School of Mathematics

Module MA2318 — Elementary projective and algebraic geometry 2011–12
(SF & JS Mathematics)

Lecturer: Prof. R. Tange

Requirements/prerequisites: MA1111/1212, MA2215, MA2321, MA2325

Duration: Hilary term, 11 weeks

Number of lectures per week: 3 lectures including tutorials per week

Assessment: Regular assignments.

ECTS credits: 5

End-of-year Examination: 2 hour examination in Trinity term.

Description:

Textbook: Miles Reid *Undergraduate Algebraic Geometry*, London Mathematical Society Student Texts.

See `http://www.maths.tcd.ie/~rtange/teaching/algebraic_geometry/algebraic_geometry.html` for more information.

Tentative syllabus: Algebraic curves. Conics (or quadrics), their euclidean and affine classification over the fields of real and complex numbers. Projective plane, homogeneous coordinates, projective transformations. Lines in projective plane. Projective classification of conics. Parametrization of nondegenerate conics.

Homogeneous polynomials or forms. Roots of polynomials and their multiplicities. Bezout's Theorem, proof when one of the curve is a line or a quadric. Factorization of forms vanishing along lines and nondegenerate conics. Spaces of d-forms vanishing at certain points and their dimensions. Applications to quadrics passing through 5 points and cubics passing through 9 points. Pascal's Theorem.

Nodal and cuspidal cubics, their parametrization. Tangent lines. Group law on a cubic. Riemann surfaces and genus.

Affine algebraic sets, their ideals. Noetherian rings. Hilbert Basis Theorem. Algebraic sets defined by ideals, their properties. Zariski topology. Termination of descending chains of algebraic sets. Irreducible algebraic sets, their relation with prime ideals. Unique decomposition of algebraic set into irreducible components.

Nullstellensatz (Hilber Zero Theorem) and Weak Nullstellensatz. Proof of the Nullstellensatz assuming Weak Nullstellensatz.

Polynomial functions on affine algebraic sets. Coordinate ring. Polynomials maps between affine algebraic sets. Relation between polynomial maps and coordinate ring homomorphisms. Polynomial isomorphisms. Affine varieties. Rational functions on affine algebraic sets. Regular points of rational functions. Rational maps. Dominant maps.

Projective algebraic sets. Homogeneous ideals and correspondence between them and projective algebraic sets. The affine cone over a projective algebraic set. Rational functions on projective algebraic sets, rational maps between them. Regular points of rational functions and maps. Morphisms and isomorphisms. Segre embedding of the product of two projective spaces into another projective space. Finite unions, finite products, and arbitrary intersections of projective algebraic sets are again projective algebraic.

Learning Outcomes: On successful completion of this module, students will be able to:

- Give the affine classification of quadrics over $\mathbb{R}$ or $\mathbb{C}$ (you may assume the degree is 2, not < 2). Give the projective classification of quadrics over $\mathbb{R}$ or $\mathbb{C}$.
- State Bezout's Theorem (the multiplicities do not have to be defined) and prove it in the case one of the curves is a line or a nondegenerate quadric using the standard parameterisation of a nondegenerate quadric and the result on factorisation of homogeneous polynomials in two variables. Use Bezout's Theorem to show that there exists a unique quadric passing trough 5 distinct points in $\mathbb{P}^2$ no 4 of which lie on a line.
- Determine the projectivisation, the point(s) at infinity, the singular (= not smooth) points of a plane curve. Check that a point of a curve is an inflection point.
- Explain what is meant by an affine or a projective algebraic set, when such a set is irreducible and what is meant by the irreducible components of such a set. Define what a (homogeneous) ideal is and describe the variety-ideal correspondence. State Hilbert's Basis Theorem and Hilbert's (Strong) Nullstellensatz.
- Give the construction of the field of rational functions on an irreducible affine or projective algebraic set. Explain what the domain of a rational function is. Explain what a rational map or a morphism is between two affine or projective algebraic sets. Explain what is meant by the Segre embedding and indicate how it can be used to show that the product of two projective algebraic sets is again a projective algebraic set.

April 1, 2012