continuum fermionic determinant should provide a nontrivial test of algorithms for the determinant on large lattices. The reason is that chiral limits and topological invariants - the chiral anomaly in this instance - are notoriously difficult to calculate on a lattice [14]. Many of the results here on low-energy scattering in static, inhomogenous magnetic fields are new and are relevant to the physical case of

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electrons in such fields in 2+1 dimensions.

In Sec.II we discuss how we will demonstrate (1.1). Section III develops the essentials of low-energy scattering in inhomogenous magnetic fields that will be required. In Sec.IV the crucial argument that central symmetry is sufficient to establish (1.1) is given. Finally, Sec.V gives the fine points of the limit (1.1).

II. PRELIMINARIES

We adopt Schwinger's proper time definition [9] of the fermionic determinant for Euclidean QED_2:

$$\ln \det = \frac{1}{2} \int_0^\infty \frac{dt}{t} \text{Tr} \ (e^{-P^2 t} - exp\{-[(\mathbf{P} - \mathbf{A})^2 - \sigma_3 B]t\})e^{-tm^2} \,. \qquad (2.1)$$

Then

$$\frac{\partial}{\partial m^2} \ln \det = \frac{1}{2} \text{Tr} \ [(D^2 - \sigma_3 B + m^2)^{-1} - (P^2 + m^2)^{-1}] \,, \qquad (2.2)$$

where $D^2 = (\mathbf{P} - \mathbf{A})^2$ and σ_3 is the Pauli matrix. Now introduce the sum rule [15]

$$\text{Tr} \ [(D^2 - B + m^2)^{-1} - (D^2 + B + m^2)^{-1}] = \frac{\Phi}{2\pi m^2} \,, \qquad (2.3)$$

where the trace is over space indices only, and assume without loss of generality that $\Phi > 0$. Then (2.2) and (2.3) give

$$m^2 \frac{\partial}{\partial m^2} \ln \det = \frac{\Phi}{4\pi} + m^2 \text{Tr} \ [(D^2 + B + m^2)^{-1} - (P^2 + m^2)^{-1}] \,. \qquad (2.4)$$

The continuum part of the spectrum of the negative chirality operator $D^2 + B$ stretches down to zero in the case of open spaces. Because $\Phi > 0$ the square-integrable zero modes are confined to the spectrum of $D^2 - B$ [16]. Although B has no definite sign, its flux does,

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and Φ, chirality and the number of square-integrable zero modes of the supersymmetric pair of operators $D^2 \pm B$ are correlated by the Aharonov -Casher theorem. At this stage the minor modifications one has to make to deal with the case when $\Phi < 0$ are already clear.

It might seem that (2.4) makes (1.1) self-evident. But if (2.3) is multiplied by m^2 and the limit $m^2 = 0$ taken, then the fractional part of the chiral anomaly is given by a difference of zero-energy phase shifts of opposite chirality [17], demonstrating that the trace difference in (2.3) develops a $1/m^2$- type singularity at the bottom of the continuum. How, then, does one know a priori that such a singularity is absent from the trace in (2.4)?

Our definition of the determinant in (2.1) leads us to define the trace in (2.4) by a difference of diagonal heat kernels,

$$\text{Tr}\,[(D^2 + B + m^2)^{-1} - (P^2 + m^2)^{-1}] = \int_0^\infty dt e^{-tm^2} \int d^2r <\mathbf{r}\,|\,e^{-(D^2+B)t} - e^{-P^2 t}\,|\,\mathbf{r}> .$$

$$(2.5)$$

Denote the scattering states of $D^2 + B$ corresponding to outgoing radial waves by $\psi^{(+)}(\mathbf{k}, \mathbf{r}) =< \mathbf{r}\,|\,\mathbf{k}, \text{in} >$ whose eigenvalues are $E = k^2$. These satisfy the normalization condition

$$\int d^2r\, \psi^{(+)\,*}(\mathbf{k}, \mathbf{r})\, \psi^{(+)}(\mathbf{k}', \mathbf{r}) = \delta(\mathbf{k} - \mathbf{k}') .\qquad (2.6)$$

Assume that $B(\mathbf{r})$ is noncentral. Let Θ denote the direction of the incident beam with momentum $\mathbf{k}$ relative to an axis fixed in the scattering center. The asymptotic behavior of $\psi^{(+)}(\mathbf{k}, \mathbf{r})$ for $kr \gg 1$ is

$$\psi^{(+)}(\mathbf{k}, \mathbf{r}) = \frac{1}{2\pi} e^{ikr\,\cos(\theta - \Theta)} + \frac{f(\theta, \Theta)}{2\pi\sqrt{r}} e^{ikr} + R\,,\qquad (2.7)$$

where f is the scattering amplitude and R is the remainder in the large-r expansion of $\psi^{(+)}$. Equation (2.7) is obtained from the Lippmann-Schwinger equation

$$\psi^{(+)}(\mathbf{k}, \mathbf{r}) = \frac{1}{2\pi} e^{i\mathbf{k}\cdot\mathbf{r}} - \frac{i}{4} \int d^3 r' \, H_0^{(+)}(k|\mathbf{r} - \mathbf{r}'|) V(\mathbf{r}')\psi^{(+)}(\mathbf{k}, \mathbf{r}') \,, \tag{2.8}$$

where $H_0^{(+)}$ is a Hankel function of the first kind and

$$V = -\mathbf{P}\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{P} + A^2 + B \,. \tag{2.9}$$

As we will show later, we can choose a gauge such that

$$\mathbf{A} = \frac{\Phi}{2\pi r}\,\hat{\theta}\,, \tag{2.10}$$

for r sufficiently large, where $\hat{\theta}$ is a unit vector orthogonal to $\mathbf{r}$. Therefore, we are dealing with a long range $(1/r^2)$ potential V, and this is what makes the proof of (1.1) nonroutine. The completeness of the "in" states for $D^2 + B$ and (2.5) give

$$\mathrm{Tr}\,[(D^2 + B + m^2)^{-1} - (P^2 + m^2)^{-1}]$$

$$= \int_0^\infty dt e^{-tm^2} \int d^2 r \int_0^\infty dk\, k e^{-k^2 t} \int_0^{2\pi} d\Theta(|\psi^{(+)}(\mathbf{k}, \mathbf{r})|^2 - |\psi_0(\mathbf{k}, \mathbf{r})|^2)\,, \tag{2.11}$$

where $\psi_0(\mathbf{k}, \mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}/2\pi$. We are interested in the small m^2, high t limit of (2.11) which is determined by the low energy end of the spectrum of $D^2 + B$. Therefore we cut off the energy integral in (2.11) at M with $MR \ll 1$ and consider, for $m^2 \to 0$,

$$\mathrm{Tr}\,[(D^2 + B + m^2)^{-1} - (P^2 + m^2)^{-1}]$$

$$= \int_0^\infty dt e^{-tm^2} \int d^2 r \int_0^M dk\, k e^{-k^2 t} \int_0^{2\pi} d\Theta(|\psi^{(+)}(\mathbf{k}, \mathbf{r})|^2 - |\psi_0(\mathbf{k}, \mathbf{r})|^2) + R(m^2)\,. \tag{2.12}$$

The remainder, R, can be put in the form

$$\int_0^\infty e^{-t(m^2+M^2)} \int d^2 r \int_0^\infty dp\, p e^{-p^2 t} \int_0^{2\pi} d\Theta(|\psi^{(+)}(\sqrt{p^2 + M^2}, \mathbf{r}, \Theta)|^2 - |\psi_0(\sqrt{p^2 + M^2}, \mathbf{r}, \Theta)|^2)\,,$$

which makes the energy gap between 0 and M evident so that $\lim\limits_{m^2 \to 0} m^2 R(m^2) = 0$. Thus, (2.4) shows that (1.1) will be established if the integral in (2.12) multiplied by m^2 vanishes in the limit $m^2 = 0$.

