A NOTE ON A COMMENT OF AXLER AND SHAPIRO

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In a recent paper of Axler and Shapiro [1], the following comment occurs:

> "Purely C^* theorems should have C^* proofs, not proofs that rely on Hilbert space ... . One of our favorite examples of this principle is the following proposition: If B is a C^*-algebra and $T \in B$ is left invertible, then T^*T is invertible. This is fairly easy to prove for operators on Hilbert space (and thus we get a proof for arbitrary C^*-algebras ...). A purely C^* proof is more difficult (and more interesting) to discover, but once you find the purely C^* proof you are likely to be convinced that it's the right proof."

Axler and Shapiro leave the carrot dangling there, and the temptation to look for a C^* proof is compelling. We found two quite different ones which form the subject of the present note.

For the non-specialist, we start with two definitions and an explanation.

Firstly, the definitions: A unital Banach algebra B with involution $*$ is called a B^*-algebra if the property $||x^*x|| = ||x||^2$ holds for each $x \in B$. If H is a Hilbert space and $\mathcal{B}(H)$ is the algebra of bounded linear operators on H, and if A is a closed subalgebra of $\mathcal{B}(H)$ with the property that $T \in A$ implies $T^* \in A$, then A is called a C^*-algebra.

Secondly, the explanation: That every C^*-algebra is a B^*-algebra is elementary. That the converse is also true, i.e. that every B^*-algebra can be represented as a C^*-algebra of operators on some Hilbert space, is a much more difficult theorem. Many theorems about operators on Hilbert space are automatically true for elements of B^*-algebras because of this representation, and the theorem in question is one of them. We should also mention that some analysts dispense altogether with the term B^*-algebra, but that we prefer to retain it when we are pretending we do not know that all B^*-algebras are, in effect, C^*-algebras. The task in hand, then, is to find a B^* proof of the above-mentioned theorem.

Let us first examine the special case for operators on Hilbert space. A proof using polar decomposition is the only one which springs immediately to mind:

TAKE 1. Let $T \in \mathcal{B}(H)$ be left invertible.
Let $T = UP$ be the polar decomposition of T, where U is a partial isometry and P is the unique positive operator such that $P^2 = T^*T$. Since T is left invertible, so is P, and so also T^*T. Being hermitian, T^*T is therefore invertible.

Short and sweet, but it uses polar decomposition. The existence of a positive square root for each positive operator is usually presented as a consequence of the spectral theorem, and to invoke such a theorem in the present context is rather like using a pile driver to crack an almond. Whilst it is true that square roots and polar decomposition can be presented without reference to the spectral theorem (see Halmos [3], p.64), there is still too much work involved for our liking; in any case the proof, as it stands, will not go over to B^*-algebras since, although square roots can be manufactured in them, polar decomposition is not always possible.

We soon found the following elementary operator proof:

TAKE 2. Let $T \in \mathcal{B}(H)$ be left invertible.
Then T has closed range, since $ST = I$ and $Tx_n \to y \Rightarrow y = TSy$.
Also T^* is right invertible, so $T^*(H) = H$.
Therefore $T^*T(H) = T^*(\mathcal{N}(T^*)^\perp) = T^*(H) = H$ where $\mathcal{N}$

stands for null space. Whence T^*T is right invertible, and, being hermitian, is invertible.

We are quite satisfied with that proof. Moreover, although it is by no means obvious, there is a way of mimicking it to produce a B* proof. (Actually, we discovered the B* argument first.) The proof requires the following B* folk lemma (see, for example, Goodearl [2], p. 148).

LEMMA A. Let A be a B*-algebra and $q = q^2 \in A$. Then there exists $p = p^* = p^2 \in A$ such that $pA = qA$.

<u>Proof</u>: Firstly, $q - q^*$ is normal, so that the B*-subalgebra of A generated by $q - q^*$ is commutative and can be represented as the algebra B of continuous complex functions on some compact Hausdorff space. Suppose $q - q^*$ corresponds to $g \in B$. The range of $g\bar{g}$, and therefore also $\sigma_A((q - q^*)(q^* - q))$, is contained in the non-negative real axis.

Put $x = 1 + (q - q^*)(q^* - q)$. Then $x = x^*$ is invertible.

Also $qx = qq^*q = xq$, so each of x and x^{-1} commutes with both q and q^*.

Put $p = qq^*x^{-1}$.

