THE CONNECTION BETWEEN NETS AND FILTERS

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1. Introduction

The fundamental theorem linking nets and filters can be stated as follows:

Theorem. Let S be a net in a non-void set Ω and $\mathcal{F} = \mathcal{F}(S)$ be its associated filter. If $\mathcal{G}$ is a refinement of $\mathcal{F}$, then there exists a net T in Ω such that:

(i) T is a subnet of S,

(ii) $\mathcal{F}(T) = \mathcal{G}$.

A theorem to this effect was stated by Bartle 1955 [1]. However, the first correct proof was given by M.F. Smiley 1957 [6]. It was again proved by Bartle 1963 [3]. The proofs of both Smiley and Bartle involve the use of the axiom of choice.

The object of this article is to prove this theorem without appeal to the axiom of choice. Moreover, instead of the usual concept of subnet, cf. [5], a simpler concept turns out to be adequate for the purposes of the theorem. It will then follow that this restricted concept of subnet is adequate for topological purposes in a sense that will be made precise later.

2. Recall that a *directed set* [5] is a nonvoid set $D = (D, \leq)$ carrying a reflexive transitive relation $\leq$ for which every two-point subset has an upper bound: we do not assume that $\alpha \leq \alpha' \leq \alpha \Longrightarrow \alpha' = \alpha$. If Ω is a non-void set then a net in is a mapping $S = \{x_\alpha\}_{\alpha \in D}$ from a directed set D into Ω. If $S = \{x_\alpha\}_{\alpha\in D}$ and $T = \{y_\beta\}_{\beta\in E}$ are nets in Ω, then to say that T is a *subnet* of S means [5] that there is $N : E \to D$ for which $y_\beta = x_{N(\beta)}$, such that if $\alpha \in D$ is arbitrary then there is $\beta \in E$ for which $\beta \leq \beta' \Longrightarrow \alpha \leq N(\beta')$. If in particular N is monotonic

in the sense that $\beta \leq \beta' \Longrightarrow N(\beta) \leq N(\beta')$, then T is called a *special subnet* of S.

A filter base $\mathcal{B}$ on a non-void set Ω is a non-void collection of sets in Ω, not containing the void set, and directed by inverse inclusion, i.e. if $B_1, B_2 \in \mathcal{B}$, there exists $B \in \mathcal{B}$ such that $B \subset B_1 \cap B_2$. If $\mathcal{F} = \{F \mid \exists B \in \mathcal{B},\ B \subset F\}$, then $\mathcal{F}$ is the filter generated by $\mathcal{B}$.

Let $\mathcal{B}_1$, $\mathcal{B}_2$ be filter bases for the filters $\mathcal{F}_1$, $\mathcal{F}_2$ respectively. We define $\mathcal{B}_1 \leq \mathcal{B}_2$ to mean that $\mathcal{F}_1 \subset \mathcal{F}_2$. It is easy to check that $\mathcal{B}_1 \leq \mathcal{B}_2$ if and only if $\mathcal{B}_2$ is cofinal in $\mathcal{B}_1$ (with respect to inverse inclusion) i.e. for each $B_1 \subset \mathcal{B}_1$, there exists $B_2 \in \mathcal{B}_2$, $B_2 \subset B_1$. The two filter bases $\mathcal{B}_1, \mathcal{B}_2$ are said to be equivalent if $\mathcal{B}_1 \leq \mathcal{B}_2$ and $\mathcal{B}_2 \leq \mathcal{B}_1$ i.e. if $\mathcal{F}_1 = \mathcal{F}_2$.

If $\mathcal{B}_1, \mathcal{B}_2$ are filter bases for the filters $\mathcal{F}_1, \mathcal{F}_2$, let $\mathcal{B} = \{B_1 \cap B_2 \mid B_1 \in \mathcal{B}_1,\ B_2 \in \mathcal{B}_2\}$. $\mathcal{B}$ is a filter base if and only if it does not contain the void set. If $\mathcal{B}_1$ is a filter base we say that $\mathcal{B}_1$ *is compositive* with $\mathcal{B}_2$, and it is clear that $\mathcal{B}$ is a base for the smallest filter refining both $\mathcal{F}_1$ and $\mathcal{F}_2$.

