FATOU'S THEOREM AND UNIVALENT FUNCTIONS

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1. INTRODUCTION

The purpose of this note is to present some results - old and new - concerning the behaviour of functions analytic and univalent in the unit disc $U = \{z:|z| < 1\}$ as the unit circle $C = \{z:|z| = 1\}$ is approached. Some open questions suggested by these results will also be discussed. The note is based on a lecture given at the December 1984 meeting of the DIAS Mathematical Symposium.

2. ANGULAR LIMITS

We begin with a simple definition. A function f, analytic (holomorphic) in U, is said to have an *angular* (or non-tangential) limit ℓ at a point ζ on C if

$$f(z) \to \ell$$

as $z \to \zeta$ inside every symmetric angle with vertex at ζ (as shown in Fig. 1) of opening less than π. This is easily

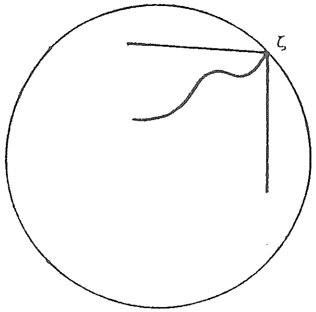


FIGURE 1

shown to be equivalent to the following: f has an angular limit ℓ at ζ if, for every fixed positive K,

$$f(z) \to \ell \quad \text{as} \quad z \to \zeta, \quad z \in \Omega$$

where

$$\Omega = \{z \in U:|\zeta - z| \le K(1 - |z|)\}$$

We also write, in the usual notation, H^∞ to denote the class of functions that are analytic and bounded in U.

The basic result connecting functions in H^∞ and angular limits is the following well-known theorem of Fatou.

<u>THEOREM</u> 1. (Fatou, 1906)

Let $f \in H^\infty$. Then f has an angular limit at all points $\exp(i\theta)$ on C except possibly for a set of θ of measure zero, that is, angular limits exist almost everywhere on C.

This important result has been generalised in a number of ways, but our interest here is in the fact that the result as stated is sharp in at least two senses. In the first place, given any subset E of C of (linear) measure zero, there is a function f in H^∞ for which the radial limit

$$\lim_{r \to 1} f(r\zeta)$$

fails to exist for all ζ in E [4], and secondly, if Γ is any curve in U that approaches the point 1 tangentially, there is a function g in H^∞ which does not approach a limit as z approaches *any* point $e^{i\theta}$ along $e^{i\theta}\Gamma$ [3]. The situation for functions in H^∞ is thus clear-cut: we cannot, in general reduce the size of the exceptional set in Fatou's theorem for such functions nor can we replace angular limit by tangential limit in any uniform sense. To obtain improvements in either of these two directions, therefore, some extra condition must be imposed on our functions, and the extra condition we consider here is that of univalence.

3. UNIVALENT FUNCTIONS

A function f is said to be *univalent* in U if it is analytic and one-to-one on U, that is,

$$f(z_1) = f(z_2), \quad z_1, z_2 \in U \Longrightarrow z_1 = z_2$$

It is almost immediate that the conclusion of Fatou's theorem holds for *all* univalent functions (and not just for bounded univalent functions). For if f is univalent in U, f(U) cannot be the entire complex plane, so there is a point w in the complement of f(U) and then, by a simple and standard argument [7, pp. 302-3] there is a complex number b such that

$$g(z) = [(f(z) - w)^{\frac{1}{2}} + b]^{-1}$$

is univalent and bounded in U. Then

$$f(z) = w + (\frac{1}{g(z)} - b)^2, \quad z \in U,$$

and so, if g has an angular limit at a point on C, f has also, unless the limit for g is zero. By a uniqueness theorem of F. and M. Riesz, this can happen on C only at a set of measure zero [2, p. 76]. Hence f has a finite angular limit almost everywhere on C.

The hypothesis of univalence is a highly restrictive one, however, so it is natural to ask (especially with the benefit of hindsight) whether a stronger result than this is true for univalent functions. Fatou's theorem can indeed be strengthened for univalent functions and this was first proved by Beurling in 1946.

<u>THEOREM 2</u>. (Beurling, [1, p. 56])

If f is univalent in U, then f has an angular limit at all points $\exp(i\theta)$ on C except possibly for a set of θ of logarithmic capacity zero.

