INTEGRAL MEANS OF UNIVALENT FUNCTIONS - A FRAGMENT

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Introduction

A notable event in the theory of the class S of normalised univalent functions on the open unit disc U occurred last summer (1984) when de Branges [3] settled the long outstanding conjecture of Bieberbach [2], to the effect that if

$$f(z) = z + \sum_2^\infty a_n z^n, \quad z \in U,$$

belongs to S, then

$$|a_n| \le n, \text{ for } n = 2,3, \ldots, \qquad (B)$$

with equality holding for one n, and so for all n, if and only if f is of the form $f(z) = k(e^{i\theta}z)$ for some real θ, where k denotes Koebe's function, i.e.

$$k(z) = z/(1-z)^2 = \sum_1^\infty nz^n, \; z \in U.$$

Not only did de Branges settle this conjecture, but he settled Robertson's conjecture [8] for the coefficients of *odd* functions in S, which was known to imply Bieberbach's, and a still stronger conjecture - Milin's [7] - into the bargain. We recall these: given f in S write

$$f(z) = z\exp(2\sum_1^\infty \gamma_n z^n), \quad z \in U,$$

and define the odd univalent function g in S as follows:

$$g(z) = \sqrt{f(z^2)} = z\exp(\sum_1^\infty \gamma_n z^{2n}) = \sum_1^\infty c_n z^n.$$

Robertson conjectured that

$$\sum_0^n |c_{2k+1}|^2 \le n + 1, \quad n = 0,1,2, \ldots \qquad (R)$$

Millin conjectured that, if

$$\Delta_n = \sum_1^n (k|\gamma_k|^2 - 1/k), \quad n = 1,2, \ldots$$

then

$$\sum_1^n (n+1-k)(k|\gamma_k|^2 - 1/k)_1^n = \sum_1^n \Delta_j \le 0, \quad n = 1,2, \ldots \qquad (M)$$

Since (in the notation just introduced) the inequalities that follow were known to hold, Bieberbach's conjecture was affirmed once Robertson's was, and Robertson's once Milin's was. Thanks to de Branges, who settled (M) affirmatively, all three are true.

$$|a_n| \le \sum_0^{n-1} |c_{2k+1}|^2, \quad n = 2,3, \ldots.$$

$$\sum_0^n |c_{2k+1}|^2 \le (n+1)\exp(\sum_1^n \Delta_j/(n+1)), \quad n = 1,2, \ldots \qquad (ML)$$

The first of these follows by expressing the coefficients of $f(z^2)$ in terms of those of $(g(z))^2$ and applying Schwarz's inequality. The second lies deeper, and is a special case of one of the celebrated Milin-Lebedev inequalities [7], which we will enunciate shortly.

The First Integral Mean

Given an analytic function h on U, and $0 < p < \infty$, we write

$$I_p(r,h) = \int_0^{2\pi} |h(re^{i\theta})|^p d\theta, \quad 0 \le r < 1.$$

In a major assault on the Bieberbach conjecture, Littlewood [6] showed in 1925 that

$$\sup\{I_1(r,f) : f \in \mathcal{S}\} \le r/(1-r), \quad 0 \le r < 1, \qquad (L)$$

and deduced from this that $|a_n| < en$, for $n = 2,3,\ldots$

Almost fifty years were to elapse before Baernstein succeeded in sharpening inequality (L): amongst other remarkable things he showed in [1] that, for any $p > 0$,

$$\sup\{I_p(r,f) : f \in \mathcal{S}\} = I_p(r,k), \quad 0 \le r < 1. \qquad (B_p)$$

In particular, then, (B_1) is equivalent to the statement that

$$I_1(r,f) \le r/(1-r^2), \quad 0 \le r < 1,$$

if $f \in \mathcal{S}$.

Here we point out that the latter inequality is a simple consequence of (R), a fact which appears to have gone unnoticed until now. To see this, let $f \in \mathcal{S}$. Keeping the notation as before, and bearing in mind that g is odd, we have, for $0 \le r < 1$,

$$\begin{aligned} I_1(r^2,f) &= \int_0^{2\pi} |f(r^2e^{2i\theta})|\,d\theta = \int_0^{2\pi} |g(re^{i\theta})|^2\,d\theta \\ &= 2\pi\sum_1^\infty |c_n|^2 r^{2n} = 2\pi r^2\sum_0^\infty |c_{2k+1}|^2 r^{4k} \\ &= 2\pi r^2(1-r^4)\sum_0^\infty\Big(\sum_0^k |c_{2j+1}|^2\Big) r^{4k} \\ &\le 2\pi r^2(1-r^4)\sum_0^\infty (k+1) r^{4k} \\ &= 2\pi r^2/(1-r^4) \\ &= I_1(r^2,k), \end{aligned}$$

using (R). It is clear that equality holds for some r in [0],1) if and only if f is the Koebe function composed possibly [wi]th a rotation.

