3. KAKUTANI, S.
"Free Topological Groups and Infinite Direct Product Topological Groups", *Proc. Imp. Acad. Japan*, **Vol. 20** (1944) 595-598.

4. MARKOV, A.A.
"O Svobodnich Topologiciskich Gruppach", *Izv. Akad. Nauk SSSR, Ser. Mat.*, **Vol. 9** (1945) 3-64; AMS Translations Series 1, No. 30 (1950) 11-88.

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HYPERBOLIC BEHAVIOUR OF GEODESIC FLOWS*

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INTRODUCTION

Geodesic flows, particularly those on manifolds of negative curvature, have been a rich source in the determination and display of possible types of macroscopic behaviour of motions in dynamical systems. Their study goes back to Hadamard and Poincare who considered the existence of periodic geodesics on some classes of surfaces. Later, in the 1930s Hedlund, Hopf and Morse studied the topological and ergodic properties of the flows on compact surfaces of negative curvature [H]. Already, they recognised the special role of the local instability of trajectories and proved that this was closely linked with the statistical (ergodic) behaviour of the flows.

One of the ways of expressing this local instability is the hyperbolic behaviour of the derivative of the flow. The central idea is that close to any fixed trajectory, the behaviour of neighbouring trajectories resembles the behaviour of trajectories in the neighbourhood of a saddle point singularity. Anosov [A] was the first to give an explicit formulation of hyperbolicity. He then used this condition as a basic assumption to study a class of dynamical systems which are now referred to as *Anosov systems*. The geodesic flow on compact manifolds of negative curvature is a very important example of these flows.

The conditions formulated by Anosov in 1967 are the strongest type of hyperbolic conditions. In 1977, Pesin [P] formulated a weaker set of hyperbolic conditions and studied the dynamical systems satisfying these conditions. Again, the geodesic flows on a class of manifolds without focal points

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provide an important example of a system satisfying Pesin's conditions.

In this article I wish to outline the study of the hyperbolic behaviour of geodesic flows.

Preliminaries

1.1 Notation

Let M be a smooth compact Riemannian manifold of dimension $n \geq 2$. The tangent and unit tangent bundles of M will be denoted by TM and SM, respectively, with corresponding fibers T_mM and S_mM at $m \in M$. The projection map from these bundles to M will be denoted by π. Finally $< , >$ and $\rho(,)$ will denote the Riemannian metric and the corresponding distance function.

1.2 Geodesics

A *geodesic* is a curve c(t) on M whose tangent vectors are parallel. This is expressed in terms of the Riemannian connection of TM by the equation

$$\nabla_{\dot{c}}\dot{c} = 0$$

or in local coordinates by

$$\frac{d^2c^i}{dt^2} + \Gamma^i_{jk}\frac{dc^j}{dt}\frac{dc^k}{dt} = 0$$

where Γ^i_{jk} are the Christoffel symbols.

If c(t) is a geodesic, then $<\dot{c}(t),\dot{c}(t)>$ is constant and we assume it has the value 1, that is, the geodesics are parameterized by arc length. Since M is compact, the geodesics are infinitely extendable in both directions so that c(t) is a curve from $\mathbb{R}$ to M. For any pair x and y of distinct points of M there exists a geodesic joining x to y (generally speaking, not unique). Among such there is always one whose length is equal to $\rho(x,y)$. If $v \in S_mM$ for some $m \in M$, there is a unique geodesic c(t) satisfying the initial conditions $c(0) = m$ and $\dot{c}(0) = v$. We will denote this geodesic by $c_v(t)$.

1.3 Geodesic Flow

The geodesic flow is defined on the unit tangent bundle SM as follows. The flow map $\Phi: \mathbb{R} \times SM \to SM$ is given by

$$\Phi(t,v) = \dot{c}_v(t).$$

Geometrically the flow map takes the tangent vector to a geodesic, and moves it a distance t along that geodesic. We will assume that the metric on M is smooth and thus the map Φ is smooth. The vector field of the geodesic flow is called the *geodesic spray* and denoted by S.