3. Every net $S = \{x_\alpha\}$ gives rise to a filter base as follows:

Definition: $\mathcal{B}(S) = \{E_\alpha\}$ where $E_\alpha = \{x_{\alpha'} \mid \alpha' \geq \alpha\}$

$\mathcal{B}(S)$ is a filter base and we denote the generated filter by $\mathcal{F}(S)$. $\mathcal{F}(S)(\mathcal{B}(S))$ will be called the filter (filter-base) associated with S. We call the nets $\{E_\alpha\}$ the *residual nets* of S.

Conversely (cf. Bartle [1], Bruns and Schmidt [4]) every filter is associated with a net. We see this as follows:

Let $\mathcal{B}$ be a filter-base. Let $D(\mathcal{B}) = \{\alpha = (x,B) \mid x \in B,\ B \in \mathcal{B}\}$. $D(\mathcal{B})$ is a directed set where $(x,B) \leq (x',B')$ is taken to mean that $B' \subset B$. We now define a net denoted by $S(\mathcal{B})$, viz:

Definition: $S(\mathcal{B}) = \{x_\alpha \mid \alpha \in D(\mathcal{B})\}$

where $x_\alpha = x$ if $\alpha = (x,B)$

It is easy to check

Lemma 3.1. $\mathcal{B}(S(\mathcal{B})) = \mathcal{B}$ i.e. the net $S(\mathcal{B})$ has $\mathcal{B}$ as its associated filter base.

4. The proof of the main theorem depends on the following preliminary lemma concerning nets:

Lemma 4.1. Let $S = \{x_\alpha\}_{\alpha \in D}$, $S' = \{x'_\beta\}_{\beta \in D'}$, be two nets in Ω such that $E_\alpha \cap E'_\beta \neq \emptyset$, $\alpha \in D$, $\beta \in D'$ where E_α, E'_β are the residual sets of S, S' corresponding to α,β respectively. Then there exists a net T which is a special subnet of both S and S'.

Proof. Let $\Lambda = \{(\alpha,\beta) \mid \alpha \in D,\ \beta \in D'$ and $x_\alpha = x'_\beta\}$

It is clear from the hypothesis that Λ is a co-final subset of the directed set $D \times D'$ (with the natural ordering).

Let $T = \{w_\lambda\}_{\lambda \in \Lambda}$ where $w_\lambda = x_\alpha = x'_\beta$ if $\lambda = (\alpha,\beta) \in \Lambda$

Now we show that T is a special subnet of S.

We define $N: \Lambda \longrightarrow D$ by $N(\alpha,\beta) = \alpha$.

Clearly N is monotone. It remains to show $N(\Lambda)$ is co-final in D.

Let $\alpha_0 \subset D$. Let β_0 be arbitrary in D'. By the co-finality of Λ in $D \times D'$, there exists $(\alpha,\beta) \in \Lambda$, $(\alpha,\beta) \geq (\alpha_0,\beta_0)$. Thus $(\alpha,\beta) \in \Lambda$ and $N(\alpha,\beta) = \alpha \geq \alpha_0$. Hence $N(\Lambda)$ is co-final in D. Let $\lambda = (\alpha,\beta) \in \Lambda$. $w_\lambda = w_{(\alpha,\beta)} = x_\alpha = x_{N(\alpha,\beta)} = x_{N(\lambda)}$ and therefore T is a special subnet of S.

Similarly, T is a special subnet of S', and the theorem is proved.

Corollary 4.1. If $\{F_\lambda\}_{\lambda \in \Lambda}$ are the residual sets of the net T constructed in lemma 4.1, then if $\lambda = (\alpha,\beta) \in \Lambda$, $F_\lambda = E_\alpha \cap E'_\beta$. The proof is obvious.

We now prove the main theorem.

Theorem 4.1. Let S be a net in a non-void set Ω and $\mathcal{F} = \mathcal{F}(S)$ be its associated filter. If $\mathcal{G}$ is a refinement of $\mathcal{F}$, i.e. $\mathcal{F} \subset \mathcal{G}$, then there exists a net T in Ω such that:

(i) T is a *special subnet* of S and

(ii) $\mathcal{F}(T) = \mathcal{G}$.

