AN ELEMENTARY NUMBER THEORY RESULT

*P. Birch and **D. Singmaster

The first two functions introduced in number theory are usually $\pi(n)$, the number of primes less than or equal to n, and $\phi(n)$, Euler's ϕ-function, the number of totitives of n (i.e. positive integers which are $\leq n$ and coprime to n). There is a simple relationship between these functions, namely: $\phi(n) > \pi(n)$ apart from a finite number of exceptional n.

This relationship, despite its simplicity, is generally unknown - it does not occur in Dickson, Hardy and Wright, or Leveque. After our discovery of it, A. Makowski referred us to [M] and [S]. The first appeared in a small unreviewed journal. The second is a proof of Erdos described by Sierpinski in the Polish original edition of his "Elementary Theory of Numbers", but omitted from the English edition. Consequently, we are presenting the result again, in the hope that it will become better known. Our proof is similar to [M].

We need one non-elementary but well known result.

Bertrand's Postulate. [H, p. 343]. For any $x > 1$, there is a prime p such that $2x > p > x$.□

From this, we obtain the following.

<u>Theorem 1.</u> If n has $r \geq 5$ distinct prime factors then $\pi(\sqrt{n}) \geq 2r$.

<u>Proof.</u> If n has more than 5 distinct prime factors, then $n \geq 2\cdot3\cdot5\cdot7\cdot11 = 2310$ and hence $\sqrt{n} > 48$. Since 43 is the 14th prime, the result is true for $r = 5$, 6 or 7. We now prove it generally for $r \geq 7$ by induction. Assume the result is true for every number which has k prime factors, for some

$k \geq 7$, and suppose n has $k+1$ prime factors. Hence $n > 16m$ which implies $\sqrt{n} > 4\sqrt{m}$. By Bertrand's Postulate, there are at least two primes between $\sqrt{n}$ and $\sqrt{m}$. By the induction hypothesis, $\pi(\sqrt{m}) \geq 2k$, hence $\pi(\sqrt{n}) \geq 2k+2$.□

Denote the number of composite totitives of n by $c(n)$. The number of prime totitives of n is the number of primes less than or equal to n minus the number of prime factors of n, i.e. $\pi(n) - r(n)$ (where we have used $r(n)$ for what we previously denoted r). Recalling that 1 is neither prime nor composite, we have:

$$(\dagger) \qquad \phi(n) = c(n) + (\pi(n) - r(n)) + 1.$$

<u>Lemma.</u> If $n \geq 92$, then $\phi(n) > \pi(n)$.

<u>Proof.</u> From (†), we see that $\phi(n) > \pi(n)$ if and only if $c(n) \geq r(n)$.

If $r(n) \leq 2$, then at least two of $\{4, 9, 25, 49\}$ are coprime to n and so $c(n) \geq 2 \geq r(n)$ for $n \geq 50$.

If $r(n) = 3$ and $2 \nmid n$, then 4, 8, 16 are coprime to n.

If $r(n) = 3$ and $3 \nmid n$, then 9, 27, 81 are coprime to n.

If $r(n) = 3$ and $n = 2^a 3^b p^c$ for some prime $p > 3$, then at least one of 25 or 49 is coprime to n and at least two of $\{35, 55, 65, 77, 91\}$ are coprime to n. So if $r(n) = 3$ and $n \geq 92$, then $c(n) \geq r(n)$.

If $r(n) = 4$, then $n \geq 2\cdot3\cdot5\cdot7 = 210$ and there are always at least four composite totitives, q_1^2, q_2^2, q_1q_2, q_1q_3, where $q_1 \leq 11$, $q_2 \leq 13$ and $q_3 \leq 17$ are prime totitives of n. So $c(n) \geq 4 = r(n)$.

If $r(n) = 5$, then since $\pi(\sqrt{n}) \geq 2r(n)$, there are at least r prime totitives of n, say $q_1, q_2, \ldots, q_r$, all less than $\sqrt{n}$. Hence $q_1^2, q_2^2, \ldots, q_r^2$ are all less than n and so $c(n) \geq r$. (In fact, we have $c(n) \geq r(r+1)/2$.)□

Either by pursuing the argument a little further or by examining every integer less than 92, we obtain our main result.

Theorem 2.

$\phi(n) < \pi(n)$ if and only if $n = 6, 10, 12, 18, 24, 30, 42$ or 60;

$\phi(n) = \pi(n)$ if and only if $n = 2, 3, 4, 8, 10, 14, 20$ or 90;

$\phi(n) > \pi(n)$ for all other n.□

Corollary. $\phi(n) > n/\log n$ except for $n = 1, 2, 3, 4, 6, 10, 12, 18$ or 30.

Proof. It is well known that $\pi(n) > n/\log n$ for $n \geq 17$ [R, p. 71 or SW, p. 106]. The Corollary follows immediately for $n > 60$ and some calculation yields the complete result.□

The following two results are immediate corollaries to our work using a little calculation.

Proposition 1. $\phi(n) \geq \sqrt{n}$ except for $n = 2$ or 6. (Vaidya [V].)

Proposition 2. An integer n has the property that all of its totitives are prime or 1 if and only if $n = 1, 2, 3, 4, 6, 8, 12, 18, 24$ or 30. (Schatunowsky (1893) and Wolfskel (1900) [D, p. 132, item 73 and p. 134, item 91].)

It is known [SW, 4.1] that

$$\underline{\lim}_{n\to\infty} \frac{\phi(n)\log\log n}{n} = e^{-\gamma}, \text{ where } \gamma \text{ is Euler's constant}$$

$(= \underline{\lim}_{r\to\infty}\{\sum_{i=1}^{r} 1/i - \log r\})$ and $\overline{\lim}_{n\to\infty} \frac{\phi(n)}{n} = 1$. The first of these is more accurate than our Corollary, but only asymptotically.

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* *Department of Mathematics, University College, Cork.*

** *Department of Mathematics, Polytechnic of the South Bank, London.*