[Th]e Integral Means for $p \ge 2$

Bieberbach's inequality (B) coupled with Parseval's iden[ti]ty tells us that

$$\sup\{I_2(r,f) : f \in \mathcal{S}\} = I_2(r,k), \quad 0 \le r < 1.$$

What about the other means? Can (B_p) be deduced from [co]efficient inequalities for other values of p as well? In [th]e remainder of this article we will answer these questions [af]firmatively for the means I_p with $p > 2$. The approach is [th]e same as the one adopted in the previous section: we first [de]rive sharp coefficient estimates for the auxiliary function $(f(z)/z)^{p/2}$; these are provided by the coefficients of $(k(z)/z)^{p/2}$. Hayman, who had often raised this question at [re]cent conferences, announced this result in the course of his [le]cture on the Fitzgerald-Pommerenke version [4] of de Branges' [pr]oof of (B) at the One-Day Function Theory Conference in [Li]verpool in September, 1984; but gave no indication of the [pr]oof. We will show that it is a consequence of (M) and some [mi]nor adaptations of the general Milin-Lebedev inequalities, [wh]ich we proceed to state.

Let

$$(k(z)/z)^{p/2} = (1-z)^{-p} = \sum_1^\infty d_n(p) z^n, \qquad z \in U,$$

[an]d set

$$(f(z)/z)^{p/2} = \exp\Big(p\sum_1^\infty \gamma_n z^n\Big) = \sum_1^\infty a_n(p) z^n,$$

[wh]ere here and from now on $f \in \mathcal{S}$.

Then (see inequality (2.37) on p. 37 of [7]) for every [n] ≥ 1 and any $p > 0$

$$|a_n(p)| \le d_n(p)\exp\Big(p\sum_{k=1}^n d_{n-k}(p-1)\Delta_k/2d_n(p)\Big). \qquad (ML_p(1))$$

Consider the sequence of real numbers

$$\sum_{k=1}^{n} d_{n-k}(p)\Delta_k, \quad n = 1,2, \ldots$$

These are the coefficients in the power series expansion of the product $(1-z)^{-p}\Sigma\Delta_n z^n$, and so of the product

$$(1 - z)^{-(p-1)}\sum_{1}^{\infty}\Big(\sum_{1}^{n}\Delta_k\Big)z^n.$$

It is easy to see that the coefficients of the first factor in the last displayed product are non-negative if $p \geq 1$. By (M), the coefficients of the second factor are non-positive. Hence

$$\sum_{k=1}^{n} d_{n-k}(p)\Delta_k \leq 0, \quad n = 1,2, \ldots$$

if $p \geq 1$.

Returning to $(ML_p(1))$, we now see that for every $n \geq 1$ and any $p \geq 2$

$$|a_n(p)| \leq d_n(p) \tag{H_p}$$

Hence, for $0 \leq r < 1$,

$$\begin{aligned} I_p(r,f) &= \int_0^{2\pi} |f(re^{i\theta})|^p d\theta = 2\pi r^p \sum_0^\infty |a_n(p)|^2 r^{2n} \\ &\leq 2\pi r^p \sum_0^\infty (d_n(p))^2 r^{2n} \\ &= I_p(r,k) \end{aligned}$$

which gives (B_p) for $p \geq 2$.

The Integral Means for $1 < p < 2$

Our results for the remaining means are incomplete; to derive them, we recall another of the Milin-Lebedev inequalities (see formula (2.33) on p. 35 in [7]):

$$\sum_{k=0}^{n} |a_k(p)|^2/d_k(p) \leq d_n(p+1)\exp\Big(p\sum_{k=1}^{n} d_{n-k}(p)\Delta_k/d_n(p+1)\Big). \tag{$ML_p(2)$}$$

As we observed in the previous section, the argument of the exponential on the right-hand side of this inequality is non-positive for $p \geq 1$. Hence we can infer that for every $n \geq 1$ and any $p \geq 1$

$$\sum_{k=0}^{n} |a_k(p)|^2/d_k(p) \leq d_n(p+1). \tag{*}$$

(In passing, we note that (*) implies (H_p), for the range $1 < p < 2$, in an average sense:

$$\begin{aligned} \Big(\sum_0^n |a_k(p)|\Big)^2 &\leq \Big(\sum_0^n |a_k(p)|^2/d_k(p)\Big)\Big(\sum_0^n d_k(p)\Big) \\ &\leq (d_k(p+1))^2 = \Big(\sum_0^n d_k(p)\Big)^2 \end{aligned}$$

i.e.

$$\sum_0^n |a_k(p)| \leq \sum_0^n d_k(p), \quad n = 0,1, \ldots).$$

As a simple consequence of (*), we deduce that

$$\begin{aligned} \sum_0^\infty |a_n(p)|^2 r^n/d_n(p) &= (1 - r)\sum_0^\infty\Big(\sum_0^n |a_k(p)|^2/d_k(p)\Big)r^n \\ &\leq (1 - r)\sum_0^\infty d_n(p+1)r^n \\ &= (1 - r)^{-p}. \end{aligned}$$