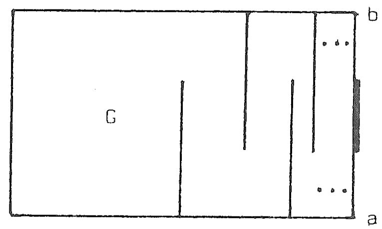
Beurling's theorem is usually proved (as it is in [1]) for functions in the classical Dirichlet space D, that is, the class of functions analytic in U for which

$$\iint_U |f'(z)|^2 dxdy \qquad (1)$$

is finite, and this is a more general result than Theorem 2. To see this, note first that D contains all bounded univalent functions since, if f is univalent, the integral (1) represents the area of f(U). Hence, by an argument similar to that used at the beginning of this section, if the conclusion of Theorem 2 holds for functions in D, it is valid also for all univalent functions.

The points on C at which angular limits fail to exist thus form a much smaller set for univalent functions than for functions in H^∞, since sets of logarithmic capacity zero are, in general, much "thinner" than sets of measure zero. It appears to be an open question whether Beurling's theorem is best - possible with regard to the size of the exceptional set but it is certainly easy to show that the exceptional set need not be empty. To do this, it is necessary only to consider a univalent function f which maps U onto the simply connected domain in Fig. 2. Such a function exists by the

FIGURE 2



Riemann mapping theorem. Note that G has infinitely many vertical slits and that these slits have, as indicated, a fixed overlap. By standard results from the theory of *prime ends* [6, §9.2]

$$\lim_{r \to 1} f(r\zeta_0)$$

fails to exist where ζ_0 is a point on C which corresponds to the boundary element ab of G.

For univalent functions, therefore, the size of the exceptional set in Fatou's theorem can be reduced. We next turn our attention to the improvements which are possible for univalent functions with regard to the *nature* of the limit which exists almost everywhere on C. To this end we now introduce the notion of a tangential limit.

4. TANGENTIAL LIMITS

Let ϕ be a decreasing continuous function on $[0,1]$ with $\phi(1) = 0$ for which

$$\frac{1-r}{\phi(r)} \to 0 \quad \text{as} \quad r \to 1. \tag{2}$$

Let $K > 0$, $\theta \in [0,2\pi]$ and set

$$\Omega(\phi,\theta,K) = \{z \in U : |e^{i\theta} - z| \leq K\phi(r)\}$$

where $r = |z|$. The region Ω makes tangential contact with C at $\exp(i\theta)$; when $\phi(r) = (1-r^2)^{\frac{1}{2}}$, for instance, $\Omega(\phi,\theta,1)$ is the disc of radius $\frac{1}{2}$ centred at $\frac{1}{2}\exp(i\theta)$.

Definition. For any ϕ satisfying (2), we say that f has a T_ϕ - limit ℓ at $\exp(i\theta)$ on C if, for every positive K,

$$f(z) \to \ell \quad \text{as} \quad z \to e^{i\theta}, \quad z \in \Omega(\phi,\theta,K)$$

We are now in a position to state our next theorem which is a special case of some recent results of Nagel, Rudin and Shapiro.

THEOREM 3. ([5]).

For $0 < r < 1$, set

$$\phi(r) = (\log \frac{1}{1-r})^{-1},$$

and let $f \in D$, the Dirichlet space. Then f has a T_ϕ - limit at almost all points on C.

Nagel *et al.* also show in [5] that, for certain other kinds of exceptional sets E - intermediate between sets of log-capacity zero and measure zero - every f in D has a T_{ϕ^ε} - limit at all points $\zeta \in C \setminus E$ where, this time, $\phi(r) = (1-r)^\varepsilon$, $0 < \varepsilon < 1$, and ε depends only on the size of the exceptional set E. All these results of course extend immediately to the full class of univalent functions. In particular, therefore, by Theorem 3, each univalent function has, at almost all points ζ on C, a limit within a region that makes tangential (indeed exponential) contact with C at ζ. This is in sharp contrast with the behaviour of functions in H^∞, described in Section 2.

