## THE IRISH MATHEMATICAL SOCIETY



## NEWSLETTER



The aim of the *Newsletter* is to inform Society members about the activities of the Society and also about items of general mathematical interest. It appears three times each year: March, September and December. Deadline for copy is six weeks prior to publication date.

The *Newsletter* also seeks articles of mathematical interest written in an expository manner. All parts of mathematics are welcome, pure and applied, old and new.

Manuscripts should be typewritten and double-spaced on A4 paper. Authors should send two copies and keep one copy as protection against possible loss. Prepare illustrations carefully on separate sheets of paper in black ink, the original without lettering and a copy with lettering added.

Correspondence relating to the *Newsletter* should be sent to:


Irish Mathematical Society Newsletter,
Department of Mathematics,
University College,
Cork.

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## IRISH MATHEMATICAL SOCIETY

Ordinary Meeting, 18th April, 1984, 12.15 pm, DIAS

1. There were 11 members present. The President sent his apologies. The Chair was taken by P. Boland. The minutes of the ordinary meeting of 22nd December, 1984, were signed.

2. On the question of Department of Education postgraduate awards for those with a B.A. degree in Mathematical Sciences, it was reported that the President had written again to the Minister. The reply was again negative. The only course open to the Society was to try and have a Mathematician selected by his College as the College's representative on the advisory committee for these awards.

3. With regard to the Aer Lingus Young Scientists Exhibition, it was reported that various members of the Society were working to encourage Mathematical projects. These included P. Boland, N. Buttimore, R. Timoney and also M. Brennan, although the latter had conveyed his regret that he could only devote a limited amount of time to the matter.

4. The Secretary reported that Professor J.L. Massera of Uruguay has been released from detention. The Society has established a policy of supporting the campaign for Massera's release (and also to support V. Kipnis of the USSR). The news of Massera's release came from the United Nations' Centre for Human Rights, from the Irish Minister for Foreign Affairs, Mr Peter Barry T.D., and from the International Campaign - Massera).

The latter Campaign has asked the Society to support a new campaign to support Yuri Orlov (Physicist) and Anatoly Scharansky (Computer Scientist) both imprisoned in the USSR. The next ordinary meeting will determine the Society's response to this request.

5. A proposal from the Committee to change the Constitution was tabled. The effects are to simplify the procedures for admission to membership and to allow committee members (other than the officers) to serve for three two-year terms consecutively instead of three sessions as at present. This is to be voted on at the next Ordinary Meeting.

6. The Committee also tabled a proposal to amend the rules about when a member is deemed to have resigned, and to delete a redundant sentence in another rule. Details of the changes to the Constitution and rules are attached.

7. M. Clancy reported that negotiations on reciprocity with the IMTA were continuing. Also he had been asked by S. Close of the IMTA to seek the assistance of members of the Society for a Seminar for gifted pupils. The Secretary agreed to circulate details to the local representatives.

8. The proposal to elect N.N. Yanenko to ordinary membership lapsed due to the death of Professor Yanenko.

In response to a proposal from S. Dineen, D. McQuillan and T. Laffey, the Committee nominated Professor M. Kennedy (an ordinary member and formerly of University College Dublin) for honorary membership of the Society.

The following were formally nominated for ordinary membership: N. Shehan, L. Leyden, B. Goldsmith, V. Ryan, F. Harary, R. Geoghegan, R. Critchley, M. Ryan, N. O'hEigeartaigh, A. Brady, T. McGrane, W. Ruckle, A. Raftery, P. Perry.

These nominations will be voted upon at the next ordinary meeting.



9. It was reported that the Committee had agreed to the Society co-sponsoring (in name only) a conference called Protext I, organised by J. Miller, among other conferences. The meeting approved of this decision.

10. The Secretary reported on correspondence he had had with the London Mathematical Society with a view to establishing co-operation between the LMS and IMS. This resulted in a proposal that there be a joint LMS/IMS two-day meeting in Ireland in 1986. The Secretary agreed to solicit suggestions about this meeting from members.

<u>Proposed Changes to the Constitution</u>

<u>Change paragraph 2 to read:</u>

"Any person may apply to the Treasurer for membership by paying one year's membership fee. His admission to membership must then be confirmed by the Committee of the Society. Candidates for honorary membership may be nominated by the Committee only, following a proposal of at least three members of the Society. Nominations for honorary membership must be made at one Ordinary Meeting of the Society and voted upon at the next, a simple majority of the members present being necessary for election."

<u>Change paragraph 5, 2nd sentence to read:</u>

"No person may serve as an additional member for more than three terms consecutively."

<u>Rules</u>

<u>Paragraph 2, change to:</u>

"Ordinary members whose subscriptions are more than eighteen months in arrears shall be deemed to have resigned from the Society."

Paragraph 5, delete first sentence.

## MEMBERSHIP LIST SUPPLEMENT 84-2

12th June 1984

Dr W.G. Tuohey, CAPTEC, Malahide, Co. Dublin

Dr Des Fanning, Maynooth College, Co. Kildare

Dr A.E. Raftery, Trinity College, Dublin

Mr Pat Perry, University College Dublin

Mr J.G. Kelleher, Regional Technical College, Cork

Dr A. Dunne, University College, Dublin

Prof. W. Ruckle, Clemson University, U.S.A.

Dr P. Dolan, Imperial College, London

Dr R. Friel, Trinity College, Dublin

Dr A.R. Pears, Queen Elizabeth College, London

Mr P. O'Murchu, Regional Technical College, Carlow

Mr P.J. O'Kane, Student, Maynooth College

Ms S. MacDonald, Student, Maynooth College

Mr P. Deeney, Student, Maynooth College

Mr M. Prendergast, Student, Maynooth College



# NEWS AND ANNOUNCEMENTS

## SUMMARY OF RESULTS OF IRISH NATIONAL MATHEMATICS CONTEST 1984

The Sixth Irish National Mathematics Contest was held on Tuesday, February 28, 1984, and attracted 1,634 entries from 84 schools as against 1,797 entries from 116 schools last year.

To judge by the results received so far, this year's contest was harder than last year's. Only 21 contestants managed to score 80 or more marks; nobody scored in excess of 99. Two of the 21 are girls.

The winner is:

Ronan Waldron,
Gonzaga College,
Sandford Road,
Ranelagh,
Dublin 6.

Ronan scored 98 marks. To note his achievement, he will be presented with an Award Pin by the Mathematical Association of America.

The highest team score - the sum of the highest three scores by individual contestants from the same school - was returned by

Presentation Brothers College,
Western Road,
Cork.

The winning team, composed of David J. Barry, Gerard Daly and Michael K. Tyrell, scored a total of 263 marks.

The ranking of the top 10 contestants is shown in the Roll of Honour overleaf:

## Roll of Honour

| Candidate | School | Score |
|---|---|---|
| Ronan Waldron | Gonzaga College, Sandford Rd, Ranelagh, Dublin 6. | 98 |
| Mark A. Gibbon | Coleraine Academical Institution, Coleraine, Co. Londonderry. | 94 |
| Conor Kiely* | O'Connell School, Dublin 1. | 90 |
| David J. Barry | Presentation Brothers College, Western Road, Cork. | 88 |
| Stephen Brady* | O'Connell School, Dublin 1. | 88 |
| Gerard Daly | Presentation Brothers College, Western Road, Cork. | 88 |
| Gillian E. Kennedy | Ballymena Academy. | 87 |
| Michael E. Tyrrell | Presentation Brothers College Western Road, Cork. | 87 |
| David J. Ambrose | Presentation Brothers College, Western Road, Cork. | 84 |
| Moira E. Hoban | Loreto College, St Stephen's Green, Dublin 2. | 83 |

* Did not participate in IIMC 1984.

## SUMMARY OF RESULTS OF IIMC 1984

The Second Irish Invitational Mathematics Contest was held on Tuesday, March 20, 1984. Twenty of the 21 top scorers in the INMC 1984 were invited to take the IIMC; in the event, only 16 sat the examination. The material for this was also supplied by the Mathematical Association of America Committee on High School Contests. Contestants had $2\frac{1}{2}$ hours in which to answer 15 questions which had integer solutions. The top two contestants were Frank Roden and Ronan Waldron who both got eight correct answers.

Some of the questions were the following.



## SAMPLE QUESTIONS

(5) Determine the value of ab if $\log_8 a + \log_4 b^2 = 5$ and $\log_8 b + \log_4 a^2 = 7$.

(7) The function f is defined on the set of integers and satisfies

$$f(n) = \begin{cases} n-3 & \text{if } n \geq 1000, \\ f(f(n+5)) & \text{if } n < 1000 \end{cases}$$

Find f(84).

(11) A gardener plants three maple trees, four oak trees and five birch trees in a row. He plants them in random order, each arrangement being equally likely. Let m/n in lowest terms be the probability that no two birch trees are next to each other. Find m+n.

(13) Find the value of

$$10\cot(\cot^{-1}3 + \cot^{-1}7 + \cot^{-1}13 + \cot^{-1}21)$$

(14) What is the largest even integer which cannot be written as the sum of two odd composite numbers? (Recall that a positive integer is said to be composite if it is divisible by at least one positive integer other than 1 and itself.)

*F. Holland*

# SPECTRAL PROJECTIONS

*Robin Harte*

1. If $T$ is a bounded linear operator on a complex Banach space $X$, and if $0 \in \mathbb{C}$ is not an accumulation point of the spectrum $sp(T)$, then the formula

$$I - P = \frac{1}{2\pi i}\oint_0 (zI - T)^{-1} dz, \qquad (1.1)$$

in which integration is conducted around a contour which winds once positively around the point 0 and winds zero times around every point of $sp(T) \setminus \{0\}$, defines a projection $P = P^2$ which is bounded and linear on $X$, and satisfies three conditions:

$$TP = PT \qquad (1.2)$$

*there are bounded linear* $U$ *and* $V$ *on* $X$ *for which* $UT=P=TV$; (1.3)

$$||T^n(I - P)||^{1/n} \to 0 \text{ as } N \to \infty. \qquad (1.4)$$

This note is in response to a feeling that while it may be tolerable to use heavy industry like the Cauchy integrals of (1.1) to *construct* a projection like $P$, it ought to be possible to *define* one in a much more elementary context. We claim in fact that the three conditions (1.2) - (1.4) determine $P$ uniquely, and then force

$$T'T = TT' \Longrightarrow PT' = T'P \qquad (1.5)$$

which we can use to show that operators $T$ for which $P$ exists are stable under certain multiplications and additions.

2. Formally,

DEFINITION 1: *The bounded linear operator* $T$ *on* $X$ *is called quasi-polar if there exists a projection* $P$ *satisfying the conditions* (1.2), (1.3) *and* (1.4).



As a first elementary observation, if $U$ and $V$ satisfy (1.3) then

$$P U P = P V P, \qquad (2.1)$$

and then, with $S = P U P$,

$$ST = TS = P. \qquad (2.2)$$

and

$$SP = PS = S. \qquad (2.3)$$

Thus if the projection $P$ is given then the conditions (2.2) and (2.3) uniquely determine an operator $S$; of course $U$ and $V$ need not themselves be unique. From (2.2) and the usual projection property we have

$$P = S^nT^n = T^nS^n \text{ for each } n \in \mathbb{N} \qquad (2.4)$$

We are now ready to prove

THEOREM 1: *If* $T$ *is quasi-polar then* $P$ *is unique and satisfies* (1.5).

Proof: Suppose $P'$ is another idempotent satisfying conditions (1.2) - (1.4): we demonstrate that

$$P = PP', \qquad (2.5)$$

which gives $P = P'$ by interchanging the roles of $P$ and $P'$. Indeed

$$P - PP' = P(I-P') = P^n(I-P') = S^nT^n(I-P') \to 0 \text{ as } n \to \infty,$$

using the condition (1.4) for $P'$. Thus $P$ is unique; to get (1.5) we demonstrate

$$T'T = TT' \Longrightarrow PT' = PT'P, \qquad (2.6)$$

and similarly $T'P = PT'P$. Indeed if $T'T = TT'$ then

$$PT' - PT'P = P^nT'(I-P) = S^nT^nT'(I-P) = S^nT'T^n(I-P) \to 0 \text{ as } n \to \infty.$$

The same arguments show that the uniquely determined $S$

satisfying (2.2) and (2.3) also commutes with every T' commuting with T. We shall write

$$S = T^{\times}. \quad (2.7)$$

If in particular (1.4) can be sharpened to

$$T^{n}(I - P) = 0 \text{ for some } n \in \mathbb{N} \quad (2.8)$$

then we shall call the operator T *polar*, and refer to $S = T^{\times}$ as the *Drazin Inverse* of T ([2], 5.1).

3. Without any contour integrals it is clear that *invertibles, quasinilpotents* and *idempotents* all satisfy the conditions of Definition 1: the projection P is either I, 0 or the operator T itself. Another familiar example is an operator T "of finite ascent and descent", in the sense that

$$cl(T^{k}X) = T^{k}X = T^{k+1}X \qquad T^{-k}0 = T^{-k-1}0 \text{ for some } k \in \mathbb{N}: \quad (3.1)$$

here $T^{k}X$ is the range and $T^{-k}0$ the null space of the projection P. If 0 is not an accumulation point of the spectrum of T then the usual contour integration theory still tells us that T is almost invertible, but we also know something new: the projection P given by the formula (1.1) is the only one around. Conversely, and without contour integration, the condition that 0 is at worst an isolated point of spectrum is necessary.

THEOREM 2: *If T is quasi-polar and if T' commutes with T then:*

*T+T' is invertible if T' and $I+T^{\times}T'$ are invertible;* (3.2)

*T+T' is quasi-polar if T' is quasinilpotent;* (3.3)

*T'T is quasi-polar if T' is quasi-polar.* (3.4)

PROOF: If T' commutes with T then by Theorem 1 it also commutes with P and therefore leaves the range and the null space of P invariant. To derive (3.2) we observe that the restriction of T+T' to P(X) is inverted by $(I+T^{\times}T')^{-1}T^{\times}$, while the



restriction of T+T' to $P^{-1}0$ is the sum of an invertible operator and a quasinilpotent which commute with one another, therefore again invertible. To derive (3.3) we observe that the restriction of T+T' to P(X) is the commuting sum of an invertible and a quasinilpotent, therefore invertible, while the restriction of T+T' to $P^{-1}0$ is the sum of two commuting quasinilpotents, therefore quasinilpotent. To derive (3.4) we consider the product of the projections P and P' associated with T and T', which by Theorem 1 commute with T, T' and one another: the restriction of T'T to the range of PP' is the product of two invertibles and therefore invertible, while the restriction of T'T to the null space of PP' is the sum of three commuting quasinilpotents and therefore quasinilpotent.

