# UNDERGRADUATE PROJECTS IN GROUP THEORY: COMMUTATIVITY RATIOS

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Many universities have tried undergraduate projects in mathematics with varying success, but often one hears that although in applied mathematics, the students' work can be creative and, to some limited extent original, in pure mathematics, the project is often to write an account of some theory which the student has searched for in "the literature". Can one do better than this? Can one provide practical and "creative" material for a project in pure mathematics? I want to suggest that one can, by describing two projects in Group Theory with which I have been involved.

Group theory like many other branches of pure mathematics taught at university level can tend to be too much in the definition-theorem-proof tradition. Students can finish up with apparently good knowledge of Sylow subgroups and the finer points of soluble groups but faced with an actual group they may not have the faintest idea where to start if required to analyse the subgroups, conjugacy classes, quotients groups etc., i.e. they do not know how to handle the more elementary theory, so their knowledge of the general deeper parts of group theory consists of a collection of statements about ill understood concepts. (If in doubt, set a group of students to work out from scratch a complete list of isomorphism types of groups up to some small given order. Can they do it?)

The situation on potential project material is similar to that on the traditional group theory course work. Of course, there are courses in group theory which have a reasonable, even an adequate, supply of examples in them and similarly there are several different solutions to the problem of designing "creative" projects in the subject. The three main approaches one can take would seem to be (i) use presentations, (ii) use representations, or (iii) concentrate on geometric symmetry groups. In the two projects that I supervised, presentations of groups were used as the basic tool since these were being treated in a 3rd year course on Knot Theory running at the same time.



In this note I will describe briefly the subject matter of a 3rd year project equivalent to a half paper in the final exams. I will describe in a further note the content of a full-paper-equivalent project taken the following year by a different student.

First a word of caution, the group theory involved is not deep, or complicated. The prerequisites were an intuitive idea of presentations and a reasonable ability to handle modular arithmetic. No claim is made for originality of the results nor for elegance of the method; what is important is that the student, once the main idea was outlined, completed the calculations by themselves. Certain pieces of theory had to be sketched out for them, but details of proofs were to be provided by them. This was not always done successfully, but the end result was some very good work by a student who was not one of the "high flyers".

For the non-group theorist, let me recall the idea of a presentation. I will give an example. The dihedral group, $D_4$, of order 8 is the group of symmetries of a square and has presentation

$$\langle x, y : x^4 = y^2 = (xy)^2 = e \rangle$$

That is the elements x and y generate $D_4$ and the relations $x^4 = e$, $y^2 = e$ and $xyxy = e$ are sufficient to give all relationships between products of powers of x's and y's in $D_4$.

The idea of the project was to calculate commutativity ratios for various families of groups. The commutativity ratio is the probability that two elements taken at random in a group G will commute (see D. MacHale [3]). This ratio R(G) can be calculated by the equation

$$R(G) = \frac{\text{number of commuting pairs}}{(\text{order of } G)^2}$$

and is closely linked to the number of conjugacy classes of G. It is however a more intuitive invariant than the latter. The families studied were the dihedral groups and generalised quaternion groups; an attempt was made at general metacyclic groups. I will give the calculations for dihedral groups and give the results for the other families.

$D_n$, the dihedral group of order 2n has presentation

$$D_n = \langle x,y: x^n = y^2 = (xy)^2 = e\rangle \quad (\text{for } n \geq 3)$$

First note that $xyxy = e$ implies $yx = x^{-1}y^{-1} = x^{n-1}y$ so a simple argument shows that any element of $D_n$ has a unique normal form $x^iy^j$ for $0 \leq i \leq n-1$, $0 \leq j \leq 1$. In this normal form, multiplication is given by

$$(x^iy^j)(x^ky^\ell) = x^ry^s$$

where

$$r \equiv i + k + jk(n-2) \pmod n$$

$$s \equiv j + \ell \pmod 2$$

(This formula and the existence and uniqueness of the normal form had to be proved by the student. Although fairly simple inductive proofs, they demand care in their presentation.)

It is now clear that $(x^iy^j)$, $(x^ky^\ell)$ is a commuting pair if and only if

$$jk(n-2) \equiv \ell i(n-2) \mod n$$

or

$$2jk \equiv 2\ell i \pmod n$$

As should come as no surprise, the cases n odd and n even are different.