To facilitate studying the hyperbolic properties of the geodesic flow, it is necessary to consider the derivative

$$T\Phi_t : T(SM) \to T(SM)$$

where $\Phi_t : SM \to SM$ is the map $\Phi_t(v) = \Phi(t,v) = \dot{c}_v(t)$. A convenient formulation of the map $T\Phi_t$ is got by considering the geometry of T(SM) and Jacobi vector fields along geodesics of M.

1.4 Geometry of T(SM)

If $v \in SM$, then the tangent space $T_v(SM)$ is decomposed into two complementary subspaces as follows. The first is the vertical subspace which is the (n-1)-dimensional subspace given by the kernel of the map $T\pi|_{T_v(SM)} : T_v(SM) \to T_{\pi(v)}(M)$ while the second is the horizontal subspace which is the n-dimensional subspace given by the kernel of the connection map $K|_{T_v(SM)} : T_v(SM) \to T_{\pi(v)}M$ [Eb]. (The connection map $K : T_v(TM) \to T_{\pi(v)}M$ is defined as follows: let $\xi \in T_v(TM)$ and let $X : (-\varepsilon,\varepsilon) \to TM$ be a curve with initial velocity ξ, then

$K\xi = \nabla_{\dot{\sigma}(0)}X(0)$ where $\sigma = \pi \circ X : (-\varepsilon,\varepsilon) \to M$ is the footpoint curve).

If $v \in S_mM$ and $v^\perp$ is the orthogonal complement in T_mM, then $K : T_v(SM) \to v^\perp$ and the map $i_v : T_v(SM) \to T_mM \oplus v^\perp$ given by

$$i_v\xi = (T\pi\xi, K\xi)$$

is a linear isomorphism. The *Sasaki metric* on SM is defined by $\langle\langle \xi,\eta \rangle\rangle = \langle T\pi\xi, T\pi\eta\rangle + \langle K\xi, K\eta\rangle$ for $\xi,\eta \in T(SM)$. Then i_v is an isometry. The Riemannian volume μ on SM defined by the Sasaki metric is called the *Louiville measure* and it is invariant under the geodesic flow [A+S].

1.5 Jacobi Fields

Let $c(t)$ be a fixed geodesic on M. A vector field $Y(t)$ on $c(t)$ is a *Jacobi field* if

$$\nabla^2 Y + R(\dot{c},Y)\dot{c} = 0$$

where ∇ is covariant differentiation along c and R is the Riemannian curvature tensor on M. Jacobi fields are the variational vector fields of variations of c by geodesics.

If $\xi \in T_v(SM)$ then ξ determines the unique Jacobi field $Y_\xi(t)$ along the geodesic $c_v(t)$ with initial conditions $Y_\xi(0) = T\pi\xi$ and $\nabla Y_\xi(0) = K\xi$.

If $\xi(t) = (T\Phi_t)\xi$, it can be shown [Eb] that

$$T\pi\xi(t) = T\pi \circ T\Phi_t\xi = Y_\xi(t)$$

and

$$K\xi(t) = K \circ T\Phi_t\xi = \nabla Y_\xi(t)$$

This gives a bijection between $T_v(SM)$ and the Jacobi fields on c_v. Further, if $Z(v)$ is the subspace of $T_v(SM)$ spanned by the geodesic spray vector field $S(v)$ we have

$$\xi \in Z(v) \Longleftrightarrow Y_\xi = a\dot{c}_v(t) \text{ for some } a \in \mathbb{R}.$$

If $T_v^\perp SM$ is the orthogonal complement of $Z(v)$ in $T_v(SM)$ with respect to the Sasaki metric, we have

$$\xi \in T_v^\perp(SM) \Longleftrightarrow Y_\xi \text{ is a perpendicular Jacobi field on } c_v \text{ [Eb].}$$

Thus the two subbundles Z and $T^\perp SM$ are $T\Phi_t$-invariant.

1.6 Stable and Unstable Jacobi Fields

We now restrict M to be a manifold without conjugate points. Thus if $Y(t)$ is a Jacobi vector field along a geodesic $c(t)$ which is not identically zero, then $Y(t) = 0$ at no more than one point along $c(t)$. This class includes manifolds of non-positive curvature and manifolds without focal points.