Then $p^* = x^{-1}qq^* = p$.

and $p^2 = x^{-1}qq^*qq^*x^{-1} = x^{-1}xqq^*x^{-1} = p$.

Also $qp = p$ and $pq = qq^*x^{-1}q = qq^*qx^{-1} = qxx^{-1} = q$.

Hence $pA = qA$.

TAKE 3. Let A be a B*-algebra and let $x \in A$ be left invertible. Suppose $yx = 1$; then $xy = q$ is an idempotent. Let $p = p^* = p^2 \in A$ be such that $qA = pA$.

Then $xA = xyxA \subseteq xyA \subseteq xA$, whence $xA = pA$, $x = px$, and $x^*p = x^*$; so $x^*xA = x^*pA = x^*A = A$, yielding x^*x invertible.

That's easy when you see it, but it could take a while to find. Perhaps it is similar to the proof which Axler and Shapiro describe as "difficult to discover". The lemma, in particular, involves some trickery; it also involves a highly non-trivial representation theorem. Our second proof, in contrast, is surprisingly elementary, no less so than Take 2 above; neither was it difficult to discover!

Firstly, we remark that there is a well-known elementary proof of the fact that every hermitian element of a B*-algebra has real spectrum. For the sake of completeness we include it here:

LEMMA B. Let A be a B*-algebra and let $x = x^* \in A$. Then $\sigma(x) \subset R$.

<u>Proof</u>: Suppose $a, b \in R$ and $a + ib \in \sigma(x)$.

Then $a + ib + ic \in \sigma(x + ic)$ for every $c \in R$.

So
$$a^2 + (b + c)^2 = |a + ib + ic|^2 \le ||x + ic||^2$$
$$= ||(x + ic)(x - ic)|| \quad \text{(B* property)}$$
$$= ||x^2 + c^2||$$
$$\le ||x^2|| + c^2.$$

Hence $a^2 + b^2 + 2bc \le ||x^2||$ for all $c \in R$.

It follows that $b = 0$.

Whereas that lemma uses the lovely classical trick of the dummy variable, our proof of the theorem uses no trick, only the elementary fact that the boundary of the set of invertible elements of a Banach algebra is contained in the set of two-sided topological divisors of zero.

TAKE 4. Let A be a B*-algebra and $x \in A$.

σ_ℓ will denote left spectrum and $\partial\sigma$ boundary of spectrum.

$$0 \in \sigma(x^*x) \Rightarrow 0 \in \partial\sigma(x^*x) \qquad \text{(by Lemma B)}$$

$$\Rightarrow \exists\ (z_n)_{n\in N} \subset A, \text{ with } ||z_n||=1\ (n \in N)$$

$$\text{such that } x^*xz_n \to 0.$$

$$\Rightarrow (xz_n)^*xz_n \to 0$$

$$\Rightarrow xz_n \to 0 \qquad \text{(by B* property)}$$

$$\Rightarrow 0 \in \sigma_\ell(x).$$

Well, you couldn't get much easier than that. We are tempted to say that it's "the right proof", but there may well be a dozen others with equal claim. Who knows?

References

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PLAYING THE NUMBERS GAME IN NEW YORK

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When I arrived in New York I was struck by the curious phenomenon of lines of people outside very small newsagents and tobacconists. It was widespread. All over Manhattan from 216th St to 14th St, from west side to east side there were snake lines of people on the pavement. They were Hispanics, blacks, whites and Chinese. They ranged from street people and bag ladies to white-coated MDs and business executives. Were they lining up to be rubber-stamped? Did I need some more identification other than my two university IDs, my social security number, my telephone and computer numbers, bank numbers and alien card number? I then saw the magic word - Lotto. Ah! The numbers game I thought. Run by the Mafia I thought. But what about those MDs? Over the next couple of days I was advised by bus drivers, at the hardware store and by my doorman to go out and buy my Lotto number. Surely, I said, not braving to reveal my ignorance. Every four-year old in New York obviously knew all about it. Strangers called to me in the street - got your Lotto number yet?

I enquired from my colleagues. A $41 million prize had accumulated and the draw was to be in two days time - August 23rd. Some of my colleagues were also buying tickets. A ststistician went on TV to declare to the public "the bigger the prize, the bigger the payoff!" I could not believe my ears. On the average I thought ..., in the long run. The front page of the *New York Times*, August 22, revealed all.

The New York Lotto 48

Select any combination of six numbers between 1 and 48.
Enter as often as you like. Minimum purchase 2 tickets.
Tickets are 50 cents each. You win: