ONE ASPECT OF THE WORK OF ALAIN CONNES

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Introduction

As we all surely know by now, the recipients of the most recently awarded Fields Medals are William P. Thurston of Princeton University, Shing-Tung Yau of the Institute for Advanced Study, Princeton and Alain Connes of Institut des Hautes Études Scientifiques, France. Fields Medals are awarded by the International Mathematical Union on the occasion of an International Congress of Mathematicians, and are the equivalent for mathematicians of the Nobel prize. The last such Congress was originally scheduled to take place in Warsaw in August, 1982, but in fact took place there one year later due to political unrest in Poland.

Thurston's work is in foliations and topology of low dimensional manifolds, Yau's is in differential geometry and partial differential equations and Connes' is in operator algebras. An appraisal of the work of each recipient was published in the Notices of the American Mathematical Society in October, 1982. In particular, Calvin Moore undertook to describe some of the fundamental achievements of Alain Connes.

Much of Connes' earlier work was concerned with the classification by "types" of factors of von Neumann algebras, and three of the five papers of Connes cited by Moore concern this area. However, the fourth (sur la théorie non-commutative de l'intégration, which is reference [2] here) and fifth concern (amongst other things) the interplay between operator algebras and foliations. This subject, which has been called "non-commutative differential geometry" is "(a) fusion of geometry and functional analysis ... likely to have a significant influence on future developments" in the words of Atiyah in his review of [2] for Mathematical Reviews.

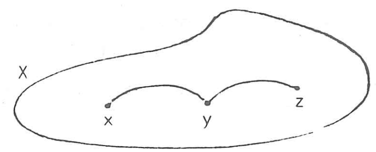
Some of my own work has been in this area, and my purpose here is to shed a little light on the sort of constructions made and the results obtained in this new and interesting area of mathematics.

1. Topological Groupoids and C*-Algebras

A *groupoid* G with *object set* X is a small category with object set X such that each element α of G has an inverse α^{-1}.

Examples

(1)



Let X be a topological space and let G(x,y) = (homotopy classes of) paths from x to y. Then $G = \bigcup_{x,y\in X} G(x,y)$ is a groupoid (the fundamental groupoid) over X with composition just the composition of (homotopy classes of) paths, identity I_x at x the trivial path and inverse "traverse the path backwards."

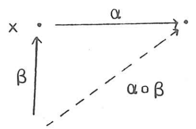
(2) A group is a groupoid with one object, i.e. X is a singleton set consisting of the identity of G. In fact, a groupoid is a group if and only if X is a singleton set.

(3) Suppose that a group H acts on the right of a set X. Then G = XxH has a natural groupoid structure over X in which the product (x',h')o(x,h) is defined if and only if x' = x.h, and is then defined to be (x,hh'). The inverse of (x,h) is $(x.h,h^{-1})$ and the identity at x is (x,e), where e is the identity of H.

Many other examples of groupoids can be given, but these three should serve to convey their nature.

A *topological groupoid* is a groupoid in which both G and X are topological spaces and all the structure maps of G are continuous, i.e. the composition, inverse map and the map $x \longmapsto I_x$ are all continuous.

Suppose from now on that G is a topological groupoid over X and G and X are both locally compact Hausdorff spaces. Given $\alpha \in G$, there exists unique $x,y \in X$ such that $\alpha \in G(x,y)$; let $\pi(\alpha) = x$, the *initial point* of α, and let $\pi'(\alpha) = y$, the *final point* of α. Let $G^x = \{\beta \in G;\ \pi'(\beta) = x\}$. Then an element $\alpha \in G(x,y)$ induces a homeomorphism $L_\alpha : G^x \to G^y$ defined by $L_\alpha(\beta) = \alpha\circ\beta$.



L_α is called *left multiplication* by α.