4. Sufficient for (3.2) is that T' is invertible with

$$||T^{\times}|| \; ||T|| < 1. \quad (4.1)$$

Specialising to the case in which

$$T' = \lambda I, \quad (4.2)$$

for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$, shows that 0 cannot be an accumulation point of the spectrum of a quasi-polar: thus the contour integral (1.1) can always be used to give I-P ([4], Prop. 50.1). The converse of (3.4) is liable to fail: for example

$$T = 0 \Longrightarrow T'T = T'T = TT' \text{ quasi-polar} \quad (4.3)$$

without restriction on T'. For Fredholm operators however the converse of (3.4) does hold:

THEOREM 3: *If T and T' are arbitrary then*

$$T \text{ Browder} \Longrightarrow T \text{ quasi-polar Fredholm}$$

*and*

$$T'T = TT' \text{ quasi-polar Fredholm} \Longrightarrow T,\ T' \text{ Browder} \quad (4.5)$$

PROOF: If we write

$$\Phi : A = BL(X,X) \to BL(X,X)/KL(X,X) = B \qquad (4.6)$$

for the "Calkin map" which quotients out the ideal $KL(X,X)$ of compact operators then it is Atkinson's theorem ([2], Thm 3.2.8) that

$$T \textit{ Fredholm} \Leftrightarrow \Phi(T) \in B^{-1} \textit{ invertible.} \qquad (4.7)$$

If in particular

$$T = S+K \textit{ with } S \in A^{-1},\ \Phi(K) = 0 \textit{ and } SK=KS \qquad (4.8)$$

we shall call T a Browder operator. One more preliminary: if $K \in KL(X,Y)$ is compact then I+K has closed range and finite ascent and descent in the sense of (3.1) ([2], Thm 1.4.5; [4], Thm 40.1): thus

$$\Phi(K) = 0 \Rightarrow I+K \textit{ quasi-polar.} \qquad (4.9)$$

Now if $T = S+K$ is Browder than $S^{-1}T = I + S^{-1}K$ is quasi-polar, and hence by (3.4) so is $T = S(S^{-1}T)$. Conversely, without using (4.9), suppose $TT'' = T'T = TT'$ is quasi-polar, with $P'' = (P'')^2$ the projection of definition 1. Then also (in an obvious sense) $\Phi(T'') \in B$ is quasi-polar, with projection $\Phi(P'') \in B$. If also $T''$ is Fredholm, so that $\Phi(T'') \in B^{-1}$ is invertible, then by the uniqueness component (2.5) of Theorem 1 we have

$$\Phi(P'') = \Phi(I) \in B. \qquad (4.10)$$

Now

$$S'' = T'TP'' + (I-P''),\ K'' = (T'T-I)(I-P'') \qquad (4.11)$$

gives a Browder decomposition for $T''$. By the doubly commuting component (2.6) of Theorem 1 both T and T' commute with $P''$: now

$$(TP'' + I-P'')(T'P'' + I-P'') = S'' = (T'P'' + I-P'')(TP'' + I-P''), \qquad (4.12)$$

so that $S = TP'' + I-P''$ and $S' = T'P'' + I-P''$ are also invertible.



Also $K = (T-I)(I-P'')$ and $K' = (T'-I)(I-P'')$ are both compact: thus $T = S+K$ and $T' = S'+K'$ are both Browder.

Theorem 3 was very nearly proved in [3] (Theorem 1, Theorem 2), using the contour integral (1.1); (4.5) is however slightly stronger than (2.8) of [3]. As in [3] the whole theory is valid for arbitrary Banach algebras A and B, or indeed general rings, provided we are content with "polar" rather than "quasi-polar" elements. It seems to be quite a delicate problem to decide what the "quasinilpotent" elements of a general ring should be.

## References


1. S.R. Caradus, Operator Theory of the Pseudo-Inverse, Queen's University Papers No. 38, Kingston-Ontario 1974.
2. S.R. Caradus, W.E. Pfaffenberger and B. Yood, Calkin Algebras and Algebras of Operators on Banach Spaces, Debhar 1974.
3. R.E. Harte, Fredholm Theory Relative to a Banach Algebra Homomorphism, *Mathematische Zeitschrift*, *179* (1982) 431-436.
4. H.G. Heuser, Functional Analysis, Wiley 1982.



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# SHIPLEY'S ALGORITHM FOR INVERTING MATRICES

*Finbarr Holland*

## 1. Introduction

This is a brief outline of a method for inverting matrices that was developed in the late fifties at the Tennessee Valley Authority. It was found to be particularly suitable for inverting matrices that describe power system impedances or admittances. The method was first reported on by Shipley and Coleman in [4]. The essentials of the method can also be found in the texts [1] and [3]; but I could find no mention of it in the mathematical literature.

I have been teaching it to a class of Third Year Electrical Engineering students at Cork for the last two years, and some readers may recall a talk I gave at the 1982 Limerick Algebra Conference in which I presented the key ideas behind the method and attempted - unsuccessfully, as it turned out - to demonstrate it on a personal computer.

The method in question is simple to apply, direct rather than iterative, easily programmable and takes full advantage of any symmetry present in the matrix under examination. Since otherwise it closely resembles the well-known Gauss Elimination method, the latter is a significant feature of the Shipley method, which is based on modifying successively the elements of the matrix according to a simple rule.

## 2. The Shipley Modification of a Square Matrix

Given an nxn matrix $A = [a_{ij}]$, with $a_{kk} \neq 0$, we define the kth Shipley modification $\sigma_k(A) = B$ by the rules:

$$b_{kk} = -1/a_{kk},$$



$$b_{ik} = -a_{ik}/a_{kk} = a_{ik}b_{kk}, \quad i \neq k$$

$$b_{kj} = -a_{kj}/a_{kk} = a_{kj}b_{kk}, \quad j \neq k$$

$$b_{ij} = a_{ij} - a_{ik}a_{kj}/a_{kk} = a_{ij} + a_{ik}b_{kj}, \quad i,j \neq k$$

Example For instance, if

$$M = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix},$$

then

$$\sigma_1(M) = \begin{bmatrix} -1 & -1 & -3 \\ -1 & -1 & -1 \\ -3 & -1 & -5 \end{bmatrix} = N, \quad \sigma_2(N) = \begin{bmatrix} 0 & -1 & -2 \\ -1 & 1 & -1 \\ -2 & -1 & -4 \end{bmatrix} = L,$$

and

$$\sigma_3(L) = \begin{bmatrix} 1 & -.5 & -.5 \\ -.5 & 1.25 & -.25 \\ -.5 & -.25 & .25 \end{bmatrix}.$$

## 3. Shipley's Algorithm

It is a simple matter to check that the last displayed matrix is the negative of the inverse of M, i.e.

$$\sigma_3(\sigma_2(\sigma_1(M))) = -M^{-1}$$

This example illustrates the essence of Shipley's algorithm.

THEOREM 1. Let A be an nxn matrix. If for some permutation $\pi$ of the integers 1,2,........,n the n-fold composition

$$\sigma_{\pi(n)} \circ \sigma_{\pi(n-1)} \circ \cdots\cdots \sigma_{\pi(2)} \cdot\cdot \circ \sigma_{\pi(1)}(A)$$

is defined, then it is equal to the negative of the inverse of A.

Perhaps the easiest way to be convinced of this is to consider the correspondence $Ax = y$, between the $n \times 1$ vectors $x$ and $y$, as a system of $n$ equations:

$$\sum_{j=1}^{n} a_{ij}x_j = y_i, \quad i = 1,2,\ldots,n$$

On the assumption that $a_{kk} \neq 0$ we can use the kth equation to express $x_k$ in terms of $y_k$ and $x_j$, $j = 1,2,\ldots,n$, $j \neq k$, and then substitute this value of $x_k$ into the other equations. After some rearrangement of terms, this leads to an equivalent system, viz.,

$$\sum_{j<k} (a_{ij}-a_{ik}a_{kj}/a_{kk})x_j + (a_{ik}/a_{kk})y_k + \sum_{j>k} (a_{ij}-a_{ik}a_{kj}/a_{kk})x_j = y_i, \quad i \neq k$$

$$\sum_{j<k} (-a_{kj}/a_{kk})x_j + (1/a_{kk})y_k + \sum_{j>k} (-a_{kj}/a_{kk})x_j = x_k.$$

Allowing for an adjustment of sign, this can be formulated as the matrix equation

$$\sigma_k(A)x' = y',$$

where $x'$ is obtained from $x$ by replacing $x_k$ by $-y_k$ and $y'$ is obtained from $y$ by replacing $y_k$ by $x_k$.

Under the conditions of the theorem we can carry out the manoeuvre just described $n$ times, in the order determined by the permutation $\pi$, and in this way replace successively each component of the vector $x$ by the negative of the corresponding component of the vector $y$; we end up with the equation

$$\sigma_{\pi(n)}(\sigma_{\pi(n-1)}(\ldots.\ (\sigma_{\pi(2)}(\sigma_{\pi(1)}(A)))\ldots.))(-y) = x.$$

This is clearly sufficient to demonstrate the truth of the theorem.



## 4. Criterion for the Algorithm to Work

Since the method is not completely general - for example it will not invert the simple 2x2 matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- it is important to know when it works. To find out, we analyse the formation of the diagonal elements under different Shipley modifications.

