School of Mathematics

Course 311 — Abstract Algebra 2007-08
(Optional JS & SS Mathematics, SS Two-subject Moderatorship)

Lecturer: Dr. D.R. Wilkins

Requirements/prerequisites:

Duration: 18 weeks

Number of lectures per week: 3

Assessment:

End-of-year Examination: One 3-hour examination

Description: See `http://www.maths.tcd.ie/~dwilkins/Courses/311/` for lecture notes and more detailed information.

Chapter 1: Topics in Group Theory. This chapter begins with a brief review of basic group theory. This is followed by the statement and proof of the *Sylow Theorems*. These theorems are then applied in order to prove that all simple groups of order less than 60 are cyclic. The chapter concludes with a discussion of solvability, a concept that is of key importance in *Galois Theory*.

Chapter 2: Rings and Polynomials. This chapter begins with a brief review of the definitions and basic properties of rings, integral domains and fields. This is followed by a detailed discussion of the basic properties of rings of polynomials in one variable with coefficients in a field. These results are very important for the development of Galois Theory. We prove *Gauss's Lemma*, which ensures that a polynomial whose coefficients are all integers can be factored as a product of polynomials of lower degree with rational coefficients if and only it can be factored as a product of polynomials of lower degree with integer coefficients. Another important result concerning irreducibility discussed here is *Eisenstein's Irreducibility Criterion*.

Chapter 3: Introduction to Galois Theory. This chapter concerns the application of concepts and results from group theory, ring theory and field theory to the study of polynomial equations. Here one seeks to express the roots of a polynomial as functions of its coefficients. To any polynomial is associated a finite group, referred to as the *Galois group* of the polynomial. The roots of a polynomial can be expressed in terms of its coefficients by means of algebraic formulae involving only the operations of addition, subtraction, multiplication, division and the extraction of nth roots if and only if the Galois group of the polynomial is 'solvable'. This result can be used to prove that there cannot exist any algebraic formula for the roots of a general quintic polynomial that involves only the algebraic operations of addition, subtraction, multiplication, division and the extraction of nth roots.

Chapter 4: Modules and Commutative Rings. This chapter covers some of the basic concepts and results of *commutative algebra*. We introduce the concept of a *module*

over a commutative ring. We study Noetherian rings and modules: a *Noetherian ring* is a unital commutative ring in which every ideal is finitely generated: a *Noetherian module* is a module over a unital commutative ring in which every submodule is finitely generated. We shall prove *Hilbert's Basis Theorem*, which ensures that any ring of polynomials with coefficients in a Noetherian ring is itself a Noetherian ring. The chapter concludes with a proof of a classification theorem for finitely generated modules over a principal ideal domain.

Chapter 5: Algebraic Varieties and Hilbert's Nullstellensatz. This chapter provides an introduction to basic concepts of algebraic geometry, which is concerned with the study of sets of common zeros of collections of polynomials in several indeterminates. Any collection of polynomials in n indeterminates (or variables) with coefficients in a field K determines a corresponding subset of K^n (the set of all ordered n-tuples of elements of K). This subset is the set of common zeros of the polynomials in the collection, and sets of this form are referred to as *algebraic sets*. We show that there is a well-defined topology on the set K^n, referred to as the *Zariski topology*, whose closed sets are the algebraic sets in K^n. We also examine the correspondence between algebraic sets in K^n and ideals of the corresponding polynomial ring. The deepest theorem in this section of the course is *Hilbert's Nullstellensatz*. The *Weak Nullstellensatz* is essentially a generalization of the Fundamental Theorem of Algebra. It asserts that the set of common zeros of a collection of polynomials in n indeterminates with coefficients in an algebraically closed field K is non-empty if and only if that collection generates a proper ideal of the corresponding polynomial ring. The *Strong Nullstellensatz* establishes a one-to-one correspondence between algebraic sets and radical ideals of the polynomial ring, in the case where the field of coefficients is algebraically closed.

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