Guided by certain analogies between group theory and ergodic theory, G.W. Mackey introduced, in 1966, the notions of measure groupoid and ergodic groupoid. In the topological context these ideas lead one naturally to formulate a concept of left invariant (or Haar) measure on a groupoid G, and to consider function spaces associated with G. In practice, the most convenient form of an invariant measure is contained in the following:

<u>Definition</u>: A *Haar measure* on G is a family of non-trivial Radon measures $\{\mu_x;\ x \in X\}$ on G such that:

(1) $\mathrm{supp}(\mu_x) \subseteq G^x$ for each $x \in X$.

(2) The μ_x are left invariant in the sense that

$$\int_G f\,d\mu_x = \int_G f\circ L_{\alpha^{-1}}\,d\mu_y$$

for all $x,y \in X$, $\alpha \in G(x,y)$ and $f \in C_c(G)$.

(3) The map $x \longmapsto \mu_x$ is vaguely continuous, i.e. the map

$x \longmapsto \int f\,d\mu_x$ is continuous for each $f \in C_c(G)$.

In this definition, and elsewhere, $C_c(G)$ denotes the space of all continuous scalar functions on G with compact support.

The relationship between two Haar measures on G is, unlike the group case, quite complicated, see [6]. However, any Haar measure on G induces a $*$-algebra structure on $C_c(G)$, as follows. Given $f,g \in C_c(G)$, we define $f*g$ on G by

$$(f*g)(\alpha) = \int_G f(\beta)g(\beta^{-1}\alpha)\,d\mu_{\pi'(\alpha)}(\beta).$$

We define also an involution $f \longmapsto f^*$ by $f^*(\alpha) = \overline{f(\alpha^{-1})}$.

Haar measures and the convolution product above were studied by the author in [6], [7] and by Renault in [5], and one of the main basic results is as follows.

<u>Theorem</u>. $C_c(G)$ is an associative $*$-algebra with these operations and is, moreover, a topological $*$-algebra in the inductive limit topology.

The remainder of this section is concerned with associating a C*-algebra with G, and the development is similar to the Effros-Hahn construction of transformation group C*-algebras, see [5].

A *representation* of $C_c(G)$ on a Hilbert space H is a $*$-homomorphism $L : C_c(G) \longrightarrow \beta(H)$ which is continuous when $C_c(G)$ has the inductive limit topology and $\beta(H)$ the weak operator topology, and is such that the linear span of $\{L(f)\xi;\ f \in C_c(G),\ \xi \in H\}$ is dense in H.

For $f \in C_c(G)$, define

$$||f||' = \sup_{x\in X} \int |f|\,d\mu_x, \qquad ||f||'' = \sup_{x\in X} \int |f|\,d(\mu_x)^{-1}$$

and finally put $||f||_1 = \max(||f||', ||f||'')$.

<u>Proposition</u> ([5])

(i) $||\ ||_1$ is a norm on $C_c(G)$ defining a topology coarser than the inductive limit topology.

(ii) $||\ ||_1$ is a $*$-algebra norm on $C_c(G)$, i.e. $||f*g||_1 \leq ||f||_1||g||_1$ and $||f^*||_1 = ||f||_1$ for all $f,g \in C_c(G)$.

<u>Definition</u> ([5]): A representation L of $C_c(G)$ is *bounded* if $||L(f)|| \leq ||f||_1$ for all $f \in C_c(G)$.

Now define, for all $f \in C_c(G)$, $||f|| = \sup ||L(f)||$ where L ranges over all bounded representations of $C_c(G)$.

It is easy to see that $||\ ||$ is a C^*-semi norm, and it is shown by exhibiting enough bounded representations (the regular representations in fact) that it is a norm. Finally, we denote by $C^*(G)$ the completion of $C_c(G)$ with respect to $||\ ||$. Then $C^*(G)$ is a C^*-algebra, i.e. a Banach algebra with conjugate linear involution $f \longmapsto f^*$ such that $||f^* * f|| = ||f||^2$ for all f, and is called the *C^*-algebra of the groupoid* G.

