Or, the Hausdorff dimension of space-time could be less than four, if we are embedded in strictly 4-dimensional space-time, all the points of that space-time are not accessible to us. Werner Heisenberg, for example, had the idea that space-time might be granular - that there might be steps in space and time.

But how to measure the Hausdorff dimension of the world? Zeilinger and Svozil have an answer to that. In the world of high-energy particle physics, particles are spread out in the "wave-particles" of quantum mechanics - and as such they probe every cranny of space-time. By reinterpreting certain very precise measurements on the magnetic moment of the electron as estimates of the Hausdorff dimension, Zeilinger and Svozil come out with their value of 3.99999947.

This is so close to 4 that it may be questioned whether the result is real. But it may just indicate the beginnings of the detection of granularity in space-time. For the measurements are limited by the quantum "wavelength" of the particles used. Experimentally, the waves cannot penetrate "crannies" or discontinuities shorter than their wavelength. And wavelength decreases with energy. The energy of the magnetic moment experiment was low, and so was perhaps like the geographer pacing the coastline with a long step.

The requirement must be now for higher energy tests (though the higher the energy, the more difficult precise measurements become). Ideally, the tests should be done at energies approaching the levels where theorists expect the 10 or 12 dimensional twists and turns to appear. Sadly, however, such energy regions are astronomically high and probably inaccessible, and we may have to content ourselves with Zeilinger and Svozil's 3.99999947.

Reference: *Phys. Rev. Lett.*, Vol. 54 P, 2553 (1985).

EIGENVECTORS AND DIAGONALIZATION

P.D. Barry

Can we motivate or encounter naturally eigenvectors of matrices, especially as occurring in the diagonalization of a quadratic form, or must they be produced apparently out of the blue?

When deriving the equation of an ellipse E in standard form we select the axes of coordinates to be along the major and minor axes of the ellipse. Thus if

$$ax^2 + 2hxy + by^2 = 1 \qquad \text{(i)}$$

represents an ellipse with centre at the origin O, to convert it to standard form we need to map its major and minor axes onto the x' and y' axes. If $P_0 = (x_0, y_0) \neq 0$ is a point on either the major or the minor axis, we then seek a linear transformation

$$\begin{aligned} x &= a_{1,1}x' + a_{1,2}y' \\ y &= a_{2,1}x' + a_{2,2}y' \end{aligned} \qquad \text{(ii)}$$

which maps the line OP_0 to the x'-axis; in matrix form (ii) is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}. \qquad \text{(iii)}$$

As then

$$\begin{aligned} x_0 &= a_{1,1}x'_0 \\ y_0 &= a_{2,1}x'_0 \end{aligned}$$

for some $x'_0 \neq 0$, we have

$$a_{1,1} = jx_0, \quad a_{2,1} = jy_0$$

for some $j \neq 0$. This gives us the two coefficients in the first column of the two-by-two transformation matrix in (iii).

We note too that if $Q_0 = (-y_0, x_0)$, then OQ_0 is perpendicular to OP_0 and so will lie along the other axis of the ellipse. Thus if our linear transformation is to preserve perpendicularity, as a rotation or an axial symmetry would, we will have the line OQ_0 mapping to the y'-axis. Then

$$-y_0 = a_{1,2}y_0'$$

$$x_0 = a_{2,2}y_0'$$

for some $y_0' \neq 0$ and so we have

$$a_{1,2} = -ky_0, \quad a_{2,2} = kx_0$$

for some $k \neq 0$. This gives us the two coefficients in the second column of the transformation matrix in (iii), which thus has been shown to have the form

$$\begin{pmatrix} jx_0 & -ky_0 \\ jy_0 & kx_0 \end{pmatrix}. \qquad \text{(iv)}$$