We will calculate in the Lorentz gauge $\partial_\mu A_\mu = 0$ which, in two dimensions, allows us to set $A_\mu = \epsilon_{\mu\nu}\partial_\nu\phi$ with

$$B(r,\theta) = -\partial^2\phi(r,\theta),\tag{2.13}$$

and $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ with $\epsilon_{12} = 1$. Assuming that B has range R we can calculate ϕ in a disk D of radius $a \gg R$ with $\phi(a,\theta) = 0$. A unique solution of Poisson's equation with Dirichlet boundary conditions requires that we also specify ϕ as $r \to \infty$, which we will do by requiring that ϕ approach the potential of a flux line through the origin. The construction of the Dirichlet Green's function for this problem is standard, with the result

$$\phi(r,\theta) = -\frac{1}{4\pi}\int_D d^2r'B(r',\theta')\ln\left(\frac{r^2 + r'^2 - 2rr'\cos(\theta'-\theta)}{a^2 + \frac{r^2r'^2}{a^2} - 2rr'\cos(\theta'-\theta)}\right), \quad r < a$$

$$= -\frac{\Phi}{2\pi}\ln(r/a), \quad r > a.\tag{2.14}$$

This potential results in a discontinuity in $\partial\phi/\partial r$ at $r = a$ that is of order R/a^2 and which vanishes in the limit of radial symmetry. This introduces a zero-flux, zero-range magnetic field on a ring at $r = a \gg R$ which cannot affect the low energy phase shifts due to B. What is gained by this is radial symmetry for $r > a$.

From (2.12) it is evident that we will need the outgoing wave solution of

$$[(\mathbf{P} - \mathbf{A})^2 + B]\psi^{(+)} = k^2\psi^{(+)},\tag{2.15}$$

for $k \to 0$. This can be solved explicitly in the exterior region $r > a$ with overall normalization fixed by (2.7). We can approximate the interior solution by the exact zero-energy solution of (2.15) because k^2 is a regular perturbation of $D^2 + B$ for $r < a$. Then an interior solution of (2.15) can be expanded as a power series in k^2. Following this the interior and exterior solutions are matched at $r = a$ [17].

III. LOW ENERGY SCATTERING STATES

Since the case of noncentral potentials may be unfamiliar we will parallel our discussion with the special case of radial symmetry in the interest of clarity.

A. Central field: $r > a$

Expand $\psi^{(+)}(\mathbf{k}, \mathbf{r})$ in partial waves,

$$\psi^{(+)}(\mathbf{k}, \mathbf{r}) = \frac{1}{\sqrt{2}\,\pi} \sum_{l=-\infty}^{\infty} \psi_l(k, r) e^{il(\theta - \Theta)}\,. \tag{3.1}$$

Equations (3.1) and (2.6) give the normalization condition

$$\int_0^\infty dr\, r\, \psi_l^*(k, r)\, \psi_l(k', r) = \delta(E - E')\,. \tag{3.2}$$

Substitution of (3.1) and (2.10) in (2.15) results in Bessel's equation for ψ_l, with l shifted to $l - \Phi/2\pi$ for $r > a$. In order to include the case when $\Phi/2\pi$ is an integer we choose as linearly independent solutions the Hankel functions $H_{|l-\Phi/2\pi|}^{(\pm)}(kr)$ whose asymptotic behavior for $r \to \infty$ is

$$H_\nu^{(\pm)}(kr) \sim \sqrt{\frac{2}{\pi k r}}\, e^{\pm i(kr - \nu\pi/2 - \pi/4)}\,. \tag{3.3}$$

Setting $W = |l - \Phi/2\pi|$ we construct ψ_l as the following linear combination

$$\psi_l(k, r) = \frac{e^{-i\pi W/2}\, e^{i\pi |l|}}{2\sqrt{2}} \left(H_W^{(-)}(kr) + e^{i\pi(W - |l|)}\, e^{2i\delta_l(k)}\, H_W^{(+)}(kr) \right)\,, \tag{3.4}$$

where $S_l = e^{2i\delta_l}$ is the S-matrix for the partial phase shift δ_l. Recalling that in two dimensions

$$e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ikr\cos(\theta - \Theta)} = \sum_{l=-\infty}^{\infty} i^l\, J_l(kr)\, e^{il(\theta - \Theta)}\,, \tag{3.5}$$

and noting (3.3) we see that the normalization factors in (3.4) ensure that (3.1) assumes the asymptotic form (2.7) as $r \to \infty$.

B. General field: r > a

Although radial symmetry is present for $r > a$, the absence of rotational symmetry for $r < a$ can cause the incident particle to scatter into a final state that is a superposition of angular momentum states. Thus the S-matrix is no longer diagonal in l : $S_l \rightarrow S_{l,L}$, where L is the initial-state angular momentum. Then (3.1) generalizes to

$$\psi^{(+)}(\mathbf{k}, \mathbf{r}) = \frac{1}{\sqrt{2}\,\pi} \sum_{l,L} \psi_{l,L}(k, r)\, e^{il\theta}\, e^{-iL\Theta} , \qquad (3.6)$$

which, together with (2.6), results in the normalization condition

$$\sum_l \int_0^\infty dr\, r\, \psi_{l,L}^*(k, r)\, \psi_{l,L'}(k', r) = \delta_{L,L'}\, \delta(E - E') . \qquad (3.7)$$

Equation (3.4) now generalizes to

$$\psi_{l,L}(k, r) = e^{-i\pi(W_l + W_L)/4}\, e^{i\pi(|l| + |L|)/2}$$

$$\times \frac{1}{2\sqrt{2}} \left(\delta_{l,L}\, H_{W_l}^{(-)}(kr) + e^{i\pi(W_l - |l|)/2}\, S_{l,L}\, e^{i\pi(W_L - |L|)/2}\, H_{W_l}^{(+)}(kr) \right) , \qquad (3.8)$$

where $W_l = |l - \Phi/2\pi|$, etc.. Unless Θ needs to be displayed, as in (3.6), we suppress it in what follows. Again, the normalization factors in (3.8) are chosen so that (3.6) assumes the asymptotic form (2.7). The scattering amplitude is given by

$$f(\mathbf{k}', \mathbf{k}) = \frac{1}{\sqrt{2\pi k}} \sum_{l,L} (S_{l,L} - \delta_{l,L})\, e^{il\theta}\, e^{-iL\Theta}\, e^{i\pi(W_L - W_l - 1)/4} . \qquad (3.9)$$

C. General field: r < a

We seek zero-energy solutions of (2.15) in the region $r < a$ that are sufficiently regular to maintain the Hermiticity of $D^2 + B$. This operator factorizes to $L^\dagger L$ so that (2.15) at $k^2 = 0$ reduces to

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$$L^\dagger L \psi = 0 \,, \tag{3.10}$$

where

$$L = e^{-i\theta} \left(\frac{1}{i} \frac{\partial}{\partial r} - \frac{1}{r} \frac{\partial}{\partial \theta} - \frac{1}{r} \frac{\partial \phi}{\partial \theta} - i \frac{\partial \phi}{\partial r} \right) . \tag{3.11}$$

One set of solutions is given by

$$L \psi = 0 \,, \tag{3.12}$$

whose solution by inspection is

$$\psi = e^{-\phi(r,\theta)} \, g\!\left(r e^{-i\theta}\right) , \tag{3.13}$$

where g is analytic in $r e^{-i\theta}$ in and on the disk D. Solutions of the form (3.13) do not give

all of the regular solutions of (3.10). This is evident in the limit of radial symmetry, for then

ψ is a superposition of only negative or zero angular momentum states.

There are irregular solutions of (3.12) and hence (3.10) of the form

$$\psi = e^{-\phi(r,\theta)} \, h\!\left(r^{-1} e^{i\theta}\right) , \tag{3.14}$$

where h may be expanded in a power series away from the origin. These solutions can be

used to find additional regular solutions of (3.10) that reduce to superpositions of positive

angular momenta in the radial symmetry limit. Thus, we look for regular solutions about

the origin of the form

$$\psi = e^{-\phi} \, h\!\left(r^{-1} e^{i\theta}\right) F(r,\theta) \,. \tag{3.15}$$

Then (3.10) gives

$$L^\dagger \left[e^{-\phi} e^{-i\theta} \, h(r^{-1} e^{i\theta}) \left(\frac{\partial F}{\partial r} - \frac{i}{r} \frac{\partial F}{\partial \theta} \right) \right] = 0 \,. \tag{3.16}$$

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Again by inspection the solution of (3.16) is

$$e^{-\phi} e^{-i\theta} h \left(\frac{\partial F}{\partial r} - \frac{i}{r} \frac{\partial F}{\partial \theta} \right) = e^{\phi} b(re^{i\theta}) , \tag{3.17}$$

where b is analytic in $re^{i\theta}$ in and on D. Actually, h is now an unnecessary complication.

