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RAZMYSLOV AND SOLVABILITY

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The exponential growth in the number of active mathematicians in the present era is sometimes illustrated by the remark that there are as many mathematicians alive today as have lived - and died - since classical times. A less picturesque but more interesting indicator of mathematical activity is the rapidity with which well known conjectures and problems, sometimes of long standing, are being resolved. A recent article in the *Newsletter* (No. 11) by David Lewis on the Merkuryev-Suslin Theorem illustrates this point, and the present article (also expository, also concerned with Russian work) provides another example.

INTRODUCTION

Many readers will be familiar with, or at least aware of, the Burnside Problem in group theory, namely: must a group be finite if it is finitely generated and has exponent k? Having exponent k means that the group elements all satisfy the law $x^k = 1$ and some element has period precisely k. The problem was stated in 1902 [1], and answered negatively in 1968; an outline of developments and a bibliography, may be found in [4] and [3]. The story is by no means complete, and many problems remain open concerning these groups, but one problem concerning solvability has been settled completely by the work of Ju. P. Razmyslov in Moscow.

Let B_k denote the Burnside Variety of all groups satisfying the law $x^k = 1$; let $B_{k,n}$ represent the free group of rank n in B_k (then the n-generator groups of exponent k are just the quotient-groups of $B_{k,n}$). It has been known for many years (> 25) that:

$B_{2,n}$ is finite and abelian

$B_{3,n}$ is finite and metabelian

$B_{6,n}$ is finite and solvable, of derived length 3

$B_{4,n}$ is finite.

Of course $B_{4,n}$ is a finite 2-group, and therefore is solvable - but what is its derived length? What Razmyslov [3] calls the Problem of Hall and Higman, under attack since the 1950s, could be put thus: Is the derived length of $B_{4,n}$ independent of n?

If this were so, then B_4 would join the varieties B_2, B_3 and B_6 in being known to be "solvable" in the sense that *all* groups in these three varieties are solvable, with bounded derived lengths.

A great deal of work on $B_{4,n}$ culminated in the proof by Razmyslov that B_4 is *not* solvable - and this, due to previous work of Gupta and Newman, determined the precise nilpotency class of $B_{4,n}$ which in turn enabled Vaughan-Lee to decide the precise derived length of $B_{4,n}$.

There is, however, much more: Bachmuth and Mochizuki a little earlier had shown that B_5 is not solvable, but Razmyslov has constructed non-solvable groups of exponent p for all primes p > 3 and also of exponent 9. A consequence of all this is the following result which we might call:

Razmyslov's Theorem: The Burnside Variety B_k is solvable *only* when k = 2, 3 or 6.

This is a satisfyingly complete result, although certainly unexpected. The work has been announced and has appeared in Russian sources during the past decade; some of the details have only recently appeared in an English translation by J. Wiegold [3]. As an introduction to the ideas involved we

will explain here a relatively easy way of producing groups of exponent p^2 which are non-solvable when p > 2. This is given in [3] as a concession to the readers really, to encourage them to persevere with the far more complex details of exponent 4.

The justification for presenting here what could be read in [3] is that hopefully our account is less Delphic in style than the original - and may perhaps achieve the aim of the original if it encourages study of the entire paper. Furthermore, the use of Lie-ring-theoretic methods has proved to be a most important tool in certain problems of combinatorial group theory; the construction given here is a nice (if not very deep) illustration of its power.

FIRST STAGE - A GROUP OF EXPONENT p^k

Let A_0 be an associative algebra with identity 1, over a field K and let A_0 be generated by an infinite set of non-commuting elements $x_1, x_2, x_3, \ldots$. This means that an element of A_0 has the form $\sum_i k_i \phi_i$ where the sum is finite, $k_i \in K$ and ϕ_i is a product of generators x_j.

In A_0 we introduce the relations

$$x_i \, w \, x_i = 0$$

for every word w in A_0; note that w may be the empty word. We consider the quotient algebra A say, and we will continue to use the symbols x_i for the images in A of the original generators x_i of A_0.

Now $(1 + x_i)(1 - x_i) = 1$ in A so the elements

$$g_i = (1 + x_i) \text{ and } g_i^{-1} = (1 - x_i), \quad 1 \leq i$$

generate a group G embedded in A.

