School of Mathematics

Module CS4002 — Category theory 2009-10
(JS & SS Mathematics, JS & SS Two-subject Moderatorship)

Lecturer: Dr. Arthur Hughes (Computer Science)

Requirements/prerequisites:

Duration: Hilary term, 11 weeks

Number of lectures per week: 2 lectures plus 1 tutorial per week

Assessment:

ECTS credits: 5

End-of-year Examination: 2 hour examination in Trinity term.

Description: What is category theory? As a first approximation, one could say that category theory is the mathematical study of (abstract) algebras of functions. Just as group theory is the abstraction of the idea of a system of permutations of a set or symmetries of a geometric object, category theory arises from the idea of a system of functions among some objects.

We think of the composition $g \circ f$ ($f; g$ often used in CS) as a sort of "product" of the functions f and g, and consider abstract "algebras" of the sort arising from collections of functions. A category is just such an "algebra", consisting of objects $A, B, C, \ldots$ and arrows $f: A \to B$, $g: B \to C$, $\ldots$, that are closed under composition and satisfy certain conditions typical of the composition of functions[1].

- Categories – functions of sets, definition of a category, examples of categories, isomorphisms, constructions on categories, free categories, foundations: large, small, and locally small.
- Abstract structures – epis and monos, initial and terminal objects, generalized elements, sections and retractions, products, examples of products, categories with products, Hom-sets.
- Duality – the duality principle, coproducts, equalizers, coequalizers.
- Groups and categories – groups in a category, the category of groups, groups as categories, finitely presented categories.
- Limits and colimits – subobjects, pullbacks, properties of pullbacks, limits, preservation of limits, colimits.
- Exponentials – exponential in a category, cartesian closed categories, Heyting algebras, equational definition, λ-calculus.
- Functors and naturality – category of categories, representable structure, stone duality, naturality, examples of natural transformations, exponentials of categories, functor categories, equivalence of categories, examples of equivalence.

[1] This description is taken from S. Awodey's (2006) introduction section of the first chapter of his book.

- Categories of diagrams – Set-valued functor categories, the Yoneda embedding, the Yoneda Lemma, applications of the Yoneda Lemma, Limits in categories of diagrams, colimits in categories of diagrams, exponentials in categories of diagrams, Topoi.
- Adjoints – preliminary definition, Hom-set definition, examples of adjoints, order adjoints, quantifiers as adjoints, RAPL, locally cartesian closed categories, adjoint functor theorem.

Bibliography: Awodey, S. (2006). Category Theory. Oxford Logic Guides 49, Oxford University Press.

Learning Outcomes: On sucessful completion of this module, students will be able to explain why:

- Many objects of interest in mathematics congregate in concrete categories.
- Many objects of interest to mathematicians are themselves small categories.
- Many objects of interest to mathematicians may be viewed as functors from small categories to the category of **Sets**.
- Many important concepts in mathematics arise as adjoints, right or left, to previously known functors.
- Many equivalence and duality theorems in mathematics arise as an equivalence of fixed subcategories induced by a pair of adjoint functors.
- Many categories of interest are the Eilenberg-Moore or Kleisli categories of monads on familiar categories2
- Many data types of interest to computing science are algebras for endofunctors.
- Many process of interest to computing science are coalgebras for endofunctors.

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