Letting $F = f(r, \theta)/h$, we get

$$\frac{\partial f}{\partial r} - \frac{i}{r} \frac{\partial f}{\partial \theta} = e^{i\theta} e^{2\phi} b(re^{i\theta}) , \tag{3.18}$$

and hence (3.15) becomes

$$\psi = e^{-\phi} f . \tag{3.19}$$

Equation (3.18) indicates that f is undetermined up to a function of the form $p(re^{-i\theta})$. But this is the same as g in (3.13), and so we set $p = 0$. Also, the value of ψ at the origin can be fixed by g. So for definiteness we require $f(0) = 0$. Noting that

$$\nabla^2 = e^{-i\theta} \left(\frac{\partial}{\partial r} - \frac{i}{r} \frac{\partial}{\partial \theta} \right) e^{i\theta} \left(\frac{1}{r} + \frac{i}{r} \frac{\partial}{\partial \theta} \right) , \tag{3.20}$$

the solution of (3.18) is, for $\mathbf{r} \in D$,

$$f(\mathbf{r}) = \frac{1}{2\pi} e^{i\theta} \left(\frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \theta} \right) \int_D d^2 r' \ln |\mathbf{r} - \mathbf{r}'| e^{2\phi(\mathbf{r}')} b(r'e^{i\theta'}) + C , \tag{3.21}$$

where C is a constant fixed by $f(0) = 0$. Since

$$e^{i\theta} \left(\frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \theta} \right) \ln |\mathbf{r} - \mathbf{r}'| = -e^{i\theta'} \left(\frac{\partial}{\partial r'} + \frac{i}{r'} \frac{\partial}{\partial \theta'} \right) \ln |\mathbf{r} - \mathbf{r}'| , \tag{3.22}$$

f takes the final form

$$f(\mathbf{r}) = -\frac{1}{2\pi} \int_D d^2 r' e^{i\theta'} e^{2\phi(\mathbf{r}')} b(r'e^{i\theta'}) \left(\frac{\partial}{\partial r'} + \frac{i}{r'} \frac{\partial}{\partial \theta'} \right) \ln \left(|\mathbf{r} - \mathbf{r}'|/r' \right) . \tag{3.23}$$

Combining (3.13), (3.19) and (3.23), the general solution of (3.10) is

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$$\psi(r,\theta) = e^{-\phi(r,\theta)}\left(g(re^{-i\theta}) + f(r,\theta)\right). \tag{3.24}$$

The functions b, f and g will be determined below when we join the region $r < a$ with $r > a$. They then acquire energy-dependent normalization factors that depend on scattering data, including the initial-state angular momentum L. Thus ψ in (3.24) has an implicit dependence on Θ.

As discussed at the end of Sec. II, k^2 is a regular perturbation of $D^2 + B$ in (2.15) in the region $r < a$. Hence the radial wave functions $\psi_{l,L}(k,r)$ in (3.6) may be expanded in k^2 for $r < a$. Inserting the implicit Θ-dependence of $\psi(r,\theta)$ in (3.24) we expand it in partial waves

$$\psi(r,\theta,\Theta) = \sum_{l,L} \psi_{l,L}(r)\, e^{il\theta}\, e^{-iL\Theta}\, . \tag{3.25}$$

We set

$$\psi_{l,L}(k,r)/\sqrt{2}\pi = \psi_{l,L}(r)\left(1 + k^2\chi_{l,L}(r) + \mathrm{O}(k^4)\right) \tag{3.26}$$

and thus

$$\psi^{(+)}(\mathbf{k},\mathbf{r}) = \psi(r,\theta,\Theta) + \frac{k^2}{(2\pi)^2}\int_0^{2\pi} d\theta' \int_0^{2\pi} d\Theta'\, \psi(r,\theta-\theta',\Theta-\Theta')\, \chi(r,\theta',\Theta') + \mathrm{O}(k^4\psi)\,,$$
$$\tag{3.27}$$

where χ can be expanded as in (3.25). We will abbreviate (3.27) as

$$\psi^{(+)}(\mathbf{k},\mathbf{r}) = \psi(r,\theta) + k^2\psi \star \chi + \mathrm{O}(k^4\psi)\,, \tag{3.28}$$

where the star denotes convolution. An equation for χ can be obtained by substituting (3.28) in (2.15) and retaining terms of order k^2. As we will see, χ is not required in the general field case.

To fix b and f define the operator

$$\mathcal{L} = \frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \theta}, \tag{3.29}$$

and let it act on $\psi^{(+)}$ in (3.28), using (3.24):

$$\mathcal{L}\,\psi^{(+)} = -\psi\,\mathcal{L}\,\phi + e^{-\phi}\,\mathcal{L}\,f + k^2\mathcal{L}\,\psi \star \chi + \mathrm{O}(k^4\mathcal{L}\psi). \tag{3.30}$$

Equation (2.14) with $R = a$ gives

$$\partial_\theta\,\phi(a,\theta) = 0, \quad (\partial_r\phi(r,\theta))_a = -\Phi/2\pi a + \mathrm{O}(R/a^2). \tag{3.31}$$

From now on the arbitrarily small correction to the radial derivative of ϕ at $r = a$ will be implicit in what follows. Thus (3.31) and (3.18) applied to (3.30) at $r = a$ give

$$\mathcal{L}\,\psi^{(+)}(k,a,\theta) = \Phi\,\psi(a,\theta)/2\pi a + e^{i\theta}\,b(ae^{i\theta}) + k^2\mathcal{L}\,\psi \star \chi + \mathrm{O}(k^4\mathcal{L}\psi). \tag{3.32}$$

Denote the wave functions on either side of a by $\psi_<^{(+)}$ and $\psi_>^{(+)}$. Continuity of $\psi^{(+)}$ and $\mathcal{L}\,\psi^{(+)}$ at $r = a$ and repeated use of (3.28) allow (3.32) to be put in the form

$$e^{i\theta}\,b(ae^{i\theta}) = (\mathcal{L} - \Phi/2\pi a)\,\psi_>^{(+)} - k^2\Phi\,\psi_>^{(+)} \star \chi/2\pi a + k^2\mathcal{L}\,\psi_>^{(+)} \star \chi + \mathrm{O}(k^4\mathcal{L}\psi_>^{(+)}). \tag{3.33}$$

Since $b(re^{i\theta})$ is analytic in and on D we can make the expansion

$$b(re^{i\theta}) = \sum_{l=0}^{\infty}\sum_{L=-\infty}^{\infty} b_{l,L}\,r^l\,e^{il\theta}\,e^{-iL\Theta}, \tag{3.34}$$

where we have anticipated the Θ-dependence of b. From (3.6), (3.33) and (3.34) we get

$$\sqrt{2}\pi b_{l-1,L}\,a^{l-1} = \left(\frac{d}{dr} + (l - \Phi/2\pi)/r\right)\psi_{l,L}^>(k,r)\,(1 + k^2\chi_{l,L}(r) + \mathrm{O}(k^4)), \tag{3.35}$$

with $r = a$ after differentiating. Referring to (3.8) it is evident from (3.35) that the expansion coefficients $b_{l,L}$ and hence b and f in (3.23) will be determined to leading order in k^2 once $S_{l,L}$ is known.

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There now remains the function g in (3.24). Equations (3.24) and (3.28) together with continuity of $\psi^{(+)}$ at $r = a$ give

$$\psi^{(+)}_>(k, a, \theta) = g(ae^{-i\theta}) + f(a, \theta) + k^2 \psi \star \chi(a, \theta) + \mathrm{O}(k^4\psi). \tag{3.36}$$

Letting

$$f(r, \theta) = \sum_{l, L} f_{l, L}(r)\, e^{il\theta}\, e^{-iL\Theta}, \tag{3.37}$$

and recalling that g is analytic in and on D so that

$$g(re^{-i\theta}) = \sum_{l=0}^{\infty} \sum_{L=-\infty}^{\infty} g_{l, L}\, r^l\, e^{-il\theta}\, e^{-iL\Theta}, \tag{3.38}$$

we obtain from (3.6), (3.36)-(3.38), for $l \geq 0$,

$$\psi^>_{-l, L}(k, a)/\sqrt{2\pi} = g_{l, L}\, a^l + f_{-l, L}(a) + k^2 \psi^>_{-l, L}(k, a)\, \chi_{-l, L}(a)/\sqrt{2\pi} + \mathrm{O}(k^4\psi^>_{-l, L}). \tag{3.39}$$

This simplifies on making the expansion

$$\ln(r^2 + r'^2 - 2rr'\cos(\theta - \theta')) = \ln r_>^2 - 2\sum_{l=1}^{\infty} \frac{1}{l}\left(\frac{r_<}{r_>}\right)^l \cos[l(\theta - \theta')], \tag{3.40}$$

in (3.23), giving $f_{-l, L}(a) = 0$, $l > 0$ and

$$f_{0, L}(a) = \frac{1}{2\pi}\int_0^R dr \int_0^{2\pi} d\theta\, e^{i\theta}\, e^{2\phi(r, \theta)} \sum_{l=0}^{\infty} b_{l, L}\, r^l\, e^{il\theta}. \tag{3.41}$$

Thus g is determined to leading order in k^2 by (3.38)-(3.39) once $S_{l, L}$ is known. We have now fully determined the low energy limit of $\psi^{(+)}(\mathbf{k}, \mathbf{r})$ for general magnetic fields in terms of the S-matrix.