We notice now that if $c(y_1, y_2, \ldots, y_k)$ is any group commutator in elements $y_1, y_2, \ldots, y_k$ we have

$$c(g_1, \ldots, g_k) = 1 + c^*(x_1, x_2, \ldots, x_k)$$

where $c^*(x_1, x_2, \ldots, x_k)$ is the corresponding Lie commutator in A.

For example, if $c(g_1, g_2) = g_1^{-1}g_2^{-1}g_1g_2$ then

$$c^*(x_1, x_2) = x_1x_2 - x_2x_1.$$

Notice also that the group G is locally nilpotent: thus for instance in the subgroup G(n) say generated by $g_1, g_2, \ldots, g_n$ if c is any single commutator in the elements g_i which is of weight n + 1, the corresponding c^* will be a homogeneous polynomial of weight n + 1 in $x_1, x_2, \ldots, x_n$ and each monomial term in c^* will have a repeated x_i and be 0 because of the relations in A; that is c = 1 and so G(n) has nilpotency class n. Of course any finitely generated subgroup of G lies in G(n) for a suitable n.

If we now stipulate that K be a field of characteristic p we have $g_i^p = 1$ for every i, and an element of G may be written as a product of positive powers of the generators g_i.

These generators all have period p, and indeed G now has the property that every element must have period a power of p - but we wish to do more than that, we want every element to satisfy the law $g^{p^k} = 1$: in characteristic p this means that $(g - 1)^{p^k} = 0$. We note that if $g \in G$ then

$$g = (1 + x_{i_1})(1 + x_{i_2}) \ldots (1 + x_{i_t})$$

and we begin by considering

$$g = (1 + x_1)(1 + x_2) \ldots (1 + x_t), \text{ where } t \geq 1.$$

Let $D = [(1 + x_1)(1 + x_2) \ldots (1 + x_t) - 1]^{p^k}$

and let $\Delta(x_1, x_2, \ldots, x_t)$ be the homogeneous component of maximum weight in the expansion of D: this weight must be t, since terms of higher weight are killed because of repetitions. Of course D = 0 when $t < p^k$.

We notice that the homogeneous component of weight t - 1 in D must be the sum of t separate components, namely:

$$\Delta(x_1, x_2, \ldots, x_{t-1}) + \Delta(x_1, x_2, \ldots, x_{t-2}, x_t) + \ldots + \Delta(x_2, x_3, \ldots, x_t).$$

Similarly for the homogeneous components of lower weight in D.

Thus finally if we let J be the ideal of A generated by *all* $\Delta(x_{i_1}, x_{i_2}, \ldots, x_{i_s})$ for all $s \geq 1$ and for all possible choices of $i_1, i_2, \ldots, i_s$ we have an ideal which must contain $(g - 1)^{p^k}$ for *all* $g \in G$.

Now if we take the quotient algebra $A_2 = A/J$ the image of G in A_2 is a (locally nilpotent and finite) group of exponent p^k.

We might remark by the way that the approach so far is not novel, and similar ideas were used in some earlier papers on groups with exponent 4.

However, we will see that in the particular algebra A which we are going to produce below, this last step is unnecessary; in other words J will be (0) already in A and so G will automatically have exponent p^k.

AN IDENTITY

We digress now to consider an identity which holds in any associative algebra B of dimension s over a field of prime characteristic p.

Consider the symmetric function

$$S_t(y_1, y_2, \ldots, y_t) = \sum_{\sigma} y_{1\sigma}\, y_{2\sigma} \cdots y_{t\sigma}$$

where $y_i \in B$ $(1 \leq i \leq t)$ and σ runs over all permutations of the set $\{1, 2, \ldots, t\}$.

S_t is multilinear in all variables y_i, so if $b_1, b_2, \ldots, b_s$ is a basis for B and

$$y_i = \sum_{j=1}^{s} a_{ij} b_j \quad \text{we get}$$

$$S_t(y_1, y_2, \ldots, y_t)$$

$$= \sum_{j_1=1}^{s} \sum_{j_2=1}^{s} \cdots \sum_{j_t=1}^{s} a_{1j_1} a_{2j_2} \cdots a_{sj_s} S_t(b_{j_1}, b_{j_2}, \ldots, b_{j_t}).$$

Suppose now that we take $t = s(p - 1) + 1$. Then in any $S_t(b_{j_1}, b_{j_2}, \ldots, b_{j_t})$ some basis element must occur at least p times in the entries b_{j_i}. Suppose, for example, that b_1 occurs $(p + \alpha)$ times. Then $S_t(b_{j_1}, \ldots, b_{j_t})$ breaks into a sum of blocks each consisting of $(\alpha + p)!$ identical products: since $(\alpha + p)! \equiv 0 \bmod p$ this means that every $S_t(b_{j_i}, \ldots, b_{j_t}) = 0$ and so we see that

$$S_t(y_1, y_2, \ldots, y_t) \equiv 0 \text{ is an identity in } B.$$

(We remark that $S_t \equiv 0$ implies $S_{t+m} \equiv 0$, all $m \geq 0$).