Given $A = [a_{ij}]$, we can form $B = \sigma_1(A)$ if $a_{11} \neq 0$. We can then form $C = \sigma_2(B) = \sigma_2(\sigma_1(A))$ if $b_{22} \neq 0$, i.e. if

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0.$$

If $c_{33} \neq 0$, i.e. if

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0,$$

we can compute $\sigma_3(C) = \sigma_3(\sigma_2(\sigma_1(A)))$, etc. This leads to the following theorem, whose proof we can omit.

THEOREM 2. If for some permutation $\pi$ of the integers $1,2,\ldots,n$, the principal minors of the matrix $[a_{\pi(i)\pi(j)}]$ are all non-zero, then

$$-A^{-1} = \sigma_{\pi(n)}(\sigma_{\pi(n-1)}(\ldots\ldots(\sigma_{\pi(1)}(A))\ldots\ldots)).$$

Thus, for example, if $A$ is strictly positive-definite or strictly dominant diagonal, that is to say if

$$x^tAx > 0 \text{ for all } x \neq 0,$$

or

$$\sum_{j=1}^{n} |a_{ij}| < 2|a_{ii}|, \quad i = 1,2,\ldots,n,$$

then we can apply the Shipley algorithm to evaluate the inverse of A, it being clear in both cases that the property of positive-definiteness or diagonal dominance is inherited by the principal minors of $[a_{\pi(i)\pi(j)}]$ for every permutation $\pi$.

## 5. A Noteworthy Feature of the Shipley Algorithm

An important property of the Shipley algorithm is that it preserves symmetry.

THEOREM 3. If $A = [a_{ij}]$ is symmetric and $a_{kk} \neq 0$, then $\sigma_k(A)$ is symmetric.

PROOF. Obvious.

The fact that symmetry is retained during the implementation of the algorithm reduces the arithmetic of computing the inverse of a symmetric matrix by approximately one half and the memory requirements of a computer by about the same amount.

## 6. The Complexity of the Shipley Algorithm

To perform one modification of a given matrix we must carry out one division, $(2(n-1) + (n-1)^2)$ multiplications and $(n-1)^2$ additions. Therefore to invert an nxn matrix we must perform about $n^3$ multiplications and the same number of additions. The order of complexity of the method is therefore $O(n^3)$. In this respect, then, there is no difference between it and the standard elimination procedure.

## 7. Computer Implementation of the Method

The method is a little unpleasant to operate by hand, but it can be easily programmed for a computer and can therefore be readily demonstrated in a classroom. The program below was designed by my son Ian for the Newbrain personal computer (on which a draft of this article was prepared). It is structured to take advantage of the savings involved in applying the algorithm to symmetric matrices. Also, for a given matrix $A = [a_{ij}]$, the program selects the sequence of operations $\sigma_{\pi(1)}, \sigma_{\pi(2)}, \ldots, \sigma_{\pi(n)}$, according to the following rule: $\pi(1)$ is chosen to be the index k corresponding to the largest non-vanishing $|a_{kk}|$; if $B = \sigma_{\pi(1)}(A)$, $\pi(2)$ is chosen to be the index $k \neq \pi(1)$ corresponding to the largest non-vanishing $|b_{kk}|$; and so on. By this means it is hoped to keep computational errors to a minimum.



## 8. Acknowledgements

It is a pleasure to record my thanks to my son Ian for the assistance he has given me with this project, to Tom Laffey who drew my attention to Schwein's theorem and its relatives [2] (which one needs to prove Theorem 2) and to Michael O'Callaghan, with whom I first discussed the Shipley Algorithm, for many useful observations.

## References

1. Brown, Homer E., Solution of Large Networks by Matrix Methods, Wiley, 1975.
2. Muir, Thomas, A Treatise on the Theory of Determinants, Dover, 1933.
3. Shipley, R.B., Matrices and Power Systems, Wiley, 1976.
4. Shipley, R.B., and Coleman, D.W., A New Direct Matrix Inversion Method, Trans. AIEE C&E, Nov. 1959, 568-572.

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   1  REM ** Program to invert a nxn matrix using the Shipley Algorithm **
   9  CLOSE£1:OPEN£1,5:CLEAR
  10  REM Open relevant screens
  20  OPEN£0,4 ,"125"
  21  d$="  "
  30  PUT 31: REM Clear screen
  40  REM Read in dimension of matrix
  50  PUT 22,10,10:?"WHAT IS THE DIMENSION OF THE MATRIX  ";
  60  INPUT n
  62  PUT 22,10,10:?"IS THE MATRIX SYMMETRIC?  Y/N                  ";:GET£1,A$
  64  IF INSTR("YyNn",a$)=0 THEN GOTO 62
  70  DIM b(n,n),a(n,n)
  80  REM Read in the elements of the matrix A
  85  PUT 31
  90  PUT 22,20,1:?"Enter the elements of the matrix"
  91  IF INSTR("nN",a$)>0 THEN GOTO 125
  95  FOR x=1 TO n
 100     FOR y=x TO n
 105        PUT 22,y*9,x*2+2:INPUT("")a(x,y):a(y,x)=a(x,y)
 110     NEXT y
 115  NEXT x
 120  GOTO 150
 125  FOR x=1 TO n
 130     FOR y=1 TO n
 135        PUT 22,y*9,x*2+2:INPUT("") a(x,y)
 140     NEXT y
 145  NEXT x
 150  REM *** THE SHIPLEY ALGORITHM ***
 155  FOR t=1 TO n:REM t counts no of modifications
 160  REM Find the largest unused diagonal element
 170  k1=0
 180  FOR x=1 TO n
 190     IF INSTR(d$,STR$(x))>0 THEN GOTO 210
 200     IF ABS(a(x,x)) > k1 THEN k=x:k1=ABS(a(x,x))
 210  NEXT x
 220  IF k1=0 THEN END:REM Singular MATRIX if k1=0
 250  D$=D$+STR$(K)
 300  REM *** THE ALGORITHM PROPER ***
 309  REM kth column
 310  FOR j=1 TO n
 320     IF j=k THEN GOTO 340
 330     b(k,j) = -a(k,j) / a(k,k)
 340  NEXT J
 399  REM kth row
 400  FOR i=1 TO n
 410     IF i=k THEN GOTO 430
 420     b(i,k) = -a(i,k) / a(k,k)
 430  NEXT i
 500  b(k,k) = -1/a(k,k)
 550  REM The modification of the remainder of the matrix
 560  FOR i=1 TO n
 570     IF i=k GOTO 620
 580     FOR j=1 TO n
 590        IF j=k GOTO 610
 600        b(i,j) = a(i,j) - a(i,k)*a(k,j) / a(k,k)
 610     NEXT j
```