D. Central Field: r < a

Now everything is diagonal. Refer back to (3.4) and define the energy-dependent part, Δ_l, of the phase shifts by

$$\delta_l(k^2) = \pi \left(|l| - |l - \Phi/2\pi| \right)/2 + \Delta_l(k^2) + m\pi \,, \tag{3.42}$$

where $m = 0, \pm 1, \ldots$ Then (3.4) reduces to, for $r > a$,

$$\psi_l(k, r) = 2^{-1/2} \, (-1)^m \, i^{|l|} \, e^{i\delta_l} \left(J_W(kr) \, \cos\Delta_l - Y_W(kr) \, \sin\Delta_l \right), \tag{3.43}$$

where Y_W is the Bessel function of the second kind. Then at $r = a$ with $ka \ll 1$ and $W \neq 0$,

$$\psi_l^>(k, a) = 2^{-1/2} \, (-1)^m \, i^{|l|} \, e^{i\delta_l} \left[(ka/2)^W / \Gamma(W+1) + \Delta_l \, \Gamma(W) \, (ka/2)^{-W} / \pi \right]$$

$$\times (1 + \mathrm{O}(k^2, \Delta_l^2)) \,, \tag{3.44}$$

$$(r\partial_r \, \psi_l^>)_a = 2^{-1/2} \, W \, (-1)^m \, i^{|l|} \, e^{i\delta_l} \left[(ka/2)^W / \Gamma(W+1) - \Delta_l \, \Gamma(W) \, (ka/2)^{-W} / \pi \right]$$

$$\times (1 + \mathrm{O}(k^2, \Delta_l^2)) \,. \tag{3.45}$$

It will be shown in Sec. IV that $\Delta_l = \mathrm{O}(ka)^{2W}$ at least. In (3.44) and (3.45) the remainder term $\mathrm{O}(k^2, \Delta_l^2)$ should be replaced with $\mathrm{O}((ka)^2 \ln ka)$ when $W = 1$.

We can now calculate b_{l-1} in (3.35). For $l > \Phi/2\pi$, (3.35), (3.44) and (3.45) give

$$b_{l-1} \, a^l = \pi^{-1} \, (l - \Phi/2\pi) \, (-1)^m \, i^l \, e^{i\delta_l} \, (ka/2)^W / \Gamma(W+1) \, (1 + \mathrm{O}(k^2, \Delta_l^2)) \,, \tag{3.46}$$

and for $1 \leq l < \Phi/2\pi$,

$$b_{l-1} \, a^l = \pi^{-2} \, (l - \Phi/2\pi) \, (-1)^m \, i^l \, e^{i\delta_l} \, \Delta_l \, \Gamma(W)(ka/2)^{-W} \, (1 + \mathrm{O}(k^2, \Delta_l^2)) \,. \tag{3.47}$$

For the case $W = 0$,

$$b_{l-1} \, a^l = -\pi^{-2} \, (-1)^m \, i^l \, e^{i\delta_l} \, \Delta_l \, (1 + \mathrm{O}(\Delta_l^2)) \,. \tag{3.48}$$

For g_l, (3.39) gives

$$g_l = \frac{\psi_{-l}^>(k, a)}{\sqrt{2\pi} \, a^l} \, (1 - k^2 \chi_{-l}(a) + \mathrm{O}(k^4)) \,, \tag{3.49}$$

since (3.41) gives $f_{o,L}(a) = 0$ for the case of radial symmetry. Combining (3.49) with (3.44), we obtain, for $l \geq 0$,

$$g_l = (-1)^m \, i^l \, e^{i\delta_{-l}} \, (2\pi a^l)^{-1} \, [(ka/2)^{l+\Phi/2\pi}/\Gamma(l+1+\Phi/2\pi)$$

$$-\pi^{-1} \, \Delta_{-l} \, \Gamma(l+\Phi/2\pi) \, (ka/2)^{-(l+\Phi/2\pi)}] \times (1 + O(k^2, \Delta_{-l}^2)) \,. \tag{3.50}$$

Referring to (3.37), (3.40) and (3.23) one finds for $r \leq a$,

$$f_l(r) = b_{l-1} \, r^{-l} \int_0^r dx \, x^{2l-1} \, e^{2\phi(x)}, \quad l \geq 1$$

$$= 0, \quad l \leq 0 \,. \tag{3.51}$$

In the radial symmetry limit $\psi_{l,L}(r)$ in (3.26) becomes diagonal, with (3.24) now giving, for $r \leq a$,

$$\psi_l(r) = b_{l-1} \, r^{-l} \, e^{-\phi(r)} \int_0^r dx \, x^{2l-1} \, e^{2\phi(x)}, \quad l \geq 1$$

$$= g_{-l} \, r^{-l} \, e^{-\phi(r)}, \quad l \leq 0 \,. \tag{3.52}$$

Since it will be needed in what follows we end this section by calculating χ_l in (3.26). Substitution of (3.25) and (3.26) in (2.15) and matching terms of $O(k^2)$ gives

$$\left(-\frac{d}{dr} + \frac{l-1}{r} + \phi' \right) \left(\frac{d}{dr} + \frac{l}{r} + \phi' \right) \psi_l \, \chi_l = \psi_l \,. \tag{3.53}$$

Requiring $\chi_l(0) = 0$, (3.52) and (3.53) fix χ_l for $r \leq a$ to be

$$\chi_l = -\int_0^r dx \, x^{-2|l|-1} \, e^{2\phi(x)} \int_0^x dy \, y^{2|l|+1} \, e^{-2\phi(y)} \,, \quad l \leq 0$$

$$= -\int_0^r dx \, x^{2l-1} \, e^{2\phi(x)} \left(\int_0^x dw \, w^{2l-1} \, e^{2\phi(w)} \right)^{-2}$$

$$\times \int_0^x dy \, y^{1-2l} \, e^{2\phi(y)} \left(\int_0^y dz \, z^{2l-1} \, e^{2\phi(z)} \right)^2 \,, \quad l \geq 1 \,. \tag{3.54}$$

It is important to bound the growth of χ_l for $|l| \to \infty$. In the limit of radial symmetry (2.14) reduces to

$$\phi(r) = -\int_0^a dr'\, r'\, B(r')\, \ln(r_>/a)\,, \tag{3.55}$$

for $r \le a$. Then some easy estimates applied to (3.54) and (3.55) yield

$$|\phi(r)| \le ||B||\, (a-r)/2\sqrt{\pi}\,, \tag{3.56}$$

and

$$
\begin{aligned}
|\chi(r)| &\le \frac{e^{2||B||a\sqrt{\pi}}}{4(|l|+1)}\, r^2\,, \quad l \le 0 \\
&\le \frac{e^{6||B||a\sqrt{\pi}}}{4(l+1)}\, r^2\,, \quad l \ge 1\,.
\end{aligned}
\tag{3.57}
$$

IV. LOW ENERGY PHASE SHIFTS

In order to calculate (2.12) in the limit $m^2 \to 0$ the leading energy-dependent behavior of $S_{l,L}$ is required. The case of central fields is dealt with first.

A. Central Fields

The calculation of Δ_l in (3.42) proceeds by matching the log-derivatives $\gamma_l = r\partial_r \ln \psi_l$ at $r = a$. Then (3.43) gives

$$\tan \Delta_l = \frac{\gamma_l J_W(ka) - ka J_W'(ka)}{\gamma_l Y_W(ka) - ka Y_W'(ka)}\,, \tag{4.1}$$

where γ_l denotes γ_l (inside). For $ka \ll 1$ this reduces to

$$\Delta_l = \pi \frac{W-\gamma_l}{W+\gamma_l}\, \frac{(ka/2)^{2W}}{\Gamma(W)\Gamma(W+1)}\, (1 + \mathrm{O}(ka)^2)\,, \tag{4.2}$$

for $W = |l - \Phi/2\pi| \ne 0, 1, \dots$ Results for integer values of W will be given below.

