SPECTRAL PROJECTIONS

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1. If T is a bounded linear operator on a complex Banach space X, and if $0 \in \mathbb{C}$ is not an accumulation point of the spectrum sp(T), then the formula

$$I - P = \frac{1}{2\pi i}\oint_0 (zI - T)^{-1}dz, \qquad (1.1)$$

in which integration is conducted around a contour which winds once positively around the point 0 and winds zero times around every point of $sp(T) \setminus \{0\}$, defines a projection $P = P^2$ which is bounded and linear on X, and satisfies three conditions:

$$TP = PT \qquad (1.2)$$

there are bounded linear U *and* V *on* X *for which* UT=P=TV; (1.3)

$$||T^n(I - P)||^{1/n} \to 0 \text{ as } N \to \infty. \qquad (1.4)$$

This note is in response to a feeling that while it may be tolerable to use heavy industry like the Cauchy integrals of (1.1) to *construct* a projection like P, it ought to be possible to *define* one in a much more elementary context. We claim in fact that the three conditions (1.2) - (1.4) determine P uniquely, and then force

$$T'T = TT' \Longrightarrow PT' = T'P \qquad (1.5)$$

which we can use to show that operators T for which P exists are stable under certain multiplications and additions.

2. Formally,

DEFINITION 1: *The bounded linear operator* T *on* X *is called quasi-polar if there exists a projection* P *satisfying the conditions* (1.2), (1.3) *and* (1.4).

As a first elementary observation, if U and V satisfy (1.3) then

$$P U P = P V P, \qquad (2.1)$$

and then, with S = P U P,

$$ST = TS = P. \qquad (2.2)$$

and

$$SP = PS = S. \qquad (2.3)$$

Thus if the projection P is given then the conditions (2.2) and (2.3) uniquely determine an operator S; of course U and V need not themselves be unique. From (2.2) and the usual projection property we have

$$P = S^nT^n = T^nS^n \text{ for each } n \in \mathbb{N} \qquad (2.4)$$

We are now ready to prove

THEOREM 1: *If T is quasi-polar then* P *is unique and satisfies* (1.5).

Proof: Suppose P' is another idempotent satisfying conditions (1.2) - (1.4): we demonstrate that

$$P = PP', \qquad (2.5)$$

which gives P = P' by interchanging the roles of P and P'. Indeed

$$P - PP' = P(I-P') = P^n(I-P') = S^nT^n(I-P') \to 0 \text{ as } n \to \infty,$$

using the condition (1.4) for P'. Thus P is unique; to get (1.5) we demonstrate

$$T'T = TT' \Longrightarrow PT' = PT'P, \qquad (2.6)$$

and similarly T'P = PT'P. Indeed if T'T = TT' then

$$PT' - PT'P = P^nT'(I-P) = S^nT^nT'(I-P) = S^nT'T^n(I-P) \to 0 \text{ as } n \to \infty.$$

The same arguments show that the uniquely determined S satisfying (2.2) and (2.3) also commutes with every T' commuting with T. We shall write

$$S = T^{\times}. \qquad (2.7)$$

If in particular (1.4) can be sharpened to

$$T^n(I - P) = 0 \textit{ for some } n \in \mathbb{N} \qquad (2.8)$$

then we shall call the operator T *polar*, and refer to $S = T^{\times}$ as the *Drazin Inverse* of T ([2], 5.1).

3. Without any contour integrals it is clear that *invertibles, quasinilpotents* and *idempotents* all satisfy the conditions of Definition 1: the projection P is either I, 0 or the operator T itself. Another familiar example is an operator T "of finite ascent and descent", in the sense that

$$cl(T^kX) = T^kX = T^{k+1}X \qquad T^{-k}0 = T^{-k-1}0 \textit{ for some } k \in \mathbb{N}: \quad (3.1)$$

here T^kX is the range and $T^{-k}0$ the null space of the projection P. If 0 is not an accumulation point of the spectrum of T then the usual contour integration theory still tells us that T is almost invertible, but we also know something new: the projection P given by the formula (1.1) is the only one around. Conversely, and without contour integration, the condition that 0 is at worst an isolated point of spectrum is necessary.