```
 620  NEXT i
 650  REM COPY B INTO A
 660  FOR i=1 TO n
 670     FOR j=1 TO n
 680        a(i,j) = b(i,j)
 690     NEXT j
 700  NEXT i
 750  REM END OF ONE COMPLETE MODIFICATION
 760  GOSUB 1910:REM Call routine to print out modified matrix
 770  PUT 22,20,23:?"THE MATRIX AFTER ";T;" MODIFICATIONS"
 780  PUT 22,20,24:?"press any key to continue";
 790  GET£1,ky$
 800  NEXT t
 900  REM PRINT OUT THE INVERSE
 910  PUT 31:PUT 22,20,1:?"The INVERSE is"
 920  GOSUB 1910
1000  END
1001  REM THE END OF THE PROGRAM
1900  REM A routine to print out the contents matrix A
1910  REM
1910  FOR x=1 TO n
1920     FOR y=1 TO n
1930        a$+STR$(a(x,y))
1940        IF LEN(a$)=15 THEN a$=LEFT$(a$,4)+RIGHT$(a$,5)
1950        a$=LEFT$(a$+"        ",9)
1960        PUT 22,y*9,x*2+2:?a$
1970     NEXT y
1980  NEXT x
1990  RETURN
```

*This program is in Newbrain BASIC.*

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# THE CONNECTION BETWEEN NETS AND FILTERS

*Maurice Kennedy*

## 1. Introduction

The fundamental theorem linking nets and filters can be stated as follows:

Theorem. Let $S$ be a net in a non-void set $\Omega$ and $\mathcal{F} = \mathcal{F}(S)$ be its associated filter. If $\mathcal{G}$ is a refinement of $\mathcal{F}$, then there exists a net $T$ in $\Omega$ such that:

(i) $T$ is a subnet of $S$,

(ii) $\mathcal{F}(T) = \mathcal{G}$.

A theorem to this effect was stated by Bartle 1955 [1]. However, the first correct proof was given by M.F. Smiley 1957 [6]. It was again proved by Bartle 1963 [3]. The proofs of both Smiley and Bartle involve the use of the axiom of choice.

The object of this article is to prove this theorem without appeal to the axiom of choice. Moreover, instead of the usual concept of subnet, cf. [5], a simpler concept turns out to be adequate for the purposes of the theorem. It will then follow that this restricted concept of subnet is adequate for topological purposes in a sense that will be made precise later.

2. Recall that a *directed set* [5] is a nonvoid set $D = (D, \leq)$ carrying a reflexive transitive relation $\leq$ for which every two-point subset has an upper bound: we do not assume that $\alpha \leq \alpha' \leq \alpha \Longrightarrow \alpha' = \alpha$. If $\Omega$ is a non-void set then a net in is a mapping $S = \{x_\alpha\}_{\alpha \in D}$ from a directed set $D$ into $\Omega$. If $S = \{x_\alpha\}_{\alpha \in D}$ and $T = \{y_\beta\}_{\beta \in E}$ are nets in $\Omega$, then to say that $T$ is a *subnet* of $S$ means [5] that there is $N : E \to D$ for which $y_\beta = x_{N(\beta)}$, such that if $\alpha \in D$ is arbitrary then there is $\beta \in E$ for which $\beta \leq \beta' \Longrightarrow \alpha \leq N(\beta')$. If in particular $N$ is monotonic



in the sense that $\beta \leq \beta' \Longrightarrow N(\beta) \leq N(\beta')$, then $T$ is called a *special subnet* of $S$.

A filter base $\mathcal{B}$ on a non-void set $\Omega$ is a non-void collection of sets in $\Omega$, not containing the void set, and directed by inverse inclusion, i.e. if $B_1, B_2 \in \mathcal{B}$, there exists $B \in \mathcal{B}$ such that $B \subset B_1 \cap B_2$. If $\mathcal{F} = \{F | \exists B \in \mathcal{B},\ B \subset F\}$, then $\mathcal{F}$ is the filter generated by $\mathcal{B}$.

Let $\mathcal{B}_1$, $\mathcal{B}_2$ be filter bases for the filters $\mathcal{F}_1$, $\mathcal{F}_2$ respectively. We define $\mathcal{B}_1 \leq \mathcal{B}_2$ to mean that $\mathcal{F}_1 \subset \mathcal{F}_2$. It is easy to check that $\mathcal{B}_1 \leq \mathcal{B}_2$ if and only if $\mathcal{B}_2$ is cofinal in $\mathcal{B}_1$ (with respect to inverse inclusion) i.e. for each $B_1 \subset \mathcal{B}_1$, there exists $B_2 \in \mathcal{B}_2$, $B_2 \subset B_1$. The two filter bases $\mathcal{B}_1, \mathcal{B}_2$ are said to be equivalent if $\mathcal{B}_1 \leq \mathcal{B}_2$ and $\mathcal{B}_2 \leq \mathcal{B}_1$ i.e. if $\mathcal{F}_1 = \mathcal{F}_2$.

If $\mathcal{B}_1, \mathcal{B}_2$ are filter bases for the filters $\mathcal{F}_1, \mathcal{F}_2$, let $\mathcal{B} = \{B_1 \cap B_2 | B_1 \in \mathcal{B}_1,\ B_2 \in \mathcal{B}_2\}$. $\mathcal{B}$ is a filter base if and only if it does not contain the void set. If $\mathcal{B}_1$ is a filter base we say that $\mathcal{B}_1$ *is compositive* with $\mathcal{B}_2$, and it is clear that $\mathcal{B}$ is a base for the smallest filter refining both $\mathcal{F}_1$ and $\mathcal{F}_2$.

3. Every net $S = \{x_\alpha\}$ gives rise to a filter base as follows:

Definition: $\mathcal{B}(S) = \{E_\alpha\}$ where $E_\alpha = \{x_{\alpha'} | \alpha' \geq \alpha\}$

$\mathcal{B}(S)$ is a filter base and we denote the generated filter by $\mathcal{F}(S)$. $\mathcal{F}(S)(\mathcal{B}(S))$ will be called the filter (filter-base) associated with $S$. We call the nets $\{E_\alpha\}$ the *residual nets* of $S$.

Conversely (cf. Bartle [1], Bruns and Schmidt [4]) every filter is associated with a net. We see this as follows:

Let $\mathcal{B}$ be a filter-base. Let $D(\mathcal{B}) = \{\alpha = (x,B) | x \in B,\ B \in \mathcal{B}\}$. $D(\mathcal{B})$ is a directed set where $(x,B) \leq (x',B')$ is taken to mean that $B' \subset B$. We now define a net denoted by $S(\mathcal{B})$, viz:

Definition: $S(\mathcal{B}) = \{x_\alpha | \alpha \in D(\mathcal{B})\}$

where $x_\alpha = x$ if $\alpha = (x,B)$

It is easy to check

Lemma 3.1. $\mathcal{B}(S(\mathcal{B})) = \mathcal{B}$ i.e. the net $S(\mathcal{B})$ has $\mathcal{B}$ as its associated filter base.

4. The proof of the main theorem depends on the following preliminary lemma concerning nets:

Lemma 4.1. Let $S = \{x_\alpha\}_{\alpha \in D}$, $S' = \{x'_\beta\}_{\beta \in D'}$, be two nets in $\Omega$ such that $E_\alpha \cap E'_\beta \neq \emptyset$, $\alpha \in D$, $\beta \in D'$ where $E_\alpha$, $E'_\beta$ are the residual sets of S, S' corresponding to $\alpha,\beta$ respectively. Then there exists a net T which is a special subnet of both S and S'.

Proof. Let $\Lambda = \{(\alpha,\beta) | \alpha \in D, \beta \in D' \text{ and } x_\alpha = x'_\beta\}$

It is clear from the hypothesis that $\Lambda$ is a co-final subset of the directed set DxD' (with the natural ordering).

Let $T = \{w_\lambda\}_{\lambda \in \Lambda}$ where $w_\lambda = x_\alpha = x'_\beta$ if $\lambda = (\alpha,\beta) \in \Lambda$

Now we show that T is a special subnet of S.

We define $N: \Lambda \longrightarrow D$ by $N(\alpha,\beta) = \alpha$.

Clearly N is monotone. It remains to show $N(\Lambda)$ is co-final in D.

Let $\alpha_0 \subset D$. Let $\beta_0$ be arbitrary in D'. By the co-finality of $\Lambda$ in DxD', there exists $(\alpha,\beta) \in \Lambda$, $(\alpha,\beta) \geq (\alpha_0,\beta_0)$. Thus $(\alpha,\beta) \in \Lambda$ and $N(\alpha,\beta) = \alpha \geq \alpha_0$. Hence $N(\Lambda)$ is co-final in D. Let $\lambda = (\alpha,\beta) \in \Lambda$. $w_\lambda = w_{(\alpha,\beta)} = x_\alpha = x_{N(\alpha,\beta)} = x_{N(\lambda)}$ and therefore T is a special subnet of S.

Similarly, T is a special subnet of S', and the theorem is proved.



Corollary 4.1. If $\{F_\lambda\}_{\lambda \in \Lambda}$ are the residual sets of the net T constructed in lemma 4.1, then if $\lambda = (\alpha,\beta) \in \Lambda$, $F_\lambda = E_\alpha \cap E'_\beta$. The proof is obvious.

We now prove the main theorem.

Theorem 4.1. Let S be a net in a non-void set $\Omega$ and $\mathcal{F} = \mathcal{F}(S)$ be its associated filter. If $\mathcal{G}$ is a refinement of $\mathcal{F}$, i.e. $\mathcal{F} \subset \mathcal{G}$, then there exists a net T in $\Omega$ such that:

(i) T is a *special subnet* of S and

(ii) $\mathcal{F}(T) = \mathcal{G}$.

