BOUNDED CONJUGACY CONDITIONS

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This article surveys the history of the study of groups with finite conjugacy conditions. It presents some new results obtained by the author. It is based on a talk given by the author to the Group Theory Conference held in Galway on 11/12 May 1984.

1. BFC Groups and Derived Groups

A group is said to be a BFC group if there is a finite upper bound on the sizes of its conjugacy classes. B.H. Neumann characterized such groups in 1954 [8].

Theorem 1 (Neumann [8]). If G is a BFC group then the derived group G' is finite. Hence the BFC groups are precisely the finite-by-abelian groups.

The BFC-number of a BFC group is the maximum of the sizes of its conjugacy classes. We write it as $n(G)$ or just n. Already in [8] Neumann wondered whether the order $|G'|$ of a BFC group G could be bounded in terms of $n(G)$. The question was answered affirmatively by Wiegold in [14]; refinement of the argument led to the following result.

Theorem 2 (Wiegold [15]). If G is a BFC group then

$$|G'| \leq n^{\frac{1}{2}n^4(\log n)^3},$$

where the logarithm is to base 2 (as will be the case throughout, unless specified otherwise).

However, even this bound is much too big. Examples of groups led Wiegold to the conjecture that $|G'| \leq n^{\frac{1}{2}(1+\log n)}$

for any BFC group, and to date no group has been found that disproves this, although there are groups known for which equality holds with arbitrarily large BFC-numbers. For p-groups the conjecture was sharpened by requiring that the logarithm be to base p.

Macdonald improved the bound of Theorem 2 in [7] to $|G'| \leq n^{6n(\log n)^3}$, but the first significant advance came with Shepperd and Wiegold's paper [10] of 1963.

Theorem 3 (Shepperd and Wiegold [10]). Let G be a BFC group.

(i) If G is soluble, then $|G'| \leq n^{q(\log n)}$, where $q(x)$ is a certain quintic polynomial.

(ii) If G is nilpotent of class 2, then $|G'| \leq n^{(\log n)^2}$.

Thus, at the price of restricting attention to special classes of BFC groups, they were able to produce bounds in which the exponent is purely logarithmic, as the Wiegold conjecture requires.

This set the pattern for the next 11 years. By using commutator calculations and Lie ring methods, workers established the Wiegold conjecture first for class 2 p-groups (Bride [2]), then for metabelian p-groups (Vaughan-Lee [11]), and finally for p-groups in general (Vaughan-Lee [12]).

Theorem 4 (Vaughan-Lee [12]). Let G be a p-group with $n(G) = p^b$. Then $|G'| \leq p^{\frac{1}{2}b(b+1)}$.

The fact that this represents the stronger version of the Wiegold conjecture has been important since. It was used in the proof of the following result, by P.M. Neumann and Vaughan-Lee in [9].

Theorem 5 (Neumann and Vaughan-Lee [9]). Let G be a BFC group.

(i) If G is soluble, then $|G'| \leq n^{\frac{1}{2}(5+\log n)}$.

(ii) In any case, $|G'| \leq n^{\frac{1}{2}(3+5\log n)}$.

The result for soluble groups improves Shepperd and Wiegold's, and indeed is just n^2 away from the Wiegold conjecture. This is still the best bound known for soluble groups.

The result for general BFC groups is not quite so good, but it is worth recalling that the previous best bound was Macdonald's, in 1961. Recently the present author has tightened this bound ([3] and [4]). This, however, depends on the Classification Theorem for finite simple groups, for it uses the fact that all such groups may be generated by two elements.

Theorem 6 (Cartwright [3]). If G is a BFC group, then $|G'| \leq n^{\frac{1}{2}(41+\log n)}$.

This is just n^{20} away from the Wiegold conjecture. The proof, as well as the result, parallels Neumann and Vaughan-Lee's bound for soluble groups.

Theorems 4, 5(i) and 6 represent the current state of knowledge.

2. Class of p-Groups

The bound given in Theorem 4 on the size of the derived group of a BFC p-group gives an immediate bound on its class; namely, a p-group with BFC number p^b has class at most $\frac{1}{2}(b^2+b)+1$. This bound is, however, much too big. The 'class-breadth conjecture' for finite p-groups was that such a group with class c and BFC-number p^b satisfies $c \leq b+1$. This would mean that the dihedral groups, for example, are a natural case where the limit is attained. In 1969 the following result was proved, which goes some way towards this.

Theorem 7 (Leedham-Green, P.M. Neumann and Wiegold [6]). Let G be a finite p-group with BFC number p^b and class c. Then $c \leq (\frac{p}{p-1})b+1$. In particular, if G is nonabelian then $c \leq 2b$.

However, in a series of papers in 1980-81, examples of groups were constructed for each integer k with $c > b+k$. In fact the groups produced in [5] have class approaching $b+b^{\frac{1}{2}}$. These examples are all 2-groups; so far, no-one has found counterexamples for odd primes.

