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"COMPLEMENTARY PIVOTING ON A PSEUDOMANIFOLD STRUCTURE WITH APPLICATIONS TO THE DECISION SCIENCES"

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In these heady days when the influence of computing on mathematics waxes ever more strongly, a natural development is the enhanced status of constructive existence proofs. For example, the standard inf/sup argument used to prove the intermediate value theorem could well be supplanted by the slightly longer (but constructive!) proof which repeatedly bisects the interval under consideration. However, it is often difficult to obtain a constructive proof of a known result. Brouwer [2] published the proof of his famous fixed point theorem in 1912, and several alternative proofs appeared in later years, but not until 1967 did Scarf [5] give an explicitly constructive proof (which enabled one to find "approximate fixed points" z, in the sense that $||f(z)-z||_\infty$ is small). Incidentally, Hirsch [3] in 1963 published a proof of (a result equivalent to) the Brouwer theorem, the constructive nature of which was only noticed in 1976 by Kellogg, Li and Yorke [4].

Scarf's paper marks a watershed in the numerical solution of nondifferentiable nonlinear systems of algebraic equations. Its basic algorithmic procedure, that of *complementary pivoting*, was the inspiration for an explosion of research activity in

the 1970s. A description of this procedure follows.

First, an analogy. Consider a house having one entrance. The house consists of a finite number of rooms. All rooms have 0, 1 or 2 doors. All doors link exactly two rooms (the exterior of the house is regarded as a room). Then if one enters the house and obeys the rule that one cannot enter and leave a room through the same door, it is not difficult to see that one's path must terminate in a room inside the house having one door.

Let's translate the analogy into mathematics (I shall cut some technical corners here). Our house becomes a closed, bounded, simply connected subset D of R^n. Each room corresponds to an *n-simplex* (a 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, etc.: an n-simplex is the closed convex hull of its n+1 extreme points or *vertices*). The set D is the union of these n-simplexes, and moreover the n-simplexes are required to fit together in a geometrically pleasing way: the intersection of each pair S_1, S_2 is either the empty set or an m-simplex, $0 \leq m \leq n$, whose vertices are vertices both of S_1 and of S_2. Every n-simplex has n+1 (n-1)-dimensional *faces* which are themselves (n-1)-simplexes. Certain of these faces will be designated as doors, in a way that corresponds to the analogy above.

To each vertex of each n-simplex we assign a *label* chosen from the set $T_n = \{0,1,\ldots,n\}$. (The way in which this assignment is carried out depends on the nature of the problem being solved.) Thus each n-simplex S has a set of n+1 labels associated with it, which may include some repetitions. If this set equals T_n, we say that S is *completely labelled* (cl). Similarly associate a set of n labels with each (n-1)-simplex S' which is a face of some n-simplex. We say that S' is *almost completely labelled* (acl) iff this set equals T_{n-1}.

Observation: if S' is an acl face of S, then either S is cl or S has exactly one other acl face.

Suppose that we choose our labelling so that among all the (n-1)-simplexes making up the boundary of D, exactly one is acl. Returning to our house analogy, this exceptional (n-1)-simplex is our entrance. The acl faces in D are the doors in the house. All the conditions described in the analogy are now seen to hold (the Observation shows that each n-simplex has 0, 1, or 2 acl faces). We can therefore start from the unique acl face on the boundary of D and move (*pivot*) from one n-simplex to another through the acl faces until we terminate at a cl n-simplex.

The labelling should be such that cl n-simplexes yield approximate solutions of the problem under consideration. For example, if $f = (f_1, f_2, \ldots, f_n) : D \to R^n$ is continuous and one wishes to solve $f(x) = (0, 0, \ldots, 0)$, an appropriate labelling of each vertex z is

$$0 \text{ if } f(z) < 0 \text{ for } i = 1, 2, \ldots n$$

$$\min\{i : f_i(z) \geq 0\} \text{ otherwise.}$$

It is easy to show that with this labelling every point y in a cl n-simplex S has $||f(y)||_\infty$ "small" (the smaller the diameter of S, the smaller the bound on $||f(y)||_\infty$). To ensure that exactly one acl (n-1)-simplex lies on the boundary of D, one can for example add extra artificial n-simplexes to D.

This, in essence, is Scarf's algorithm. Many improvements to it have been put forward, and many other algorithms incorporating some form of complementary pivoting have been published since it first appeared.

The objectives of Gould and Tolle's book are (i) "to present a unified framework into which most of these algorithms can fit" and (ii) "to provide instructive examples of their

application". The book is easily divided into two halves reflecting these objectives: Chapters 1-4 cover (i) while Chapters 5-8 examine applications of complementary pivoting.

Although Chapters 1-4 do reach objective (i), the treatment is a little disappointing. It is a determined rigorous abstraction of the way in which our n-simplexes were assembled to form the set D (this generalization is called a *pseudomanifold*) followed by a description of the complementary pivoting algorithm as it operates on this structure. The style of these early chapters is adequate but unexciting. (I note that I was baffled by the "definition" of "basic solutions" on page 24!). The book's choice of examples leans towards operations research, in particular linear programming, and end-of-chapter exercises are good though perhaps too few. My main complaint is that the basic algorithm is not reached until page 70, at the end of Chapter 4. This is surely much too late, considering the publisher's claim that books in this series "introduce the reader to a field of mathematics". A novice should hardly be asked to wait this long to learn about what is, more or less, the algorithm of Scarf we described earlier. One can see the authors' dilemma, caught between the above claim and their own objective (i), but I feel that a page or two in Chapter 2 (where pseudomanifolds are defined) giving a sketch of the algorithm would have been a good investment. Alternatively, Chapter 3 ("examples and constructions of pseudomanifolds") and Chapter 4 ("complementary pivoting") could have been interchanged.

Happily, I can be much more positive about Chapters 5 to 8. These deal with the application of the complementary pivoting algorithm to linear complementarity theory, Brouwer and Kakutani fixed points, unconstrained nondifferentiable minimization, and nondifferentiable programming. This is mostly operations research material which many mathematicians never encounter. The presentation seems somewhat smoother than in the earlier chapters, perhaps because there is less effort at abstraction. All in all it makes for interesting reading.

Merrill's significant extension of Scarf's algorithm is hidden in section 6.3, "fixed points of upper semicontinuous point-to-set mappings", and deserves a much more prominent place (it isn't even mentioned in the index, which is far too short and selective). Furthermore, there is no description of any of the several new algorithms developed since the mid-70s which now dominate the field; all that is offered is a list of references. This is difficult to justify, and greatly lessens the value of the book to current researchers.

For beginners, my advice is to look first at the attractive review article of Allgower and Georg [1], which is fairly up-to-date and discusses the implementation and efficiency of complementary pivoting algorithms as well as covering nicely much of the book's early material.

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(This review was written while the author was at Waterford Regional Technical College)

"FINITE FIELDS" (Encyclopaedia of Mathematics and its Applications, Volume 20)

By *Rudolf Lidl and Harald Niederreiter*

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Recently, before breakfast one morning, I received a telephone call from a former student of mine who is now teaching at a Regional Technical College. He wanted to know if a certain polynomial of degree 16 was irreducible over GF(2). This question was motivated by a project one of his students was hoping to carry out in coding theory. For me, this incident epitomises the sudden rush of interest from many sides in the classical theory of finite fields.

It is difficult to know where to begin describing the book in hand. Perhaps some raw statistics may serve to convey something of the scope of this massive work. It has 755 (+xx) pages in all handsomely bound. There is a comprehensive bibliography of 160 pages of detailed references to finite fields