Proof. Let $S = \{x_\alpha\}_{\alpha \in D}$ and $\mathcal{B}(S)$ be its associated filter-base. By lemma 3.1, there exists a net $S' = \{x'_\beta\}_{\beta \in D'}$ such that $\mathcal{B}(S') = \mathcal{G}$. By hypothesis $\mathcal{B}(S) \leq \mathcal{G} = \mathcal{B}(S')$ or $\mathcal{F}(S) \subset \mathcal{G} = \mathcal{B}(S')$. Thus $\mathcal{B}(S)$ and $\mathcal{B}(S')$ are trivially compositive and generate $\mathcal{G}$. A base for $\mathcal{G}$ is $\mathcal{B}' = \{E_\alpha \cap E'_\beta | \alpha \in D, \beta \in D'\}$. But by lemma 4.1 and corollary 4.1 there exists a net T which is a special subnet of both S and S' and whose associated filter base $\mathcal{B}(T) = \{E_\alpha \cap E'_\beta | (\alpha,\beta) \in \Lambda\}$, where $\Lambda$ is defined as in lemma 4.1. Since $\Lambda$ is co-final in DxD', $\mathcal{B}(T) \sim \mathcal{B}'$. Since $\mathcal{B}'$ generates $\mathcal{G}$ so does $\mathcal{B}(T)$. Hence $\mathcal{F}(T) = \mathcal{G}$.

5. Let S be net in $\Omega$. Let T be a subnet in the usual sense (cf. J.L. Kelley [5]). Since $\mathcal{F}(S) \subset \mathcal{F}(T)$ we may use theorem 4.1 to construct a special subnet T' of S such that $\mathcal{F}(T') = \mathcal{F}(T)$. Thus, in any topology on $\Omega$, the cluster points of the special subnet T' coincide with the cluster points of the subnet T.

The author wishes to express his thanks to M.F. Smiley for this observation, which would suggest that in general topology it is more natural and as adequate to confine the notion of subnet to the simpler notion of special subnet.

## References


1. R.G. Bartle, "Nets and Filters in Topology", *American Math. Monthly*, *62* (1955), 551-557.
2. R.G. Bartle, "A Correction for 'Nets and Filters in Topology'", *American Math. Monthly*, *70* (1963), 52-53.
3. R.G. Bartle, "Relations between Nets and Indexed Filter Bases", *Colloq. Math.*, *10*(1963), 211-215.
4. G. Bruns and J. Schmidt, "Zur Aquivalence von Moore-Smith Folgen und Filtern", *Math. Nachr.*, *13*(1955), 169-186.
5. J.L. Kelley, "General Topology", Van Nostrand (1955).
6. M.F. Smiley, "Filters and Equivalent Nets", *American Math. Monthly*, *64* (1957), 336-338.


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# THE MERKURYEV-SUSLIN THEOREM

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This article reports on one of the most important, and to many people, astonishing results in algebra so far this decade. In 1981, a Russian mathematician Merkuryev, virtually unknown in the west, proved a theorem concerning the algebraic K-theory and the Brauer group of a field. This result is now known as Merkuryev's theorem and not long afterwards Merkuryev, together with Suslin, a famous Russian mathematician, generalized the result to what is commonly called the Merkuryev-Suslin theorem. These theorems at once provide answers to some very hard problems in the theory of simple algebras, in the theory of quadratic forms and in algebraic geometry. Thus it seems worthwhile to try and explain, in as elementary a way as possible, what the Merkuryev-Suslin theorem is all about. A good source of background information for this article is [5].

We start with that well-known Dublin product, the real quaternions, discovered in 1843 by Hamilton and usually denoted $\mathbb{H}$. A quaternion is an expression of the form $a+bi+cj+dij$ where $a,b,c,d \in \mathbb{R}$, the real numbers, and quaternions can be added in the obvious way and multiplied together using the famous equations $i^2=j^2=-1$, $ij=-ji$. Hamilton's construction may be generalized to give quaternion algebras over any field $F$. We simply choose non-zero elements $a,b$ in $F$, ($a=b$ is allowed), and do exactly as in $\mathbb{H}$ except that we require $i^2=a$, $j^2=b$. For $F=\mathbb{R}$, $a=b=-1$, we have $\mathbb{H}$ of course. A quaternion algebra defined as above is usually denoted $\left(\frac{a,b}{F}\right)$ as it depends on the choice of $a,b$ and on the base field $F$. It is always four-dimensional as an $F$-vector space and it turns out always to be either a skewfield as $\mathbb{H}$ is (i.e. a field except that multiplication lacks commutativity) or else is isomorphic to the ring of all 2x2 matrices with entries in $F$. (In fact it fails to be a skewfield precisely when there exist $x,y$ in $F$ such that $ax^2+by^2 = 1$.) For $F=\mathbb{R}$, $\mathbb{H}$ is the only skewfield that occurs as a quaternion algebra but for other fields things can be quite different. For example, if F=Q, the rationals, there exist indefinitely many non-isomorphic quaternion algebras which are skewfields.

Any quaternion algebra has a natural involution - on it induced by $\bar{i} = -i$, $\bar{j} = -j$. (By an involution on an algebra A we mean a map $A \to A$, $x \to \bar{x}$ such that $\overline{x+y} = \bar{x}+\bar{y}$, $\overline{xy} = \bar{y}\,\bar{x}$ and $\bar{\bar{x}} = x$, i.e. an anti-automorphism of period two.) On $\mathbb{H}$ this involution is the usual conjugation operation. This kind of involution is called an involution of the *first kind* because elements of F are fixed by it. We view F as being contained in $(\frac{a,b}{F})$ in the same way as $\mathbb{R}$ lies inside $\mathbb{H}$. An involution of the *second kind* is one which is non-trivial on F.

We must now say a few words about tensor products of algebras. An F-algebra is a ring which also is an F-vector space, the ring and vector space operations being compatible. Given two F-algebras $A_1, A_2$, there exists a unique F-algebra T and a map $i : A_1 x A_2 \to T$ with the following property:

Given any bilinear map $f : A_1 x A_2 \to W$ into any F-vector space W there exists a unique algebra homomorphism $g : W \to T$ such that $gf = i$. T is called the tensor product and is denoted $A_1 \otimes A_2$.

For example if $A_1$ and $A_2$ are each quaternion algebras then there are three possibilities for $A_1 \otimes A_2$. Firstly $A_1 \otimes A_2$ may be a division algebra (i.e. an F-algebra which is a skewfield). Secondly $A_1 \otimes A_2$ may be the ring of all 2x2 matrices with entries in a quaternion division algebra and thirdly, $A_1 \otimes A_2$ could be the ring of all 4x4 matrices with entries in F. Generally the dimension of $A_1 \otimes A_2$ over F is the product of the dimensions of $A_1$ and $A_2$.

Quaternion algebras are special cases of central simple algebras. A central simple algebra A over F is a finite dimensional F-algebra whose centre is F, i.e. $\{x \in A : xy=yx$ for



all $y \in A\} = F$, and which has no proper two sided ideals when viewed as a ring. For short, we will write c.s. algebra from now on. The tensor product of two c.s. algebra over F is a c.s. algebra. A celebrated theorem of Wedderburn says that any c.s. algebra over F is isomorphic to $M_nD$, the ring of nxn matrices with entries in a skewfield D. Moreover n is unique and D is unique up to isomorphism. D is a division algebra over F. We say that two c.s. algebras are similar if their skewfield parts from Wedderburn's theorem are isomorphic. Similarity is an equivalence relation on the set of c.s. algebras over F. In 1929, Brauer discovered that the set of similarity classes of c.s. algebras over F has a group structure, tensor product being the group operation. The class of F itself is the identity element of the group and the inverse of A is the opposite algebra, denoted $A^{op}$, which is identical with A as a set but with multiplication reversed, i.e. $A^{op}$=A as a set with multiplication $*$ defined by $a*b = ba$, ba being the usual multiplication in A. Then $A \otimes A^{op}$ is isomorphic to the ring of all F-homomorphisms from A to A and this ring is isomorphic to a full matrix ring $M_nF$, n = dimension of A over F, and thus $A \otimes A^{op}$ is similar to F. This group is usually called the Brauer group of the field F and is denoted B(F). For F finite B(F) is trivial since finite skew-fields are commutative (by another theorem of Wedderburn). $B(\mathbb{R})$ is cyclic of order 2, $\mathbb{H}$ being the generator. For a local field $B(F) = Q/\mathbb{Z}$, the rationals modulo one and for an algebraic number field, i.e. a finite algebraic extension of Q, B(F) is extremely large, its calculation being the culmination of work involving Brauer, Hasse, Noether and Albert. See [1] for details.