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Now suppose $l \geq 1$. Then (3.35) reduces to

$$\gamma_l = \Phi/2\pi - l + \sqrt{2\pi}\, b_{l-1}\, a^l/\psi_l^<(k,a) + k^2[(\Phi/2\pi - l - \gamma_l)\, \chi_l(a) - (r\partial_r\chi_l)_a] + \mathrm{O}(k^4)\,. \quad (4.3)$$

From (3.6), (3.24) and (3.26),

$$\psi_l^<(k,a)/\sqrt{2\pi} = f_l(a)(1 + k^2\chi_l(a) + \mathrm{O}(k^4))\,, \quad (4.4)$$

which, together with (3.51) and (4.3), gives

$$\gamma_l = \Phi/2\pi - l + \frac{a^{2l}}{\int_0^a dr\, r^{2l-1}\, e^{2\phi(r)}} - k^2\, a\chi_l'(a) + \mathrm{O}(k^4)\,. \quad (4.5)$$

Note that $\gamma_l \sim l$ as $l \to \infty$, as it should.

Next, let $l \leq 0$. Equations (3.6), (3.24) and (3.26) give

$$\gamma_l = \Phi/2\pi - l + k^2\, a\chi_l'(a) + \mathrm{O}(k^4)\,, \quad (4.6)$$

since $f_l = 0$ for $l \leq 0$ by (3.51). Equations (3.42), (4.2), (4.5) and (4.6) determine the leading energy dependence of the phase shifts for $W \neq 0, 1, \dots$

Finally, let $W = 0, 1, \dots$ There is nothing new in principle here; only the expansion of Y_W for $ka \ll 1$ has to be modified. The whole calculation goes forward as above with the result that for $W = 2, 3, \dots, \Delta_l$ is still given by (4.2); for $W = 1$ replace $\mathrm{O}(ka)^2$ in (4.2) by $\mathrm{O}((ka)^2 \ln(ka))$, and for $W = 0$,

$$\Delta_l = \frac{\pi}{2\ln(ka)} + \mathrm{O}(1/\ln^2(ka))\,. \quad (4.7)$$

It is interesting that the energy dependence of Δ_l for $W = 0$ specialized to $l = 0$ is exactly the same as that derived by Chandon et al. [18] for a large class of non-magnetic Schrödinger operators in 2+1 dimensions. Note that there is no smooth interpolation of Δ_l from $W \neq 0$ to $W = 0$. This case will, therefore, have to be considered seperately in what follows.

B. General Fields

The S-matrix $S_{l,L}$ appearing in (3.8) is obtained from

$$S_{l,L} = \delta_{l,L} + \sqrt{2\pi}\, i^{-l-1} \sum_m \int_0^\infty dr\, r\, J_l(kr)\, V_{l-m}(r)\, \psi_{m,L}(k,r)\,, \tag{4.8}$$

with $\psi_{l,L}$ given by (3.6) and (2.8) and where

$$V_{l-m}(r) = \frac{1}{2\pi} \int_0^{2\pi} d\theta\, V(r,\theta)\, e^{-i(l-m)\theta}\,, \tag{4.9}$$

with V as in (2.9). An infinite set of coupled equations must be solved to extract the phase shifts in the general field case. In practice, only a few off-diagonal elements of $S_{l,L}$ are required to obtain the phase shifts in the low energy limit.

Consider an incident low-energy particle ($ka \ll 1$) with angular momentum L with respect to the scattering center. It will encounter a high centrifugal barrier $(l - \Phi/2\pi)^2/r^2$ to the spatially asymmetric region $r < a$ where $B(r,\theta) \neq 0$. For values of $L \sim \Phi/2\pi$ the barrier is minimized, and so we expect the magnitude of the energy-dependent corrections to the Aharonov-Bohm phase shifts, Δ_L, will assume their maximum values, as (4.2) and (4.7) illustrate in the centrally symmetric case. The intuition is that $S_{l,L}$ only has significant off-diagonal elements for values of l, L clustered about $\Phi/2\pi$ and that otherwise $S_{l,L}$ can be assumed diagonal with small error.

To test this hypothesis we will assume that

$$L < \frac{\Phi}{2\pi} < L+1\,, \tag{4.10}$$

and take $S_{l,L}$ to be a 2×2 matrix to include the transitions $L \longleftrightarrow L+1$, and diagonal otherwise. This S-matrix can be calculated for all Φ satisfying (4.10) using the results of

the previous sections. As one would expect, the mixing of angular momentum states L and $L+1$ is maximum at the mid-interval value $\Phi/2\pi = L + 1/2$. Instead of reproducing this calculation it is more instructive to set $\Phi/2\pi = L + 1/2$, where the Hankel functions assume a simple form, and show that Δ_L and Δ_{L+1} have the same energy dependence as in the centrally symmetric case; only the numerical coefficients are modified. In the case of higher order transitions $|\Delta_L| > 1$ involving larger matrices, we find that the relevant mixing parameters (see below) compared to the $|\Delta_L| = 1$ case are smaller by factors of order $k^{|\Delta_L|-1}$.

The calculation begins by noting that the potential in (2.9) is not time-reversal invariant for a fixed magnetic field. Therefore, $S_{l,L}$ is not symmetric. We choose the parameterization

$$
S = \begin{pmatrix} e^{2i\delta_L} \cos 2\epsilon & ie^{i\alpha} \sin 2\epsilon \\ ie^{i\beta} \sin 2\epsilon & e^{2i\delta_{L+1}} \cos 2\epsilon \end{pmatrix} ,
\tag{4.11}
$$

where we expect the mixing parameter ϵ to vanish as $k \to 0$. Unitarity requires α and β to be real with

$$
\alpha + \beta = 2(\delta_L + \delta_{L+1}) .
\tag{4.12}
$$

The definition of the phase shifts in (4.11) is that of Stapp et al. [19], generalized here to include T-violation. Referring to (3.42) we can rewrite (4.11) as

$$
S = \begin{pmatrix} e^{2i(\Delta_L - \Phi/4)} \cos 2\epsilon & ie^{i(\Delta_L + \Delta_{L+1} + \lambda)} \sin 2\epsilon \\ ie^{i(\Delta_L + \Delta_{L+1} - \lambda)} \sin 2\epsilon & e^{2i(\Delta_{L+1} + \Phi/4)} \cos 2\epsilon \end{pmatrix} ,
\tag{4.13}
$$

which introduces a real T-violating parameter λ. From (3.8) and (4.12) with $\Phi/2\pi = L+1/2$, the matching of the interior and exterior log-derivatives at $r = a$ gives

$$
\gamma_{L,L} = \frac{kaH_{1/2}^{(-)\prime} + e^{2i\Delta_L} kaH_{1/2}^{(+)\prime} \cos 2\epsilon}{H_{1/2}^{(-)} + e^{2i\Delta_L} H_{1/2}^{(+)} \cos 2\epsilon} ,
\tag{4.14}
$$

with $\gamma_{L+1,L+1} = \gamma_L(\Delta_L \rightarrow \Delta_{L+1})$. Also

$$\gamma_{L,L+1} = \gamma_{L+1,L} = ka H_{1/2}^{(+)\prime}/H_{1/2}^{(+)} \,. \tag{4.15}$$

Recalling that

$$H_{1/2}^{(\pm)}(z) = \mp i \sqrt{\frac{2}{\pi z}}\, e^{\pm iz} \,, \tag{4.16}$$

(4.13) becomes,

$$\gamma_{L,L} = \frac{\left(\frac{1}{2} + ika\right) e^{-2ika} + \left(-\frac{1}{2} + ika\right)e^{2i\Delta_L}\cos 2\epsilon}{e^{2i\Delta_L}\cos 2\epsilon - e^{-2ika}} \,, \tag{4.17}$$

$$\gamma_{L,L+1} = -\tfrac{1}{2} + iz \,. \tag{4.18}$$

To get the interior values of $\gamma_{l,L}$, refer to (3.35). Then

$$\gamma_{L,L} = \Phi/2\pi - L + \frac{\sqrt{2}\,\pi\,a^L\,b_{L-1,L}}{\psi_{L,L}^{<}(k,a)} + \mathrm{O}(ka)^2 \,. \tag{4.19}$$