2. <u>Foliations and the Holonomy Groupoid</u>

Foliations have a long history even though the definition and subject matter were not formalised until the 1940s by Ehresmann and Reeb. One encounters foliations in:

(a) Submersions of manifolds (here the leaves are the components of the fibres).

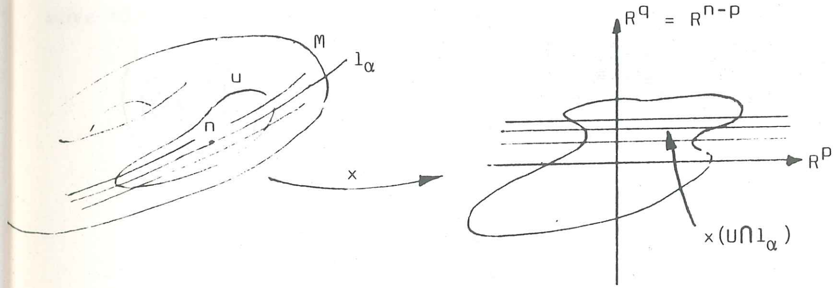
(b) Bundles with discrete structure group.

(c) Actions of Lie groups (here the leaves are the orbits).

(d) Differential equations (here the solutions are the leaves).

Definition ([1], [4]): Let M be an n-dimensional manifold and let p,q be natural numbers such that p+q = n. A *p-dimensional class C^r foliation* of M is a decomposition of M into a union of disjoint connected subsets $\{l_\alpha\}_{\alpha \in A}$, called the *leaves* of the foliation, with the following property: every point m of M has a neighbourhood U and a system of local class C^r coordinates $x = (x^1, x^2, \ldots, x^n): U \to R^n$ such that for each $\alpha \in A$ the components of $U \cap l_\alpha$ are described by the equations

$$x^{p+1} = \text{constant}, \ \ldots, \ x^n = \text{constant}.$$



We denote such a foliation by $\mathcal{F} = \{l_\alpha\}_{\alpha \in A}$. p is called the *dimension* and q = n-p the *codimension* of $\mathcal{F}$.

Note that every leaf of $\mathcal{F}$ is a p-dimensional embedded submanifold of M but this embedding need not be proper as the leaves can be dense in M.

Local coordinates with the property mentioned in the definition above are said to be *distinguished* by the foliation. If x,y are two such coordinate systems defined on an open set $U \subset M$, then yx^{-1} is a local C^r diffeomorphism: $R^n \rightarrowtail R^n$ giving the "change of coordinates" and is expressed by the equations

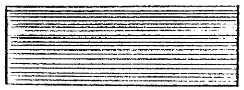
$$y^i = y^i(x^1, x^2, \ldots, x^n), \quad i = 1,2,\ldots,n$$

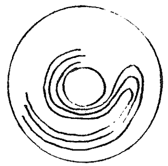
and these must satisfy the differential equations

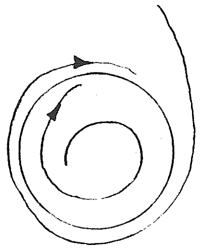
$$\frac{\partial y_i}{\partial x_j} = 0 \quad 1 \le j \le p < i \le n$$

in U. This means that yx^{-1} maps leaves into leaves. Thus, whilst an n-dimensional manifold looks locally like R^n, an n-dimensional manifold with p-dimensional foliation looks locally like $R^n = R^{n-p}xR^p$ trivially foliated by p-dimensional hyperplanes parallel to R^p.