The most recent result in this area (in [4]) improves the bound of Theorem 7 to $c \leq \frac{5}{3}b+1$.

3. Derived Length of Soluble Groups

It is well-known that the derived length of a nilpotent group of class c is at most $1 + \log.c$. From Theorem 7 we may therefore deduce the following.

Theorem 8. Let G be a nonabelian finite p-group with BFC-number p^b and derived length d. Then $d \leq 2 + \log b$.

Thus if H is a nilpotent group of BFC-number n and derived length d, we have $d \leq 2 + \log \log n$.

Despite the gap in Theorem 7, this bound is very nearly the best possible. For if H is taken to be a Sylow 2-subgroup of S_{2^k}, the symmetric group of degree 2^k, then H has derived length precisely k and order (and therefore BFC-number) less than 2^{2^k}, so that here we have $d > \log \log n$.

Perhaps more surprising than this is that a similar result can be proved for soluble groups in general, as P.M. Neumann and Vaughan-Lee showed in [9].

Theorem 9 (Neumann and Vaughan-Lee [9]). Let G be a nonabelian soluble BFC group with BFC-number n and derived length d. Then $d < a \log \log n + 8$, where $a = 1+(5/\log 9) \approx 2.58$.

Among known examples, those with largest derived length relative to their BFC-number are the 2-groups mentioned above.

4. Generalisations

Baer [1] generalised the concept of a BFC group in the following way. Suppose G is a group and $H,K \leq G$. We call (H,K) a BFC pair in G if there are integers m,n such that $|H:C_H(y)| \leq m$ for all $y \in K$ and $|K:C_K(x)| \leq n$ for all $x \in H$. Theorem 1 then generalises as follows.

Theorem 10 (Vaughan-Lee [13]). Suppose (H,K) is a BFC pair in a group G. Let M and N be the normal closures in $\langle H,K\rangle$ of H and K respectively. Then the commutator subgroup $[H,K]$ is finite if and only if both $|M:H|$ and $|N:K|$ are finite.

Moreover, the size of $[H,K]$ may be bounded in a way that parallels Theorems 2-6. Vaughan-Lee in [13] proves the following result, which deals with the case where both H and K are normal in $\langle H,K\rangle$; but it is easy to see how this may be adapted to the more general case. Naturally, the parameters $|M:H|$ and $|N:K|$ must also be used.

Theorem 11 (Vaughan-Lee [13]). Suppose (H,K) is a BFC-pair in G, with $H,K \trianglelefteq \langle H,K\rangle$, and suppose m and n are upper bounds for $\{|H:C_H(y)| : y \in K\}$ and $\{|K:C_K(x)| : x \in H\}$ respectively. Then $|[H,K]| \leq m^{\frac{1}{2}(3+13\log n)}$.

Corresponding to the Wiegold conjecture for BFC groups, there is a conjecture in this more general situation. It is thought that $|[H,K]| \leq m^{\log n}$ always holds. (This is a symmetric bound, for $m^{\log n} = 2^{\log m.\log n}$.) Again, cases are known in which this value is attained, but no example has come to light which disproves the conjecture.

Unlike the situation with BFC groups, the conjecture has been established for relatively few special cases. Vaughan-Lee in [13] verified the conjecture in the case that $[H,K]$ is in the centre of $\langle H,K\rangle$, and improved on Theorem 11 in the case where $[H,K]$ is central in H. The present author has extended this slightly.

Theorem 12 (Cartwright [4]). Let (H,K) be a BFC pair in a p-group G, with $H,K \trianglelefteq \langle H,K\rangle$ and $[H,K] \leq Z(H)$. Let $\max\{|H:C_H(y)| : y \in K\} = p^a$ and $\max\{|K:C_K(x)| : x \in H\} = p^b$. Then $|[H,K]| \leq p^{ab}$.

Also in [4], improved results are obtained for more general cases.

Theorem 13 (Cartwright [4]). Let (H,K) be a BFC pair in G, with $H,K \trianglelefteq \langle H,K\rangle$. Let $\max\{|H:C_H(y)| : y \in K\} = m$ and $\max\{|K:C_K(x)| : x \in H\} = n$.

(i) If G is a p-group then $|[H,K]| \leq m^{\frac{1}{2}+(7/4)\log_p n}$.

(ii) In any case, $|[H,K]| \leq m^{(41/2)+(25/8)\log n}$.

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5. References

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OBVIOUS

We in the Math. Faculty in the University of Cork were alarmed to discover recently that our students had opened a file on us. Apparently:

'If Professor Barry says it is obvious then it is obvious at once. If Professor Twomey says it is obvious then it will be obvious after a few minutes' thought. If Professor Holland says it is obvious then it will become obvious after about three weeks' intensive study. If Professor Harte says it is obvious then it is probably not true.'

Robin Harte