Let $v \in SM$, let $w \in v^\perp$, and let $Y_{w,s}(t)$ be the unique Jacobi field on $c_v(t)$ such that

$$Y_{w,s}(0) = w \text{ and } Y_{w,s}(s) = 0.$$

Then the limit $Y_w^-(t) = \lim_{s\to\infty} Y_{w,s}(t)$ exists and is a Jacobi vector field on $c_v(t)$ [Eb]. Clearly $Y_w^-(0) = w$ and $Y_w^-(t) \neq 0$ for $t > 0$. We call Y_w^- a *stable Jacobi field*.

The *unstable Jacobi fields* $Y_w^+(t)$ along $c_v(t)$ are got by considering the limits as $s \to -\infty$,

$$Y_w^+(t) = \lim_{s\to-\infty} Y_{w,s}(t)$$

For each $w \in v^\perp$, there is a unique $\xi^-(w) \in T_v^\perp(SM)$ for which $Y_{\xi^-(w)}(t) = Y_w^-(t)$ and a unique $\xi^+(w)$ such that $Y_{\xi^+(w)}(t) = Y_w^+(t)$ [Eb].

Using these limiting Jacobi fields we now get the *stable* and *unstable subspaces* of $T_v(SM)$ which are defined as follows:

$$X_s(v) = \{\xi \in T_v^\perp(SM) : Y_\xi(t) \text{ is stable, i.e. } Y_\xi(t) = Y_w^-(t)$$

where $w = T\pi\xi\}$.

$X_u(v) = \{\xi \in T_v^{\perp}(SM : Y_\xi(t)$ is unstable, i.e. $Y_\xi(t) = Y_w^+(t)$

where $w = T\pi\xi\}$.

The subspaces $X_s(v)$ and $X_u(v)$ are $(n-1)$-dimensional subspaces of $T_v(SM)$ which are invariant under the geodesic flow. The two subspaces coincide and consist of the space of perpendicular parallel vector fields on $c_v(t)$ in the case of M having sectional curvature $K \equiv 0$. If M has no focal points, then a Jacobi field $Y(t)$ is stable (unstable) if and only if $||Y(t)||$ is bounded for $t \geq 0$ $(t \leq 0)$ [Eb], [Es]. (If M has no focal points, then for any Jacobi field $Y(t)$ along a geodesic $c(t)$ such that $Y(t_o) = 0$, we have $||Y(t)||$ strictly increasing as $t \to \infty$.) We will show later that in the case of manifolds with strictly negative curvature, $X_s(v) \cap X_u(v) = \{0\}$.

2.1 Anosov Flows

The strongest type of hyperbolic condition is the following which was first formulated by Anosov [A].

Let N be a smooth manifold and let $\phi : \mathbb{R}\times N \to N$ be a complete flow which is smooth. Then it is an *Anosov flow* if the following holds: there are two continuous nontrivial distributions E^- and E^+ of TN such that

(i) $T_nN = E^-(n) \oplus E^+(n) \oplus Z(n)$, where $Z(n)$ is the subspace of T_nN generated by the flow vector field.

(ii) $T\phi_t(E^+(n)) = E^+(\phi_t(n))$ and $T\phi_t(E^-(n)) = E^-(\phi_t(n))$ for any $n \in N$, $t \in \mathbb{R}$.

(iii) there exist constants $a \geq 1$, $b > o$ such that for $n \in N$

$$||T\phi_t(v)|| \leq a||v||e^{-bt} \quad \text{if } v \in E^-(n)$$

$$||T\phi_t(v)|| \geq a^{-1}||v||e^{bt} \quad \text{if } v \in E^+(n)$$

The subspaces $E^-(n)$ and $E^+(n)$ are called the *stable* and *unstable subspaces.*

These conditions mean that at each point $n \in N$ the tangent space T_nN can be decomposed in an invariant way into three subspaces $E^-(n)$, $E^+(n)$ and $Z(n)$ such that $T\phi t|_{E^-(n)}$ is a contraction, $T\phi t|_{E^+(n)}$ is an expansion and $Z(n)$ is the subspace generated by the flow vector field. Furthermore the coefficients of contraction or expansion are uniform on N. Near any fixed trajectory $\{\phi_t(n)\}$ the behaviour of neighbouring trajectories resembles the behaviour of trajectories in the neighbourhood of a saddle point.