Examples







trajectories of a differential equation

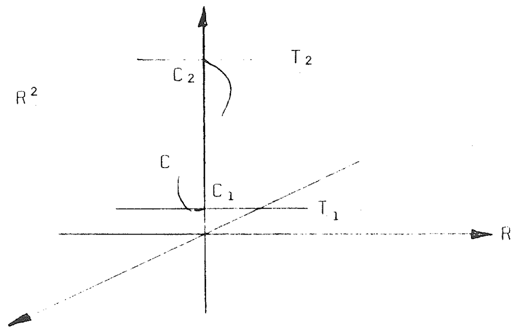
The Holonomy Groupoid

Let $(M,\mathcal{F})$ be a foliated manifold as above, and let (U,x) be a distinguished local coordinate. Then the *plaques* of U are given by the equation $(p_2x)(m)$ = constant, where $p_2 : R^pxR^q \to R^q$ is the projection. Give M the "topology of leaves", i.e. the topology on M which has the plaques of distinguished open sets as a basis, and call the resulting space F A continuous function $U \to R^q$ is called *distinguished* if it is locally of the form hop_2ox, where h is a local homeomorphism of R^q. Let D be the sheaf of germs of distinguished function and $\sigma: D \to F$ the map sending a germ to its source; σ is a covering map. [$f \sim g$ if there exists $m \in M$ and a neighbourhood V of m such that $f|_V = g|_V$, then $\sim$ is an equivalence relation and an equivalence class of $\sim$ is called a *germ* at m.] It can

be shown that the fundamental groupoid of F acts on D and we elaborate a little on this below. Finally, on identifying elements of the fundamental groupoid which give the same action we get the *holonomy groupoid* G of $(M,\mathcal{F})$. G is a topological groupoid in a natural way, in fact a locally trivial topological groupoid. It is this construction together with the results of §1 which bring about the sort of application of functional analysis to differential geometry that we have in mind.

Before considering such applications we will look a little more closely at the notion of holonomy.

Consider a curve C lying in the plane R^2 as shown:



Suppose C_1 has coordinates $(0,(0,\alpha_1))$ and C_2 has coordinates $(0,(0,\alpha_2))$, and that T_1 and T_2 are perpendicular, and hence transverse, to R^2 and passing through C_1 and C_2 respectively.

Any neighbourhood U of C in R^3 intersects T_1 and T_2 in neighbourhoods of C_1 and C_2 in T_1 and T_2 respectively, and hence induces a C^r-diffeomorphism $(x,(0,\alpha_1)) \longmapsto (x,(0,\alpha_2))$ of a neighbourhood of C_1 in T_1 onto a neighbourhood of C_2 in T_2. Clearly the same statement is true for general transversals T_1 and T_2, though the required C^r-diffeomorphism is then more complicated to write down.

Now suppose, generally, that $C:[0,1] \to M$ is a path lying in a leaf l of a foliation $\mathcal{F}$ of M, and that T_0 and T_1 are two submanifolds of M transverse to $\mathcal{F}$ and containing $z_0 = C(0)$ and $z_1 = C(1)$ (a submanifold W is transverse to $\mathcal{F}$ if for each $z \in M$, we have $T_zM = T_zW \oplus T_zL$, where L is the leaf passing through z and "T_z" denotes the tangent space at z). Then to each neighbourhood U of C in M there corresponds a C^r-diffeomorphism Φ_C of a neighbourhood of z_0 in T_0 onto a neighbourhood of z_1 in T_1 such that:

(i) If Φ_C is defined at $z \in T_0$, then $\Phi_C(z)$ belongs to $T_1 \cap$ leaf containing z.

(ii) The germ of Φ_C at z_0 does not depend on U nor on the choice of C up to homotopy.