If n is odd, $2jk \equiv 2\ell i$ if and only if $jk \equiv \ell i$. An attack case by case follows:



If $j = 0$ and $\ell = 0$, this works for all i and k. (This, of course, corresponds to $x^ix^k = x^kx^i$ - not surprising!)

This gives $n^2$ commuting pairs.

Similarly $j = 0$, $\ell = 1$ gives n more.

If $j = 1$ and $\ell = 1$, then $i = k$ giving another n.

Thus for n odd

$$R(D_n) = \frac{n^2 + 3n}{4n^2}$$

For n even, one gets some additional solutions, namely when

$$jk - \ell i \equiv \frac{n}{2} \mod n.$$

As is easily checked, this gives 3n more commuting pairs and

$$R(D_n) = \frac{n^2 + 6n}{4n^2}$$

if n is even.

It should be noted that the student using group tables for small values of n found the patterns for n odd and n even by themselves. I then pointed out that the presentation should give one those patterns in general. They then went away and produced the calculation summarised above.

For the dicyclic group of order 4n,

$$\langle 2,2,n\rangle = \langle x,y : x^n = y^2,\ y^{-1}xy = x^{-1}\rangle$$

the calculations are similar, giving

$$R(\langle 2,2,n\rangle) = \frac{n^2 + 3n}{4n^2}$$

As both dihedral and dicyclic groups are metacyclic groups, I then suggested that the same methods would perhaps work for all metacyclic groups. For the non-group theorist a metacyclic group G is a group with a cyclic normal subgroup, whose corresponding quotient group is also cyclic. One easily checks that such a group must have presentation

$$G = \langle x,y : x^m = e,\ y^{-1}xy = x^r,\ y^n = x^s \rangle$$

where m, n, r, s are positive integers, r,s ≤ m and $r^n \equiv 1$ and $rs \equiv s \pmod m$. (A discussion of this can be found in [2], p. 65.) Any element in G can be written uniquely in the form $y^i x^j$ (the reverse order being adopted to accord with [2]). Multiplication in this form gives

$$(i,j)(k,\ell) = \begin{cases} (i+k,\ \ell + jr^k) & \text{if } i+k < n \\ (i+k,\ \ell + jr^k + s) & \text{if } i+k \geq n \end{cases}$$

The condition for commutativity between (i,j) and (k,ℓ) is

$$j(r^k - 1) \equiv \ell(r^i - 1) \pmod m$$

This is as far as I can go. The student, in fact, failed to get to this point due to a slip earlier in their final calculations. I had hoped for some indication of the number of solutions, at least for special values of r and s as this is exactly what happens for the $D_n$ and <2,2,n>, but apart from obvious cases such as s = 0, r = 1 ($G \simeq C_n \times C_m$) or the dihedral and dicyclic families themselves, no particularly nice families were apparent. I did not look very far into this and in retrospect I should have looked at some of the other families of metacyclic groups such as Coxeter and Moser's ZS-metacyclic groups (see [1]). Perhaps someone would like to set this as an undergraduate project on modular arithmetic and group theory.

My own view of the project was that the student obtained a remarkable feeling for the calculations involved, their sense of enjoyment was obvious and the benefit to their general understanding of other group theory based courses: "Groups and Knots" (2 joined units), "Rings, Modules, and Linear Algebra" (1 unit), and "Group Theory" (1 unit) was considerable even though the use of presentations as such was only a part of the Groups and Knots course and none of the material in the



project was directly useful in that.

The point may be that presentations provide one means by which students can "do" group theory. "Doability" would seem to be a useful concept in teaching mathematics. You only really learn mathematics by "doing", i.e. by handling examples until you feel what a theorem says, by recreating in some small way the original *raison d'etre* of a concept and so on. The problem is that one must balance such ideas with a need to cover a reasonable amount of ground so as to satisfy the external examiner. In a project one can sometimes avoid this pressure to some extent, since the process of discovery, the accuracy of calculation and, that which is of great importance, the presentation, are what will be looked for by the examiner. Perhaps one should hope that some step in a similar direction might be made in the conventional exam. setting as well.

## REFERENCES

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