Proof. Let $S = \{x_\alpha\}_{\alpha \in D}$ and $\mathcal{B}(S)$ be its associated filter-base. By lemma 3.1, there exists a net $S' = \{x'_\beta\}_{\beta \in D'}$ such that $\mathcal{B}(S') = \mathcal{G}$. By hypothesis $\mathcal{B}(S) \leq \mathcal{G} = \mathcal{B}(S')$ or $\mathcal{F}(S) \subset \mathcal{G} = \mathcal{B}(S')$. Thus $\mathcal{B}(S)$ and $\mathcal{B}(S')$ are trivially compositive and generate $\mathcal{G}$. A base for $\mathcal{G}$ is $\mathcal{B}' = \{E_\alpha \cap E'_\beta \mid \alpha \in D,\ \beta \in D'\}$. But by lemma 4.1 and corollary 4.1 there exists a net T which is a special subnet of both S and S' and whose associated filter base $\mathcal{B}(T) = \{E_\alpha \cap E'_\beta \mid (\alpha,\beta) \in \Lambda\}$, where Λ is defined as in lemma 4.1. Since Λ is co-final in $D \times D'$, $\mathcal{B}(T) \sim \mathcal{B}'$. Since $\mathcal{B}'$ generates $\mathcal{G}$ so does $\mathcal{B}(T)$. Hence $\mathcal{F}(T) = \mathcal{G}$.

5. Let S be net in Ω. Let T be a subnet in the usual sense (cf. J.L. Kelley [5]). Since $\mathcal{F}(S) \subset \mathcal{F}(T)$ we may use theorem 4.1 to construct a special subnet T' of S such that $\mathcal{F}(T') = \mathcal{F}(T)$. Thus, in any topology on Ω, the cluster points of the special subnet T' coincide with the cluster points of the subnet T.

The author wishes to express his thanks to M.F. Smiley for this observation, which would suggest that in general topology it is more natural and as adequate to confine the notion of subnet to the simpler notion of special subnet.

References

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THE MERKURYEV-SUSLIN THEOREM

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This article reports on one of the most important, and to many people, astonishing results in algebra so far this decade. In 1981, a Russian mathematician Merkuryev, virtually unknown in the west, proved a theorem concerning the algebraic K-theory and the Brauer group of a field. This result is now known as Merkuryev's theorem and not long afterwards Merkuryev, together with Suslin, a famous Russian mathematician, generalized the result to what is commonly called the Merkuryev-Suslin theorem. These theorems at once provide answers to some very hard problems in the theory of simple algebras, in the theory of quadratic forms and in algebraic geometry. Thus it seems worthwhile to try and explain, in as elementary a way as possible, what the Merkuryev-Suslin theorem is all about. A good source of background information for this article is [5].

We start with that well-known Dublin product, the real quaternions, discovered in 1843 by Hamilton and usually denoted $\mathbb{H}$. A quaternion is an expression of the form $a+bi+cj+dij$ where $a,b,c,d \in \mathbb{R}$, the real numbers, and quaternions can be added in the obvious way and multiplied together using the famous equations $i^2=j^2=-1$, $ij=-ji$. Hamilton's construction may be generalized to give quaternion algebras over any field F. We simply choose non-zero elements a,b in F, (a=b is allowed), and do exactly as in $\mathbb{H}$ except that we require $i^2=a$, $j^2=b$. For $F=\mathbb{R}$, $a=b=-1$, we have $\mathbb{H}$ of course. A quaternion algebra defined as above is usually denoted $(\frac{a,b}{F})$ as it depends on the choice of a,b and on the base field F. It is always four-dimensional as an F-vector space and it turns out always to be either a skewfield as $\mathbb{H}$ is (i.e. a field except that multiplication lacks commutativity) or else is isomorphic to the ring of all 2x2 matrices with entries in F. (In fact it fails to be a skewfield precisely when there exist x,y in F such that $ax^2+by^2 = 1$.) For $F=\mathbb{R}$, $\mathbb{H}$ is the only skewfield