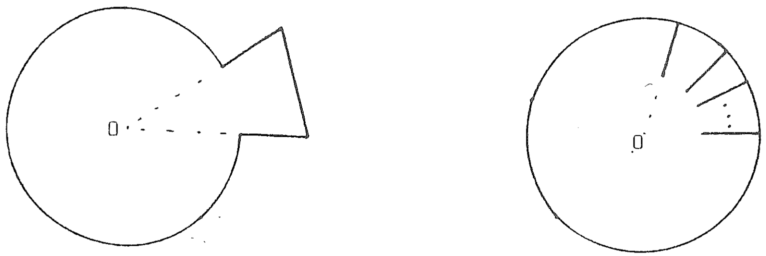
The results we have discussed so far all relate to the existence of certain kinds of restricted limits at points on C and one might ask whether, for a univalent f, there must always be some point ζ on C at which f has an *unrestricted limit*, that is, at which

$$\lim_{z \to \zeta} f(z)$$

exists as $z \to \zeta$ in *any* way from inside U. Such a point ζ, however, would correspond to a prime end of f(U) whose *impression* [6, p. 276] consists of a single point and Caratheodory [1, p. 184] has given an example of a bounded simply connected domain G which has no such prime ends. Then, by the Riemann mapping theorem again, there is a univalent function f with f(U) = G and this function cannot have an unrestricted limit at any point on C. There is an interesting subclass of univalent functions, however, the members of which always have unrestricted limits at some points on C.

5. STARLIKE UNIVALENT FUNCTIONS

A univalent function f, with $f(0) = 0$, is said to be *starlike* if the image domain $f(U)$ is starshaped with respect to 0, that is, $f(U)$ contains the line segment $[0,w]$ whenever it contains w. We give two examples of starshaped regions in Fig. 3; note that the region in (b) may have infinitely many slits.



(a) (b)

FIGURE 3

If f is starlike and bounded in U, then

$$\lim_{r\to 1} f(re^{i\theta}) = \hat{f}(\theta) \qquad (3)$$

exists for *every* θ in $[0,2\pi]$. (Indeed it can be shown that such functions have, for every ε in $(0,1)$, a T_ϕ - limit with $\phi(r) = (1-r)^\varepsilon$ at all points on C. Details, the reader will be relieved to learn, to appear elsewhere.) By classical results of Baire on pointwise limits of sequences of continuous functions, it follows from (3) that the set A of points of discontinuity of $\hat{f}$ is a set of the first category. Hence $B = [0,2\pi]\setminus A$, the set of points on which $\hat{f}$ is continuous, is a set of the second category and is thus uncountable and everywhere dense in $[0,2\pi]$. Next, by the usual Poisson representation formula,

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} P(r,\ \theta-t)\hat{f}(t)dt,$$

$$z = re^{i\theta} \in U,$$

where P is the Poisson kernel, and it follows from this, by a standard result, that if f is continuous at $t = \theta_0$,

$$\lim_{z\to\zeta_0} f(z) = \hat{f}(\theta_0)$$

as $z \to \zeta_0 = \exp(i\theta_0)$ in any way from inside U. Noting finally that the set of points at which any function is discontinuous is of type F_σ, we have thus proved one part of our concluding theorem; the second part is an easy consequence of (the proof of) [3, Theorem 1].

THEOREM 4.

A subset E of C is the set of points at which some bounded starlike function f does not have unrestricted limits if and only if E is of type F_σ and of first category.

A bounded starlike function thus has unrestricted limits at a set of points on C which may have measure zero but is uncountable and dense on C.

6. SOME OPEN QUESTIONS

A number of questions arise naturally from the results discussed above, and we conclude this note with a brief selection.

(a) Can we reduce the size of the exceptional set in Theorem 2 or in Theorem 3 for functions in D or for univalent functions?

In this context we note that the existence of an angular limit at a point on C does not imply the existence of a T_ϕ - limit with $\phi(r) = (1-r)^\varepsilon$ for any ε in $(0,1)$ at that point either for functions in D or for univalent functions. Details, again, to appear elsewhere.

(b) Is the conclusion of Theorem 3 sharp, with respect to the type of tangential limit obtained, for star-like functions, for univalent functions or for functions in D? Is there a function ϕ (satisfying the conditions in Section 4) such that there exists a univalent function f which does not have a T_{ϕ} - limit at any point on C?

Answers on a postcard, please.

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BOOK REVIEWS

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