From (3.6) and (3.36) - (3.38),

$$\psi_{L,L}^{<}(k,a)/\sqrt{2}\pi = f_{L,L}(a) + \mathrm{O}(ka)^2 \,, \tag{4.20}$$

so that

$$\gamma_{L,L} = \Phi/2\pi - L + \frac{a^L\,b_{L-1,L}}{f_{L,L}(a)}\left(1 + \mathrm{O}(ka)^2\right) + \mathrm{O}(ka)^2 \,. \tag{4.21}$$

Referring back to (3.23), (3.37) and (3.40) it follows that, for $L > 0$,

$$f_{L,L}(a) = \frac{1}{2\pi\,a^L} \int_0^a dr\, r^L \int_0^{2\pi} d\theta\, e^{i(1-L)\theta}\, e^{2\phi(r,\theta)} \sum_{m=0}^{\infty} b_{m,L}\, r^m\, e^{im\theta} \,. \tag{4.22}$$

In the sum over $b_{m,L}$ in (4.22), only $b_{L-1,L}$ and $b_{L,L}$ are nonzero as seen from (3.8), (3.35) and (4.13) since mixing is only assumed for $L \longleftrightarrow L+1$. Then (4.21) and (4.22) give

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$$\gamma_{L,L} = \tfrac{1}{2} + 2\pi\, a^{2L} \left(\int_0^a dr\, r^{2L-1} \int_0^{2\pi} d\theta\, e^{2\phi(r,\theta)} + \frac{b_{L,L}}{b_{L-1,L}} \int_0^a dr\, r^{2L} \int_0^{2\pi} d\theta\, e^{i\theta}\, e^{2\phi(r,\theta)} \right)^{-1}$$

$$\times (1 + \mathrm{O}(ka)^2) + \mathrm{O}(ka)^2\,. \tag{4.23}$$

Repeating the above steps we find

$$\gamma_{L+1,L+1} = -\tfrac{1}{2} + 2\pi\, a^{2L+2} \left(\int_0^a dr\, r^{2L+1} \int_0^{2\pi} d\theta\, e^{2\phi(r,\theta)} \right.$$

$$\left. + \frac{b_{L-1,L+1}}{b_{L,L+1}} \int_0^a dr\, r^{2L} \int_0^{2\pi} d\theta\, e^{-i\theta}\, e^{2\phi(r,\theta)} \right)^{-1} \times (1 + \mathrm{O}(ka)^2) + \mathrm{O}(ka)^2\,. \tag{4.24}$$

We now calculate the ratios of $b_{l,L}$ in (4.23) and (4.24). From (3.35),

$$\sqrt{2\pi}\, a^l\, b_{l-1,L} = (\gamma_{l,L} + L - \Phi/2\pi)\, \psi_{l,L}^>(k,a)\, (1 + \mathrm{O}(ka)^2)\,. \tag{4.25}$$

Then (4.25), (4.15) and (4.18) give

$$b_{L,L}/b_{L-1,L} = \frac{ik\psi_{L+1,L}^>(k,a)}{(\gamma_{L,L} - \tfrac{1}{2})\, \psi_{L,L}^>(k,a)}\, (1 + \mathrm{O}(ka)^2)\,. \tag{4.26}$$

Equations (3.8), (4.13) and (4.16) give

$$\psi_{L+1,L}^>(k,a)/\psi_{L,L}^>(k,a) = \frac{e^{i(\Delta_L + \Delta_{L+1} - \lambda - \pi L)} \sin 2\epsilon}{e^{-2ika} - e^{2i\Delta_L} \cos 2\epsilon}\,. \tag{4.27}$$

Likewise,

$$b_{L-1,L+1}/b_{L,L+1} = \frac{(ika - 1)\, a\, \psi_{L,L+1}^>(k,a)}{(\gamma_{L+1,L+1} + \tfrac{1}{2})\, \psi_{L+1,L+1}^>(k,a)}\, (1 + \mathrm{O}(ka)^2)\,, \tag{4.28}$$

$$\psi_{L,L+1}^>(k,a)/\psi_{L+1,L+1}^>(k,a) = \frac{e^{i(\Delta_L + \Delta_{L+1} + \lambda - \pi L)} \sin 2\epsilon}{e^{2i\Delta_{L+1}} \cos 2\epsilon - e^{-2ika}}\,. \tag{4.29}$$

Then solving (4.23), (4.26) and (4.27) for $\gamma_{L,L}$ and matching the result with $\gamma_{L,L}$ in (4.17)

gives

$$(1 + ika)\, e^{-2ika}\, I_L + (ika - 1)\, I_L\, e^{2i\Delta_L} \cos 2\epsilon =$$

$$e^{2i\Delta_L} \cos 2\epsilon - e^{-2ika} + ika\, J_L\, e^{i(\Delta_L + \Delta_{L+1} - \lambda - \pi L)} \sin 2\epsilon + \mathrm{O}((ka)^3, (ka)^2 \Delta_L)\,, \tag{4.30}$$

where

$$I_L = (2\pi\, a^{2L})^{-1} \int_0^a dr\, r^{2L-1} \int_0^{2\pi} d\theta\, e^{2\phi(r,\theta)}$$

$$J_L = (2\pi\, a^{2L+1})^{-1} \int_0^a dr\, r^{2L} \int_0^{2\pi} d\theta\, e^{i\theta}\, e^{2\phi(r,\theta)}\,. \tag{4.31}$$

Similarly, (4.24), (4.28), (4.29) and (4.17) with Δ_L replaced with Δ_{L+1} give

$$ika\left(e^{-2ika} + e^{2i\Delta_{L+1}}\cos 2\epsilon\right) I_{L+1} =$$

$$e^{2i\Delta_{L+1}}\cos 2\epsilon - e^{-2ika}(1 - ika)\, J_L^\star\, e^{i(\Delta_L + \Delta_{L+1} + \lambda - \pi L)}\,\sin 2\epsilon + \mathrm{O}((ka)^3, (ka)^2\Delta_{L+1})\,. \tag{4.32}$$

Equations (4.30) and (4.32) can be solved for Δ_L, Δ_{L+1}, ϵ and λ with

$$\lambda = \lambda_0 + \lambda_1 ka + \mathrm{O}((ka)^2)\,. \tag{4.33}$$

The results are

$$\Delta_L = -ka/(1 + I_L) + \mathrm{O}((ka)^3)$$

$$\Delta_{L+1} = \left[I_{L+1} - 1 - (\cos\lambda_0 \mathrm{Im}J_L - \sin\lambda_0 \mathrm{Re}J_L)^2/(1 + I_L)\right] ka + \mathrm{O}((ka)^3)$$

$$0 = \cos\lambda_0 \mathrm{Re}J_L + \sin\lambda_0 \mathrm{Im}J_L$$

$$\lambda_1 = 0$$

$$\epsilon = (-1)^L \left(\sin\lambda_0 \mathrm{Re}J_L - \cos\lambda_0 \mathrm{Im}J_L\right) ka/(1 + I_L) + \mathrm{O}((ka)^2)\,. \tag{4.34}$$

Note that Δ_L and Δ_{L+1} have the same energy dependence as in the centrally symmetric case

given by (4.2).

V. CHIRAL LIMIT

We concluded in Sec. IV that general magnetic fields only modify the numerical coeffi-

cients of the energy-dependent part of the phase shifts calculated in the centrally symmetric

case. It was also noted that the off-diagonal elements of $S_{l,L}$ fall off as powers of k. There-fore, inhomogenous fields of the type considered here do not result in special cases that are not already included in the central symmetry limit. We therefore confine our discussion to central symmetry from here on and proceed to show that (1.1) is true by demonstrating that the integral in (2.12) satisfies

$$\lim_{m^2 \to 0} m^2 \int_0^\infty dt\, e^{-tm^2} \int d^2r \int_0^M dk\, k\, e^{-k^2 t} \int_0^{2\pi} d\Theta \left(|\psi^{(+)}(\mathbf{k}, \mathbf{r})|^2 - |\psi_0(\mathbf{k}, \mathbf{r})|^2 \right) = 0. \quad (5.1)$$

This integral can be divided between contributions from the regions $r < a$ and $r > a$.