We wish to use this result where B is the algebra M of all 2×2 matrices over an infinite field K of characteristic p: then for $t = 4(p - 1) + 1$ we have $S_t(y_1, y_2, \ldots, y_t) \equiv 0$ in M.

FINAL STAGE

Let us now return to the construction of a non-solvable group. For the algebra A_0 we choose the free algebra, on free generators $x_1, x_2, x_3, \ldots$, in the *variety* of algebras generated by the matrix algebra M referred to above. A_0 comes furnished with characteristic p; we construct the quotient algebra A containing the group G as before, $A = A_0/R$ where R is the ideal in A_0 generated by all expressions $x_i w x_i$, w being any (possibly empty) word in A_0

Now in the group G (generated by all $g_i = 1 + x_i$, $1 \leq i$ in A) let (u,v) denote the group commutator $u^{-1}v^{-1}uv$. Let $\delta_1 = (g_1,g_2)$, $\delta_2 = ((g_1,g_2),(g_3,g_4))$, $\delta_3 = (\delta_2,((g_5,g_6),(g_7,g_8)))$ and so on; then δ_k involves 2^k generators and lies in the kth derived subgroup of G. For every $k \geq 1$ there is a corresponding δ_k^* where $\delta_k = 1 + \delta_k^*$; clearly δ_k^* is a homogeneous polynomial in $x_1, \ldots, x_{2k}$ of degree 2^k where no term has a repeated factor x_i. There is a preimage of δ_k^* in A_0 having exactly the same form, call it d_k^*; the x_i which appear in d_k^* are the free generators of A_0

Since M contains the Lie algebra $s\ell(2,K)$ which is simple when $p > 2$ there is a Lie commutator $\gamma(a_1, \ldots, a_{2^k}) \neq 0$ in M, where γ has the same form as d_k^*, for every k; the mapping $x_j \mapsto a_j$ $(1 \leq j \leq 2^k)$ induces a homomorphism of A_0 into M (we might map all other x_j onto 0) which shows that $d_k^* \neq 0$ for any k. The form of d_k^* now shows that it does not lie in the ideal R in A_0 so the image δ_k^* is not 0 in A. Thus finally $\delta_k \neq 1$ in G for any k and so G is non-solvable.

Notice here that we need $p \geq 3$: also that we have yet to show (as we promised) that G has exponent p^2 (where we are now fixing $k = 2$).

What makes this work is the observation that the expression $\Delta(x_1, x_2, \ldots, x_t)$ is a sum of terms $S_m(u_1, u_2, \ldots u_m)$ where $m = p^k$ and the u_i are certain monomials in the elements

x_j when $t \geq p^k$; when $t < p^k$ we have $\Delta(x_1, \ldots, x_t) = 0$.

This is easy to see - an example will suffice - if, for example, $p^k = 3$ and $t = 5$ we would have

$$\Delta(x_1,x_2,x_3,x_4,x_5) = \sum_{i<j<k} S_3(x_i x_j x_k, x_r, x_s) + \sum_{\substack{i<j\\ r<s}} S_3(x_i x_j, x_r x_s, x_k)$$

where $\{i, j, k, r, s\} = \{1, 2, 3, 4, 5\}$.

Now since A above is in the variety generated by M we have the identity $S_t(y_1, \ldots, y_t) = 0$ in A whenever $t \geq 4(p-1) + 1$.

But $p^2 - (4p-3) = (p-2)^2 - 1 \geq 0$ if $p \geq 3$. This means that already in A the relation $(g-1)^{p^2} = 0$ is satisfied for all g in G.

Thus finally we have arrived at a non-solvable group G of exponent p^2 which is also locally a finite p-group.

FOCAL SCUIR

In consonance with the didactic tendency of this journal, we end with an exercise for the reader: taking 3x3 matrices and applying techniques similar to those used above, construct a non-solvable group of exponent 8.

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