It still remains to locate P_0, and we recall the manifest property of a point at the end of a diameter of E that it maximizes or minimizes the distance of a point P on E from the centre O. We thus seek a point at which $x^2 + y^2$ is maximized or minimized, subject to the condition (i). Recalling the method of Lagrange multipliers, we wish to locate a point $(x,y) = (x_0,y_0) \neq (0,0)$ at which

$$F(x,y) = x^2 + y^2 + \mu(ax^2 + 2hxy + by^2 - 1)$$

satisfies

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0.$$

This gives

$$\mu(ax + hy) + x = 0,$$

$$\mu(hx + by) + y = 0,$$

that is,

$$ax + hy = \lambda x,$$

$$hx + by = \lambda y,$$

where $\lambda = -1/\mu$, at the point $(x,y) = (x_0,y_0) \neq (0,0)$. Thus

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

is an eigenvector of the matrix

$$\begin{pmatrix} a & h \\ h & b \end{pmatrix}, \qquad \text{(v)}$$

corresponding to the eigenvalue λ. This eigenvector can now be inserted in the first column in (iv) and so in the first column in the transformation matrix in (iii), and a similar conclusion for

$$\begin{pmatrix} -y_0 \\ x_0 \end{pmatrix}$$

gives the second column in (iv) and (iii).

We have hitherto presented this as converting the expression on the left-hand side of (i) to standard form. In matrix notation it can be written as

$$(x,y)\begin{pmatrix} a & h \\ h & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

and this exhibits the matrix (v) for which we have been encountering eigenvectors. Our work can also be presented as diagonalizing this matrix.

The above algebra and geometry can surely be asked of anyone at this level of linear algebra, but perhaps Lagrange multipliers seem a late topic in some unappetising course on calculus. Supposing more elementarily that E has been parameterized, with variable t, to find P we need

$$\tfrac{1}{2}\frac{d}{dt}(x^2 + y^2) = x\frac{dx}{dt} + y\frac{dy}{dt} = 0 \qquad \text{(vi)}$$

where, on differentiating (i),

$$(ax + hy)\frac{dx}{dt} + (hx + by)\frac{dy}{dt} = 0. \qquad \text{(vii)}$$

Now (vi) and (vii) have a non-trivial solution in dx/dt, dy/dt if

$$\det\begin{pmatrix} ax + hy & hx + by \\ x & y \end{pmatrix} = 0,$$

which is a condition that

$$\begin{aligned} 1(ax + hy) - \lambda x &= 0 \\ 1(hx + by) - \lambda y &= 0 \end{aligned} \qquad \text{(viii)}$$

have a solution in λ. In (vi) and (vii) we are looking for a solution (x,y) which is not $(0,0)$. Then (viii) brings in the eigenvectors of (v), so with some loss of immediacy we can omit Lagrange multipliers and still reach our objective.

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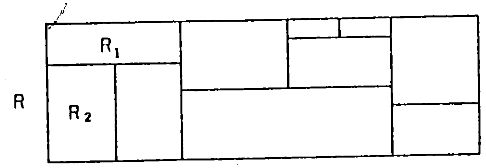
PROBLEM PAGE

The first problem this time is 'going around' at the moment. I heard it from several different people within a period of a week, and it has a remarkable solution.

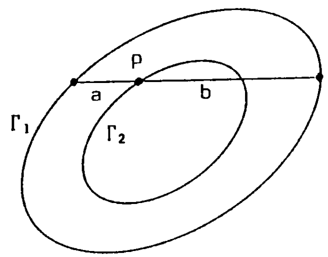
1. A rectangle R is partitioned into a finite number of rectangles $R_1, R_2, \ldots, R_n$, each of which has the property that at least one side is of integer length.



Prove that R has the same property.

The next problem came from Jim Clunie who learnt it from Tom Willmore.

2. A rod moves so that its endpoints lie on a convex curve Γ_1 in $\mathbb{R}^2$ and a point P, which divides the rod into lengths a and b, then describes a closed curve Γ_2.



Prove that the region lying between Γ_1 and Γ_2 has area πab.