2.2 Geodesic Flows of Anosov Type

Returning to the geodesic flow, we see that the subspaces $X_s(v)$ and $X_u(v)$ of $T_v^{\perp}(SM)$ are candidates for the subspaces $E^-(v)$ and $E^+(v)$ required by the Anosov conditions. If M has negative sectional curvature they do satisfy the condition.

Theorem [A]. Let M be a compact manifold with negative sectional curvature. Then the geodesic flow satisfies the Anosov conditions.

Proof. Since M is compact there are constants r_1 and r_2 such that

$$-r_1^2 \leq K_m(P) \leq -r_2^2$$

for all sectional curvatures $K_m(P)$. Then for any $v \in SM$, $w \perp v$, the stable Jacobi field $Y_w^-(t)$ along the geodesic $c_v(t)$ satisfies the inequalities

$$||w||e^{-r_1 t} \leq ||Y_w^-(t)|| \leq ||w||e^{-r_2 t} \quad \dots (1)$$

[H + H].

We also have the following bound for the covariant derivative of $Y_w^-(t)$ [Eb]:

$$||\nabla_w Y^-(t)|| \le r_1 ||Y_w^-(t)|| \quad \ldots (2)$$

Now let $\xi \in X_s(v)$ and $w = T\pi\xi$. Then

$$||T\Phi_t\xi||^2 = ||(Y_w^-(t), \nabla Y_w^-(t))||^2$$

$$= ||Y_w^-(t)||^2 + ||\nabla Y_w^-(t)||^2$$

which, by (1) and (2) $\quad \le ||w||^2 e^{-2r_2 t} + (r_1)^2 ||w||^2 e^{-2r_2 t}$

$$\le ||\xi||^2 e^{-2r_2 t}(1 + r_1^2).$$

Thus $||T\Phi_t\xi|| \le \sqrt{1+r_1^2}\, e^{-r_2 t}||\xi||$ and so we have the required contraction for $X_s(v)$. The required inequality for $X_u(v)$ is got by using the fact that $X_u(v)$ may be identified with $X_s(-v)$ [Eb].

Finally if $\xi \in X_s(v) \cap X_s(u)$, then Y_ξ is a parallel Jacobi field [Esch], i.e. $\nabla Y_\xi = 0$. Then

$$||T\Phi_t\xi|| = ||\xi|| \quad \text{for } t \in \mathbb{R}$$

and so

$$X_s(v) \cap X_u(v) = \{0\}.$$

Thus the geodesic flow is an Anosov Flow.

While the above theorem shows that strict negative curvature is sufficient to ensure that the geodesic flow is Anosov, it is not a necessary condition. Eberlein [Eb] gave an example of a manifold, with non-positive curvature containing open subsets where the sectional curvature is zero on all tangent planes, and yet the geodesic flow is Anosov. Klingenberg [K] proved that if the geodesic flow is Anosov, then M has no conjugate points, and Eberlein then gave the following necessary and sufficient conditions.

Theorem [Eb]. Let M be a compact manifold without conjugate points. Then the following are equivalent.

(a) The geodesic flow is Anosov.

(b) $X_s(v) \cap X_u(v) = \{0\}$ for all $v \in SM$.

(c) There exists no nonzero perpendicular Jacobi vector field $Y(t)$ on a geodesic $c(t)$ of M such that $||Y(t)||$ is bounded for all $t \in \mathbb{R}$.

3.1 Weaker Hyperbolicity

The hyperbolicity condition due to Anosov is the strongest type in the sense that the subspaces $E^+(n)$ and $E^-(n)$ of the tangent space T_nN generate the complement of $Z(n)$ in T_nN (2.1) and the expansion and contraction of the flow are uniform with respect to n. By relaxing either or both of these requirements we get partial rather than complete hyperbolicity (when the subspaces $E^+(n)$ and $E^-(n)$ do not span $T_nN \setminus Z(n)$) and/or nonuniform rather than uniform hyperbolicity.