THEOREM 2: *If T is quasi-polar and if T' commutes with T then:*

T+T' is invertible if T' and $I+T^{\times}T'$ are invertible; (3.2)

T+T' is quasi-polar if T' is quasinilpotent; (3.3)

T'T is quasi-polar if T' is quasi-polar. (3.4)

PROOF: If T' commutes with T then by Theorem 1 it also commutes with P and therefore leaves the range and the null space of P invariant. To derive (3.2) we observe that the restriction of T+T' to P(X) is inverted by $(I+T^{\times}T')^{-1}T^{\times}$, while the

restriction of T+T' to $P^{-1}0$ is the sum of an invertible operator and a quasinilpotent which commute with one another, therefore again invertible. To derive (3.3) we observe that the restriction of T+T' to P(X) is the commuting sum of an invertible and a quasinilpotent, therefore invertible, while the restriction of T+T' to $P^{-1}0$ is the sum of two commuting quasinilpotents, therefore quasinilpotent. To derive (3.4) we consider the product of the projections P and P' associated with T and T', which by Theorem 1 commute with T, T' and one another: the restriction of T'T to the range of PP' is the product of two invertibles and therefore invertible, while the restriction of T'T to the null space of PP' is the sum of three commuting quasinilpotents and therefore quasinilpotent.

4. Sufficient for (3.2) is that T' is invertible with

$$||T^{\times}|| \; ||T|| < 1. \qquad (4.1)$$

Specialising to the case in which

$$T' = \lambda I, \qquad (4.2)$$

for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$, shows that 0 cannot be an accumulation point of the spectrum of a quasi-polar: thus the contour integral (1.1) can always be used to give I-P ([4], Prop. 50.1). The converse of (3.4) is liable to fail: for example

$$T = 0 \Longrightarrow T'T = T'T = TT' \textit{ quasi-polar} \qquad (4.3)$$

without restriction on T'. For Fredholm operators however the converse of (3.4) does hold:

THEOREM 3: *If T and T' are arbitrary then*

$$T \textit{ Browder} \Longrightarrow T \textit{ quasi-polar Fredholm}$$

and

$$T'T = TT' \textit{ quasi-polar Fredholm} \Longrightarrow T, T' \textit{ Browder} \quad (4.5)$$

PROOF: If we write

$$\Phi : A = BL(X,X) \to BL(X,X)/KL(X,X) = B \quad (4.6)$$

for the "Calkin map" which quotients out the ideal $KL(X,X)$ of compact operators then it is Atkinson's theorem ([2], Thm 3.2.8) that

$$T \textit{ Fredholm} \Leftrightarrow \Phi(T) \in B^{-1} \textit{ invertible}. \quad (4.7)$$

If in particular

$$T = S+K \textit{ with } S \in A^{-1},\ \Phi(K) = 0 \textit{ and } SK=KS \quad (4.8)$$

we shall call T a Browder operator. One more preliminary: if $K \in KL(X,Y)$ is compact then I+K has closed range and finite ascent and descent in the sense of (3.1) ([2], Thm 1.4.5; [4], Thm 40.1): thus

$$\Phi(K) = 0 \Rightarrow I+K \textit{ quasi-polar}. \quad (4.9)$$

Now if $T = S+K$ is Browder than $S^{-1}T = I + S^{-1}K$ is quasi-polar, and hence by (3.4) so is $T = S(S^{-1}T)$. Conversely, without using (4.9), suppose $TT'' = T'T = TT'$ is quasi-polar, with $P'' = (P'')^2$ the projection of definition 1. Then also (in an obvious sense) $\Phi(T'') \in B$ is quasi-polar, with projection $\Phi(P'') \in B$. If also T'' is Fredholm, so that $\Phi(T'') \in B^{-1}$ is invertible, then by the uniqueness component (2.5) of Theorem 1 we have

$$\Phi(P'') = \Phi(I) \in B. \quad (4.10)$$

Now

$$S'' = T'TP'' + (I-P''),\ K'' = (T'T-I)(I-P'') \quad (4.11)$$

gives a Browder decomposition for T''. By the doubly commuting component (2.6) of Theorem 1 both T and T' commute with P'': now

$$(TP'' + I-P'')(T'P'' + I-P'') = S'' = (T'P'' + I-P'')(TP'' + I-P''), \quad (4.12)$$

so that $S = TP'' + I-P''$ and $S' = T'P'' + I-P''$ are also invertible.

Also $K = (T-I)(I-P'')$ and $K' = (T'-I)(I-P'')$ are both compact: thus $T = S+K$ and $T' = S'+K'$ are both Browder.

Theorem 3 was very nearly proved in [3] (Theorem 1, Theorem 2), using the contour integral (1.1); (4.5) is however slightly stronger than (2.8) of [3]. As in [3] the whole theory is valid for arbitrary Banach algebras A and B, or indeed general rings, provided we are content with "polar" rather than "quasi-polar" elements. It seems to be quite a delicate problem to decide what the "quasinilpotent" elements of a general ring should be.

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