But for $p > 1$

$$1/d_n(p) = \Gamma(n+1)\Gamma(p)/\Gamma(n+p)$$

$$= (p-1)\int_0^1 t^n(1-t)^{p-2}dt$$

and

$$I_p(r,f) = 2\pi r^p \sum_0^\infty |a_n(p)|^2 r^{2n}, \quad 0 \le r < 1.$$

Hence combining these facts we see that

$$\int_0^r I_p(t,f)(r^2-t^2)^{p-2}t^{1-p}dt = 2\pi\sum_0^\infty |a_n(p)|^2 \int_0^r t^{2n+1}(r^2-t^2)^{p-2}dt$$

$$= \pi r^{2p-2}\sum_0^\infty |a_n(p)|^2 r^{2n}/(p-1)d_n(p)$$

$$\le \pi r^{2p-2}/(p-1)(1-r^2)^p$$

$$= \int_0^r I_p(t,k)(r^2-t^2)^{p-2}t^{1-p}dt$$

if $0 \le r < 1$ and $p > 1$.

In particular, if $1 < p < 2$ and $0 \le r < 1$, then

$$\int_0^r I_p(t,f)(r^2-t^2)^{p-2}t^{1-p}dt \le \int_0^r I_p(t,k)(r^2-t^2)^{p-2}t^{1-p}dt,$$

which is the closest we can come to (B_p) for this range of p.

The Integral Means for $0 < p < 1$

Something similar holds for p in (0,1). Indeed, if we utilise another of the Milin-Lebedev inequalities - this time formula (2.36) on p. 36 of [7] - we find that for any $p > 0$

$$\sum_0^\infty |a_k(p)|^2 r^{2k}/d_k(p) \le \exp(p\sum_1^\infty k|\gamma_k|^2 r^{2k})$$

$$= \exp(p(1-r^2)^2\sum_1^\infty(\sum_1^n(n+1-k)k|\gamma_k|^2)r^{2n})$$

$$\le \exp(p(1-r^2)^2\sum_1^\infty(\sum_1^n(n+1-k)/k)r^{2n})$$

$$= \exp(p\sum_1^\infty r^{2n}/n)$$

$$= (1-r^2)^{-p},$$

on using (M) again. Equality holds if and only if $f = k$, apart possibly from a rotation.

This inequality is a weak substitute for (B_p) in case $0 < p < 1$. Together with Schwarz's inequality it implies that

$$\sum_0^\infty |a_k(p)|r^k \le (1-r)^{-p}, \quad 0 \le r < 1,$$

which can be viewed as (H_p) in an average sense. We remark too that it forces

$$\sum_0^\infty |a_k(p)|^2/(k+1)d_k(p) \le 1/(1-p)$$

with equality holding only when $f = k$.

Concluding Remarks

It remains open whether (M) implies (B_p) for p in $(0,1) \cup (1,2)$. The implication would follow if the following inequality were true:

$$\sum_0^n |a_k(p)|^2 \le \sum_0^n (d_k(p))^2, \quad n = 0,1,2, \ldots$$

This holds true for $p = 1$ and for all $p \ge 2$. It is surely true for $1 < p < 2$, but I do not see how to prove it. It is even possible that it holds for the remaining values of p as well.

(When the first draft of this note was finished, Hayman very kindly sent me a copy of his joint work [5] with Hummel, in which (H_p) is also proved, for $p \geq 2$, in substantially the same way as that outlined above, the major difference being that a stronger inequality than (M), also obtained by de Branges, is used to show that the argument of the exponential in ($ML_p(1)$) is non-positive when $p \geq 2$. They mention too that Grinzpan and Aharonov have apparently made the same observation, and point out that (H_p) is false for $0 < p < 2$.)

REFERENCES

1. BAERNSTEIN II, A.
"Integral Means, Univalent Functions and Circular Symmetrization", *Acta Math.*, 133 (1974) 139-169.

2. BIEBERBACH, L.
"Uber die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln". *S.B. preuss. Akad. Wiss.* (1916) 940-945.

3. de BRANGES, L.
"A Proof of the Bieberbach Conjecture", *Acta Math.*, 154 (1985) 137-152.

4. FITZGERALD, C.H. and POMMERENKE, Ch.
"The de Branges Theorem on Univalent Functions", *Trans. Amer. Math. Soc.*, to appear.

5. HAYMAN, W.K. and HUMMEL, J.A.
"Coefficients of Powers of Univalent Functions", preprint.

6. LITTLEWOOD, J.E.
"On Inequalities in the Theory of Functions", *Proc. London Math. Soc.*, 23 (1925) 481-519.

7. MILIN, I.M.
"Univalent Functions and Orthonormal Systems", English Translation, *Amer. Math. Soc.*, Providence, R.I. (1977).

8. ROBERTSON, M.S.
"A Remark on the Odd Schlicht Functions", *Bull. Amer. Math. Soc.*, (1936) 366-370.

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