It should be mentioned that in general, quaternion division algebras are by no means the only kind of division algebras appearing as c.s. algebras. For example a division algebra may be a cyclic algebra defined as follows:

Let L be a cyclic extension of F, i.e. a Galois extension field of F such that the group of all automorphisms of L that fix elements of F is a cyclic group of order n. Let $\sigma$ be a generator of this group. Choose some element $b \in F$. Introduce a symbol u such that $u^n = b$. A typical element of the cyclic algebra determined by L and b is an L-linear combination

$$\sum_{i=0}^{n-1} x_i u^i,$$

each $x_i \in L$, with addition defined in the natural way and multiplication by $u^n = b$ and $ux = \sigma(x)u$ for all $x \in L$. The resulting algebra is c.s. and if b is suitably chosen it can be a division algebra for certain kinds of field F. (Note that for $F = \mathbb{R}$ if we choose $L = C$, $b = -1$ we obtain $\mathbb{H}$, $\sigma$ on C then being complex conjugation.) All division algebras over Q are cyclic. However there exist fields with central division algebras that are not cyclic algebras. See [1], also [2].

For a positive integer n we write $B_n(F) = \{x \in B(F) : x^n = 1\}$. Merkuryev's theorem implies that, for any field F of char $\neq 2$, the subgroup $B_2(F)$ is generated by the quaternion algebras and the Merkuryev-Suslin theorem implies that, provided F contains a primitive $n^{th}$ root of unity, $B_n(F)$ is generated by cyclic algebras. $B_2(F)$ in fact consists exactly of those classes of algebra which admit an involution of the first kind. (An involution of the first kind gives an isomorphism $A \simeq A^{op}$ and hence $\{A\}$ has order two in $B(F)$, and conversely $\{A\}$ has order two means there exists an isomorphism $A \simeq A^{op}$ which yields an involution of the first kind on A.) The degree of a c.s. algebra is defined to be the square root of the F-dimension of the skewfield part of A. A theorem in [1] shows that the order of $\{A\}$ in $B(F)$ divides the degree of A and also that order and degree have the same prime factors apart from multiplicity. It follows that c.s. algebras admitting an involution of the first kind must have degree a power of two.

Tensor products of quaternion algebras give elements of $B_2F$ and fundamental conjectures studied by some algebraists



were those as to whether an algebra with involution of the first kind is isomorphic to or else is similar to a tensor product of quaternion algebras. In 1978, Amitsur, Rowen and Tignol [3] produced an example of a division algebra over Q(t), a transcendental extension of Q, which has an involution of the first kind but is not isomorphic to a tensor product of quaternion algebras. Merkuryev's theorem however gives an affirmative answer to the above conjecture for similarity. So any division algebra D with an involution of the first kind must be such that, for some n, $M_nD$ is isomorphic to a tensor product of quaternion algebras. For the example of [3] $n = 2$ will do, but in general it is not known what the least value of n be.

So far we have only given part of Merkuryev's theorem. To describe it fully we must first define the group $K_2F$ occurring in algebraic K-theory. $K_2F$ is defined as the additive abelian group generated by all symbols $\{a,b\}$, a,b non-zero elements of F, with relations

$$\{ab,c\} = \{a,c\} + \{b,c\}, \quad \{a,bc\} = \{a,b\} + \{a,c\}$$

and

$$\{a,1-a\} = 0 \text{ for all } a,b,c \text{ in } F.$$

Group theorists may be more familiar with an equivalent definition of $K_2F$ as the Schur multiplier of the group E(F) generated by the elementary matrices in F, [11]. An elementary matrix is one which coincides with the identity matrix except for a single off-diagonal entry. Assume char $F \neq 2$. There is a natural map $K_2F \to B(F)$ sending $\{a,b\}$ to the quaternion algebra $(\frac{a,b}{F})$. This map is easily seen to be trivial on the subgroup $2K_2F = \{2x : x \in K_2F\}$ and Merkuryev's theorem says that the induced map $\alpha : \frac{K_2F}{2K_2F} \to B(F)$ is injective and its image is precisely $B_2F$. The surjectivity of $\alpha$ proves the conjecture stated above.

The injectivity of $\alpha$ also answers a long-standing question in quadratic form theory dating back to work of Pfister [12] in 1966. We describe this briefly. The set of isometry classes of non-singular quadratic forms over a field F can be given a ring structure, the addition (resp. multiplication) being induced by the direct sum (resp. tensor product) of the underlying vector spaces. The quotient, on factoring out by the so-called hyperbolic forms, is known as the Witt ring W(F) of F. See [6] for details. Let I denote the ideal of forms defined on even dimensional spaces. Then powers of this ideal exist, i.e. $I^2$, $I^3$, etc. and clearly $I^{n+1} \subset I^n$ for all n. The significant connection between algebraic K-theory and quadratic forms was shown by Milnor [10] in 1970 when he proved that $I^2/I^3$ is isomorphic to $\frac{K_2F}{2K_2F}$. There is a natural map $\beta : \frac{I^2}{I^3} \to B(F)$ gotten by taking the class in B(F) of the Clifford algebra of a quadratic form representing an element of $I^2$. (For anything in $I^3$ the Clifford algebra class can be shown to be trivial.) The map $\beta$ corresponds, under the Milnor isomorphism, to the map mentioned above which Merkuryev showed to be injective. Pfister [12] in 1966 had studied $\beta$ and had shown that in some cases it was injective but since then nobody had come near to a proof in general until Merkuryev's breakthrough. Thus Merkuryev solved a problem which had been regarded by quadratic form theorists as extremely difficult.

We finish by describing the Merkuryev-Suslin theorem. Let $\mu_n$ be the group of all $n^{th}$ roots of unity. We assume for simplicity that $\mu_n$ lies in F. There exists a unique homomorphism, for each n, $\frac{K_2F}{nK_2F} \to B_n(F)\otimes\mu_n$ induced by sending $\{a,b\}$ to $A\otimes\omega$ where $\omega$ is a chosen primitive $n^{th}$ root of unity and A is an algebra, called a norm residue algebra, defined as follows: A is generated by elements u and v with the properties $u^n = a$, $v^n = b$ and $vu = \omega uv$. The tensor product is of abelian groups, defined in a similar fashion to our earlier one, and tensoring on by $\mu_n$ is necessary in order to obtain a homomorphism which is independent of the choice of $\omega$. The name 'norm residue algebra' occurs because A will be isomorphic



to $M_nF$ precisely when b is a norm from $F(\sqrt[n]{a})$. A will indeed always be similar to a cyclic algebra. Merkuryev and Suslin proved that the above map is in fact an isomorphism. The surjectivity implies that, provided F contains $\mu_n$, each element of $B_nF$ is represented by a tensor product of cyclic algebras, a result that was somewhat surprising to some algebraists. Another consequence of the Merkuryev-Suslin theorem is in the realm of algebraic geometry where it leads to new finiteness results about the Chow groups of a rational surface.