A. r < a

By inspection, the term $|\psi_0|^2$ in (5.1) gives a contribution proportional to $\ln[(M^2 + m^2)/m^2]$ and so vanishes in the indicated limit. This leaves an integral over partial waves obtained by substituting (3.1) in (5.1). We can obtain a bound on $\psi_l(k, r)$ for all l from (3.26) and (3.52) with b_{l-1}, g_{-l} fixed by (3.46)-(3.47) and (3.49), respectively. These equations and the estimates

$$r^{-l} e^{-\phi(r)} \int_0^r dx\, x^{2l-1}\, e^{2\phi(x)} \leq \frac{a^l}{2l}\, e^{3a||B||/\sqrt{2\pi}},$$

$$\int_0^a dx\, x^{2l-1}\, e^{2\phi(x)} \geq a^{2l}\, e^{-a||B||/\sqrt{\pi}}/2l, \quad (5.2)$$

following from (3.56) give

$$|\psi_l(k, r)| \leq \frac{e^{5a||B||/\sqrt{2\pi}}}{\sqrt{2}\Gamma(W+1)} \left(\frac{ka}{2} \right)^W (1 + \mathrm{O}(k^2)), \quad (5.3)$$

where the $\mathrm{O}(k^2)$ term symbolizes a remainder term that vanishes as $k \to 0$ for all l and that falls off as $k^2/|l|$ for $|l| \gg \Phi/2\pi$. Recall that $W = |l - \Phi/2\pi|$. From (5.3),

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$$\int_0^\infty dt\, e^{-tm^2} \int_0^a dr\, r \int_0^M dk\, k\, e^{-k^2 t} \sum_l |\psi_l(k,r)|^2$$

$$\leq \frac{a^2}{8} e^{5a\|B\|/\sqrt{\pi}} \ln\left(\frac{M^2 + m^2}{m^2}\right) \sum_l \frac{(Ma/2)^{2W}}{\Gamma^2(W+1)}, \tag{5.4}$$

for $Ma \ll 1$, and hence the indicated limit in (5.1) is satisfied. The special case when $l = \Phi/2\pi = 1, 2, \ldots$ is dealt with by (3.48) and (4.7). For these special values of l the integrals on the left-hand side of (5.4) contribute a term of order $M^2 a^2 [(M^2 + m^2) \ln^2(Ma)]^{-1}$, which vanishes in the limit indicated in (5.1)

B. r > a

Equations (3.1), (3.43) and the expansion (3.5) substituted into (5.1) result in the following integral:

$$I = \int_0^\infty dt\, e^{-tm^2} \int_a^\infty dr\, r \int_0^M dk\, k\, e^{-k^2 t} \sum_l \Big[J_W^2(kr) - J_l^2(kr)$$

$$+ J_W(kr)\, Y_W(kr) \sin 2\Delta_l + (Y_W^2(kr) - J_W^2(kr)) \sin^2 \Delta_l \Big]. \tag{5.5}$$

Consider the sum over the first two Bessel functions. Entries 5.7.11.6 of Ref.20 and 6.538.1 of Ref.21 give the result

$$\sum_l [J_W^2(kr) - J_l^2(kr)] = 1/2\, J_f^2(kr) + 1/2\, J_{1-f}^2(kr)$$

$$- \int_{kr}^\infty dt\, t^{-1} \left[f\, J_f^2(t) + (1-f)\, J_{1-f}^2(t) \right].$$

$$\equiv g(kr), \tag{5.6}$$

where $\Phi/2\pi = N + f$, $0 < f < 1$ and $N = 0, 1, \ldots$ Next (5.6) has to be integrated over k following (5.5). For this we apply the weighted mean value theorem [22]: Assume f and g are continuous on $[a, b]$. If f never changes sign on $[a, b]$ then, for some c in $[a, b]$,

$$\int_a^b f(k)\, g(k)\, dk = g(c) \int_a^b f(k)\, dk.$$

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Let $f = k e^{-k^2 t}$ and g equal the right-hand side of (5.6). Then for some μ satisfying $0 < \mu \leq$

M,

$$\int_0^M dk\, k\, e^{-k^2 t} \sum_l (J_W^2 - J_l^2) = (1 - e^{-M^2 t})\, g(\mu r)/2t\,. \tag{5.7}$$

The value $\mu = 0$ is excluded since the energy integral is manifestly r- dependent. The t-integral in (5.5) can be done immediately, resulting in an overall factor of $\ln[(M^2 + m^2)/m^2]$.

It remains to be shown that the integration over r is bounded. The definition of g in (5.6) for large argument gives

$$g(z) = -\frac{\sin \pi f \, \cos 2z}{\pi z} + \frac{(f^2 - f - \frac{1}{4}) \sin \pi f \, \sin 2z}{\pi z^2} + \mathrm{O}\left(\frac{\cos 2z, \sin 2z}{z^3}\right)\,. \tag{5.8}$$

Subsitution of (5.8) into (5.7) and performing the r-integral in (5.5) results in

$$\int_0^\infty dt\, e^{-t m^2} \lim_{L \to \infty} \int_a^L dr\, r \int_0^M dk\, k\, e^{-k^2 t} \sum_l [J_W^2 - J_l^2]$$
$$= \ln\left(\frac{M^2 + m^2}{m^2}\right) \lim_{L \to \infty} \left[-\frac{\sin \pi f}{4\pi\mu^2} \sin 2\mu L + \text{convergent as } L \to \infty\right]\,. \tag{5.9}$$

The leading term, although oscillating, is bounded and that is all that is required to satisfy the limit indicated in (5.1). For the special case when $\Phi/2\pi = 1, 2, ..$ the sum in (5.6) is zero.

Next we consider the $J_W\, Y_W$ terms in (5.5). Since $J_W^2(kz) + Y_W^2(kz)$ is a decreasing function of z for any value of W [23], then for $r \geq a$,

$$Y_W^2(kr) \leq J_W^2(ka) + Y_W^2(ka)\,. \tag{5.10}$$

It follows that

$$\left|\sum_l J_W(kr)\, Y_W(kr) \sin(2\Delta_l)\right| \leq \sum_l |\sin(2\Delta_l)|\, [1 + Y_W^2(ka)]^{1/2}\,, \tag{5.11}$$

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where we used $|J_W(z)| \leq 1$ for $W \geq 0$. From (4.2) together with (4.5), (4.6) and (3.56) one obtains for all l and $\Phi/2\pi \neq 1, 2, \ldots$

$$|\Delta_l| \leq \pi \left(e^{a||B||/\sqrt{\pi}} + 1 \right) \frac{(ka/2)^{2W}}{\Gamma(W)\,\Gamma(W+1)} \left(1 + \mathrm{O}(k^2)\right), \qquad (5.12)$$

where the $\mathrm{O}(k^2)$ terms fall off for large $|l|$ at least like $1/|l|$. Since $ka \ll 1$ and

$$Y_W(ka) \sim -\frac{1}{\pi}\,\Gamma(W)\,(ka/2)^{-W}\,,$$

every term in the series on the left-hand side of (5.11) is bounded by a constant

$$2^{3/2} \left(e^{a||B||/\sqrt{\pi}} + 1 \right) \frac{(Ma/2)^W}{\Gamma(W+1)}\,,$$

for all $r > a$ and $0 \leq k \leq M$. It is, therefore, a uniformly covergent series of continuous functions of k and can be integrated term by term. Applying the weighted mean value theorem again we obtain

$$\int_0^\infty dt\, e^{-tm^2} \lim_{L\to\infty} \int_a^L dr\, r \int_0^M dk\, k \sum_l J_W(kr)\,Y_W(kr)\,\sin 2\Delta_l(k)$$

$$= \frac{1}{2} \ln\left(\frac{M^2 + m^2}{m^2}\right) \lim_{L\to\infty} \int_a^L dr\, r \sum_l J_W(\mu r)\,Y_W(\mu r)\,\sin 2\Delta_l(\mu)\,, \qquad (5.13)$$

for some μ in the interval $0 < \mu \leq M$. For $|l| \gg \Phi/2\pi$, $J_W(\mu r)\,Y_W(\mu r) \sim -(\pi|l|)^{-1}$ which, together with (5.12), implies each term in the series on the right-hand side of (5.13) is dominated by a constant whose l-dependence is $(\mu a/2)^{2|l|}/(|l|!)^2$ for all $r > a$. For all finite $L > a$ it is a uniformly convergent series of continuous functions of r that can be integrated term by term. From entry 5.11.10 of Ref.23,

$$\int_a^L dr\, r\, J_W(\mu r)\,Y_W(\mu r) = \frac{1}{4}\,L^2\,[2J_W(\mu L)\,Y_W(\mu L) - J_{W-1}(\mu L)\,Y_{W+1}(\mu L)$$

$$- J_{W+1}(\mu L)\,Y_{W-1}(\mu L)] - (L \to a)$$

$$\equiv h_l(L) - h_l(a)\,. \qquad (5.14)$$

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There remains $\lim\limits_{L \to \infty} \sum\limits_l h_l(L) \sin 2\Delta_l(\mu)$. For $|l| \gg \Phi/2\pi$,

$$h_l(L) = \frac{|l|}{\pi \mu^2} + \mathrm{O}\left(\frac{1}{|l|}\right), \tag{5.15}$$

which, together with (5.12), implies $h_l(L) \sin 2\Delta_l(\mu)$ is dominated by a term whose l-dependence is $(\mu a/2)^{2|l|}/[(|l| - 1)!]^2$ for all finite L and $\mu a \ll 1$. Therefore the series $\sum\limits_l h_l(L) \sin 2\Delta_l$ is uniformly and absolutely convergent for all finite values of $L > a$. But it does not necessarily converge to a function continuous at the point $L = \infty$ since $h_l(L)$ is not continuous at $L = \infty$. In fact, for fixed l and $\mu L \gg 1$,

$$h_l(L) = -\frac{\sin(2\mu L - \pi W)}{2\pi \mu^2} + \mathrm{O}\left(\frac{1}{L}\right). \tag{5.16}$$