To construct Φ_C we proceed as follows. Consider a sequence of distinguished functions f_i, $i = 0,1,2,\dots,r$ defined on open sets V_i and an ordered set of points t_i of $[0,1]$ such that $t_0 = 0$, $t_r = 1$ and $C([t_k,t_{k+1}]) \subset V_k$ for $k = 0,\dots,r-1$. Let T^i, for each i, be a submanifold transverse to $\mathcal{F}$ containing the points $C(t_i)$, $i = 0,1,2,\dots,r$, and such that $T^0 = T_0$, $T^r = T_1$. We can suppose that $F_i(C(t_i)) = 0$ and that f_i is of the form $h_i \circ p_2 \circ x_i$, where x_i is a distinguished local coordinate, for all i. For each $i < r$, x_i carries the portion of the curve C between $C(t_i)$ and $C(t_{i+1})$, together with U, onto a curve in R^n lying in the hyperplane R^p essentially as depicted above, together with a neighbourhood of this curve in R^n. Hence, applying x_i^{-1} to the diffeomorphism described there, we see that for each $i < r$ there is a C^r-diffeomorphism Φ_i of a neighbourhood of $C(t_i)$ in T^i onto a neighbourhood of $C(t_{i+1})$ in T^{i+1} such that $\Phi_i(z)$ belongs to the leaf of $V_i \cap U$ passing through z for each z where $\Phi_i(z)$ is defined. Then Φ_C is simply the composite $\Phi_{r-1} \circ \Phi_{r-2} \circ \dots \circ \Phi_1 \circ \Phi_0$, and it is clear from statements (i) and (ii) that the fundamental groupoid of F does act on D, as required.

By means of general results of [6], Haar measures exist on the holonomy groupoid G, even though G is not Hausdorff in

general. A natural, geometric construction of a Haar measure on G can be found in [3].

To date, most of the results obtained have concerned the ideal structure of $C^*(G)$ or rather the ideal structure of the reduced C^*-algebra $C^*(G)/k$, where k denotes the kernel of the regular representations of $C^*(G)$. It is important to know whether any/all leaves of $\mathcal{F}$ are dense in M, and we have the following criteria.

Theorem (Fack and Skandalis [3]). $C^*(G)/k$ is simple (i.e. has no non-trivial closed two sided ideals) if and only if every leaf of $\mathcal{F}$ is dense in M.

A C^*-algebra A is called *primitive* if it has a faithful irreducible representation on a C^*-algebra $\beta(H)$ (i.e. a $*$-homomorphism $A \to \beta(H)$ = bounded linear operators on Hilbert space H).

Theorem (Fack and Skandalis [3]). $C^*(G)/k$ is primitive if and only if at least one leaf of $\mathcal{F}$ is dense in M.

I have only touched on one small part here of the circle of ideas involved in this subject, a subject which embraces transverse measures on foliations, Connes' generalisation of the Atiyah-Singer index theorem, non-commutative integration in general, to name only a few topics. There is as yet, as far as I know, no general account of this material, and the interested reader will have to consult [2] and subsequent papers/preprints. There is, however, a detailed account of some of the measure theory of [2] to be found in Daniel Kastler's paper "On A. Connes' Non-Commutative Integration Theory", *Commun. Math. Phys.*, *85* (1982) 99-120.

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NON-LINEAR DIFFERENTIAL EQUATIONS IN BIOLOGY*

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1. Introduction

In recent years there has been considerable growth in the range of mathematical sciences applied to biology and medicine. For many years the statistics of experimental design had been regarded as the main application in the life sciences, but with the advent of mathematical modelling, both deterministic and stochastic models (see Raymond Flood's lecture to the Easter 1983 Symposium [4]) are gaining widespread acceptance. The introduction of biotechnology courses in Ireland has led to interest in the partial differential equations which arise in biological process engineering, such as the reaction-diffusion equation. Workers in fluid dynamics have linked with medical doctors to consider the equations governing the flow of blood through the heart. Stochastic differential equations arise in population dynamics and interesting problems in branching of solutions of non-linear differential equations have come from transmission in nerve axons and from the study of reversible reactions.

The mathematics involved in biological problems can range from the very recent and sophisticated, such as the sledge-hammer of topological degree theory applied to branching problems, to the ingenious application of the most elementary *ad hoc* methods of classical analysis and geometry, as we shall see in Section 2. But whatever mathematics is used, the final results are only as good as the modelling process employed.

A typical modelling scheme is shown in Figure 1 overleaf. It is rare for this process to flow smoothly from one end to the other. Often the mathematical problem cannot be solved in its original form. A solution may be possible by adding

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