Pesin studied these various weaker hyperbolicity conditions and gave the connections with Lyapunov exponents [P]. The geodesic flow on manifolds with no focal points satisfying certain geometric conditions are complete nonuniform hyperbolic flows [P], [B]. The theory of the weaker hyperbolicity conditions is much more complex than the Anosov case and is beyond the scope of this article.

REFERENCES

[A] ANOSOV, D.V.
"Geodesic Flows on Riemannian Manifolds of Negative Curvature", *Proc. Steklov Institute Math.*, **90** (1969).

[A+S] ANOSOV, D.V. and SINAI, Ya. I.
"Some Smooth Ergodic Systems", *Russian Math. Surveys*, **22** (1967) 109-167.

[B] BURNS, K.
"Hyperbolic Behaviour of Geodesic Flows on Manifolds With No Focal Points", *Ergodic Th. and Dynamical Systems*, **3** (1983) 1-12.

[Eb] EBERLEIN, P.
"When is a Geodesic Flow of Anosov Type? I", *J. Diff. Geom.*, **8** (1973) 437-463.

[Es] ESCHENBURG, J.
"Horospheres and the Stable Part of the Geodesic Flow", *Math. Z.*, **153** (1977) 237-251.

[H] HEDLUND, G.A.
"The Dynamics of Geodesic Flows", *Bull. Amer. Math. Soc.*, **45** (1939) 241-260.

[H+H] HEINTZE, E. and HOF, H.Ch. Im.
"Geometry of Horospheres", *J. Diff. Geom.*, **12** (1977) 481-491.

[K] KLINGENBERG, W.
"Riemannian manifolds with Geodesic Flow of Anosov Type", *Ann. of Math.*, **99** (1974) 1-13.

[P] PESIN, Ya. B.
"Characteristic Lyapunov Exponents and Smooth Ergodic Theory", *Russian Math. Surveys*, 32 (1977) 55-114.

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REMARKS ON 'AN ELEMENTARY NUMBER THEORY RESULT'

David Singmaster

In a joint note published in this Newsletter (No. 12, December 1984, pp. 10-13), Peter Birch and I showed that $\phi(n) > n/\log n$ except for $n = 1,2,3,4,6,10,12,18$ or 30. For convenience, let us set $\Phi(n) = n^{-1}\phi(n)\log n$, so the above says $\Phi(n) > 1$, except for the values given. Our proof used Bertrand's Postulate, so it was not entirely elementary. I have just found that Alan Baker gives an entirely elementary proof that $\Phi(n) > \frac{1}{4}$ for $n > 1$ [1, p. 12]. Further care with his argument shows the asymptotic result $\Phi(n) > \frac{1}{2} - \varepsilon$ for all large enough n and explicit calculation would show $\Phi(n) > 2/5$ for all $n > 2$.

Baker's argument, in more detail, is as follows. First consider $\sigma(n)$, the sum of the divisors of n. Then

$$\sigma(n) = \Sigma_{d|n} d = \Sigma_{d|n} n/d = n\Sigma_{d|n} 1/d \leqq n\Sigma_{d|n} d^{-1}, \text{ so}$$

$$\sigma(n) \leqq n(1 + \log n). \qquad (1)$$

Consider now $f(n) = \sigma(n)\phi(n)n^{-2}$. This is multiplicative and $f(p^j) = 1 - p^{-j-1}$. Then

$$f(n) = \Pi_{p^j|n}(1-p^{-j-1}) \geqq \Pi_{p^j \leqq n}(1-p^{-j-1}) \geqq \Pi_{p^2 \leqq n}(1-p^{-2})$$

$$\geqq \Pi_{1 \leqq m^2 \leqq n}(1-m^{-2}) = \tfrac{1}{2}(1+[\sqrt{n}]^{-1}), \text{ for } n \geqq 4, \text{ so that}$$

$$\sigma(n)\phi(n)n^{-2} \geqq \tfrac{1}{2}(1+n^{-\frac{1}{2}}), \qquad (2)$$

and this is seen to hold for $n \geqq 3$.

From (1) and (2), we have

$$\phi(n) \geqq \tfrac{1}{2}n(1+n^{-\frac{1}{2}})(1+\log n)^{-1}, \text{ for } n \geqq 3. \qquad (3)$$