The remedy is now clear. Consider instead the two series

$$\sum_l \left[h_l(L) + \frac{\sin(2\mu L - \pi W)}{2\pi \mu^2}\right] \sin 2\Delta_l$$

$$- \frac{1}{2\pi \mu^2} \sum_l \sin(2\Delta_l) \sin(2\mu L - \pi W).$$

Now the limit $L \to \infty$ and the sum can be interchanged in the first series, giving zero. The second series is bounded for all $L > a$, and so the limit in (5.1) is true for the $J_W Y_W$ terms.

The special case $\Phi/2\pi = 1, 2, \ldots$ requires (4.7) when $l = \Phi/2\pi$. This results in a term $J_0(kr) Y_0(kr)/\ln(ka)$, which is continuous for k on $[0, M]$. Therefore the weighted mean value theorem may be applied again to the k-integral, resulting in an overall factor of $\ln[(M^2 + m^2)/m^2]$. There remains an integration over $r J_0(\mu r) Y_0(\mu r)$ between a and L, giving an oscillating but bounded term $\sin(2\mu L)$ as $L \to \infty$. Again, (5.1) is satisfied for this special term.

Finally, consider the last two terms in (5.5). Note that

$$|Y_W^2(kr) - J_W^2(kr)| \leq Y_W^2(ka) + J_W^2(ka), \tag{5.17}$$

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for $r \geq a$ since $J_W^2(z) + Y_W^2(z)$ is a decreasing function of z [23]. Hence, for $ka \ll 1$, $0 \leq k \leq M$,

$$|Y_W^2(kr) - J_W^2(kr)| \sin^2 \Delta_l \leq \left(e^{a\|B\|/\sqrt{\pi}} + 1\right)^2 \frac{(Ma/2)^{2W}}{\Gamma^2(W+1)} \left(1 + \mathrm{O}((Ma)^2, (Ma)^{2W})\right), \quad (5.18)$$

so that the sum over l of the terms on the left-hand side of (5.18) converges uniformly for all $r \geq a$, $0 \leq k \leq M$. As it is also a sum of continuous functions of k for $0 \leq k \leq M$ it can be integrated term by term over k. Application of the weighted mean value theorem gives

$$\int_0^\infty dt\, e^{-tm^2} \lim_{L \to \infty} \int_a^L dr\, r \int_0^M dk\, k \sum_l [Y_W^2(kr) - J_W^2(kr)] \sin^2 \Delta_l(k)$$

$$= \frac{1}{2} \ln\left(\frac{M^2 + m^2}{m^2}\right) \lim_{L \to \infty} \int_a^L dr\, r \sum_l [Y_W^2(\mu r) - J_W^2(\mu r)] \sin^2 \Delta_l(\mu), \quad (5.19)$$

for some μ in the interval $0 < \mu \leq M$. For $|l| \gg \Phi/2\pi$ each term in the series (5.19) is dominated by a r- independent constant whose l-dependence is $(\mu a/2)^{2|l|}/(|l|!)^2$, for $r \geq a$ and $\mu a \ll 1$. It is therefore a uniformly convergent series of continuous functions for all finite $L > a$ that can be integrated term by term. Entry 5.54.2 in Ref.21 gives

$$\int_a^L dr\, r\, [Y_W^2(\mu r) - J_W^2(\mu r)] = \frac{L^2}{2} \left[Y_W^2(\mu L) - Y_{w-1}(\mu L)\, Y_{w+1}(\mu L) \right.$$

$$\left. - J_W^2(\mu L) + J_{W+1}(\mu L)\, J_{w-1}(\mu L) \right] - (L \to a)$$

$$\equiv k_l(L) - k_l(a). \quad (5.20)$$

Next, consider $\lim_{L \to \infty} \sum_l k_l(L) \sin^2 \Delta_l(\mu)$. For $\mu L \gg 1$,

$$k_l(L) = \frac{\cos(2\mu L - \pi W)}{\pi \mu^2} + \mathrm{O}\left(\frac{1}{L}\right), \quad (5.21)$$

and hence consider the series

$$\sum_l \left[k_l(L) - \frac{\cos(2\mu L - \pi W)}{\pi \mu^2} \right] \sin^2 \Delta_l$$

$$+ \quad \frac{1}{\pi \mu^2} \sum_l \cos(2\mu L - \pi W) \sin^2 \Delta_l.$$

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The first series is a sum of continuous functions for all $L > a$. It is also a uniformly and absolutely convergent series for $L > a$, first because $|J_w| \leq 1$ and, secondly, the combinations $Y_w^2 \sin^2 \Delta_l$ and $Y_{w-1} Y_{w+1} \sin^2 \Delta_l$ are dominated by L-independent constants whose l-dependence is $(\mu a2)^{2|l|}/(|l|!)^2$ for $|l| \gg \Phi/2\pi$. Hence the limit $L \to \infty$ and the sum can be interchanged in the first series, giving zero. The second series is bounded for all $L > a$, verifying (5.1) for the last series of terms in (5.5).

The special case $\Phi/2\pi = 1, 2 \ldots$ is dealt with in the same way as in the case of $J_w Y_w \sin \Delta_l$ and gives a contribution that vanishes in the limit indicated in (5.1). Thus, (1.1) is demonstrated.

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REFERENCES

[1] M. P. Fry, Phys. Rev. D **54**, 6444 (1996).

[2] J. Schwinger, Phys. Rev. **128**, 2425 (1962).

[3] M. P. Fry, Phys. Rev. D **51**, 810 (1995).

[4] J. E. Avron and R. Seiler, Phys. Rev. Lett. **42**, 931 (1979).

[5] E. Seiler, Phys. Rev. D **22**, 2412 (1980).

[6] B. Simon, Adv. Math. **24**, 244 (1977).

[7] M. P. Fry, Phys. Rev. D **45**, 682 (1992).

[8] W. Heisenberg and H. Euler, Z. Phys. **98** , 714 (1936);

V. Weisskopf, K. Dan. Vidensk. Selsk. Mat.-fys. Medd. **14**, No. 6 (1936).

[9] J. Schwinger, Phys. Rev. **82**, 664 (1951).

[10] G. Dunne and T. Hall, Phys. Lett. B **419**, 322 (1998).

[11] H. W. Lee, P. Y. Pac, and H. K. Shin, Phys. Rev. D **40**, 4202 (1989);

V. P. Gusynin and I. A. Shovkovy, Can. J. Phys. **74**, 282 (1996).

[12] M. P. Fry, Phys. Rev. D **55**, 968 (1997).

[13] M. P. Fry, Phys. Rev. D **56**, E6714 (1997).

[14] C. R. Gattringer, I. Hip and C. B. Lang, Nucl. Phys. **B508**, 329, (1997).

[15] M. P. Fry, Phys. Rev. D **47**, 2629 (1993).

[16] Y. Aharonov and A. Casher, Phys. Rev. A **19**, 2461 (1979).

[17] R. Musto, L. O'Raifeartaigh and A. Wipf, Phys. Lett. B **175**, 433 (1986).

[18] K. Chandon, N. N. Khuri, A. Martin and T. T. Wu, Phys. Rev. D **58**, 025014 (1998).

[19] H. P. Stapp, T.J. Ypsilantis and N. Metropolis, Phys. Rev. **105**, 302 (1957).

[20] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, *Integrals and Series* (Gordon and Breach, New York, 1988), Vol. 2.

[21] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, New York, 1965).

[22] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, 1958), 4th ed.

[23] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, 1958), 2nd ed.