We have so far not mentioned the proof of these theorems and to do so would be beyond the scope of this article. The original announcements and proofs are in [7], [8], [9]. The proof requires the Galois cohomological interpretation of B(F). It uses some difficult techniques from Quillen's version of algebraic K-theory and from algebraic geometry, in particular an analysis of the Severi-Brauer varieties corresponding to division algebras [13]. There is also now a more elementary proof of the general Merkuryev-Suslin theorem which has been presented in some notes by Merkuryev [7a]. This proof requires much less higher algebraic K-theory. In fact, Merkuryev's theorem (i.e. for n = 2) can now be more easily done in a couple of ways. Merkuryev himself found an easier proof using Milnor K-theory instead of Quillen K-theory. (We should explain that algebraic K-theory for a field F defines groups $K_nF$ for all non-negative integers n, Milnor and Quillen K-theory are the same for $K_2$ although it is non-trivial to prove this fact. However, for higher n the two K-theories are not always the same.) The Milnor $K_2F$ is much easier to handle and this simpler proof has been very well written up by Wadsworth [14]. Also the quadratic form version of Merkuryev's theorem has been proved by Arason [4] avoiding K-theory altogether but using some technical results from Galois cohomology.

References

1. A. Albert, 'Structure of Algebras', A.M.S. Colloquium publications, revised edition 1961.

2. S.A. Amitsur, 'On Central Division Algebras', *Israel J. Math.*, *12* (1972), 408-420.

3. S.A. Amitsur, L.H. Rowen and J.P. Tignol, 'Division Algebras of Degree 4 and Degree 8 with Involution', *Israel J. Math.*, *33* (1979), 133-148.

4. J.K. Arason, 'A Proof of Merkuryev's Theorem', to appear in Proc. of Conference on Quadratic Forms and Hermitian K-Theory, McMaster University, July 1983.

5. P.K. Draxl, 'Skew Fields', London Math. Soc. Lecture Note Series, No. 81, Camb. Univ. Press, 1982.

6. T.Y. Lam, 'Algebraic Theory of Quadratic Forms', Benjamin 1973.

7. A.S. Merkuryev, 'On the Norm Residue Symbol of Degree 2', *Doklady Akad. Nauk. SSR*, *261* (1981), 542-547, English translation, *Soviet Math. Doklady*, *24* (1981) 546-551.

7a. A.S. Merkuryev, '$K_2$ of Fields and the Brauer Group', to appear in Proc. of K-Theory Conference at Boulder, Colorado, Summer 1983. A.M.S. Contemporary Math. Series (Editor K. Dennis).

8. A.S. Merkuryev and A.A. Suslin, 'K-Cohomology of Severi-Brauer Varieties and Norm Residue Homomorphism', *Doklady Akad. Nauk. SSR*, *264*(1982), 555-559, English translation, *Soviet Math. Doklady*, *25* (1982), 690-693.

9. A.S. Merkuryev and A.A. Suslin, 'K-Cohomology of Severi-Brauer Varieties and Norm Residue Homomorphism', *Izv. Akad. Nauk. SSR*, *46* (1982), 1011-1046.

10. J. Milnor, 'Algebraic K-Theory and Quadratic Forms', *Inventiones Math.*, *9* (1970), 318-344.

11. J. Milnor, 'Introduction to Algebraic K-Theory', *Annals of Maths. Studies*, No. 72, Princeton 1971.



12. A. Pfister, 'Quadratische Formen in Beliebiegen Korpern', *Inventiones Math.* *1*, (1966), 116-132.

13. J.P. Serre, 'Local Fields', Springer, New York, 1979.

14. A. Wadsworth, 'Merkuryev's Elementary Proof of Merkuryev's Theorem', to appear in Proc. of K-Theory Conference at Boulder, Colorado, Summer 1983, A.M.S. Contemporary Math. Series (Editor K. Dennis).

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# ONE ASPECT OF THE WORK OF ALAIN CONNES

*Anthony Karel Seda*

## Introduction

As we all surely know by now, the recipients of the most recently awarded Fields Medals are William P. Thurston of Princeton University, Shing-Tung Yau of the Institute for Advanced Study, Princeton and Alain Connes of Institut des Hautes Études Scientifiques, France. Fields Medals are awarded by the International Mathematical Union on the occasion of an International Congress of Mathematicians, and are the equivalent for mathematicians of the Nobel prize. The last such Congress was originally scheduled to take place in Warsaw in August, 1982, but in fact took place there one year later due to political unrest in Poland.

Thurston's work is in foliations and topology of low dimensional manifolds, Yau's is in differential geometry and partial differential equations and Connes' is in operator algebras. An appraisal of the work of each recipient was published in the Notices of the American Mathematical Society in October, 1982. In particular, Calvin Moore undertook to describe some of the fundamental achievements of Alain Connes.

Much of Connes' earlier work was concerned with the classification by "types" of factors of von Neumann algebras, and three of the five papers of Connes cited by Moore concern this area. However, the fourth (sur la théorie non-commutative de l'intégration, which is reference [2] here) and fifth concern (amongst other things) the interplay between operator algebras and foliations. This subject, which has been called "non-commutative differential geometry" is "(a) fusion of geometry and functional analysis ... likely to have a significant influence on future developments" in the words of Atiyah in his review of [2] for Mathematical Reviews.



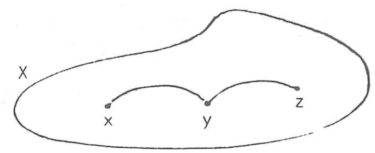
Some of my own work has been in this area, and my purpose here is to shed a little light on the sort of constructions made and the results obtained in this new and interesting area of mathematics.

## 1. Topological Groupoids and C*-Algebras

A *groupoid* G with *object set* X is a small category with object set X such that each element $\alpha$ of G has an inverse $\alpha^{-1}$.

### Examples

(1)




Let X be a topological space and let $G(x,y)$ = (homotopy classes of) paths from x to y. Then $G = \bigcup_{x,y \in X} G(x,y)$ is a groupoid (the fundamental groupoid) over X with composition just the composition of (homotopy classes of) paths, identity $I_x$ at x the trivial path and inverse "traverse the path backwards."

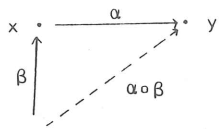
(2) A group is a groupoid with one object, i.e. X is a singleton set consisting of the identity of G. In fact, a groupoid is a group if and only if X is a singleton set.

(3) Suppose that a group H acts on the right of a set X. Then $G = X \times H$ has a natural groupoid structure over X in which the product $(x',h')o(x,h)$ is defined if and only if $x' = x.h$, and is then defined to be $(x,hh')$. The inverse of $(x,h)$ is $(x.h,h^{-1})$ and the identity at x is $(x,e)$, where e is the identity of H.

Many other examples of groupoids can be given, but these three should serve to convey their nature.

A *topological groupoid* is a groupoid in which both $G$ and $X$ are topological spaces and all the structure maps of $G$ are continuous, i.e. the composition, inverse map and the map $x \mapsto I_x$ are all continuous.

Suppose from now on that $G$ is a topological groupoid over $X$ and $G$ and $X$ are both locally compact Hausdorff spaces. Given $\alpha \in G$, there exists unique $x,y \in X$ such that $\alpha \in G(x,y)$; let $\pi(\alpha) = x$, the *initial point* of $\alpha$, and let $\pi'(\alpha) = y$, the *final point* of $\alpha$. Let $G^x = \{\beta \in G;\ \pi'(\beta) = x\}$. Then an element $\alpha \in G(x,y)$ induces a homeomorphism $L_\alpha : G^x \to G^y$ defined by $L_\alpha(\beta) = \alpha \circ \beta$.




$L_\alpha$ is called *left multiplication* by $\alpha$.

Guided by certain analogies between group theory and ergodic theory, G.W. Mackey introduced, in 1966, the notions of measure groupoid and ergodic groupoid. In the topological context these ideas lead one naturally to formulate a concept of left invariant (or Haar) measure on a groupoid $G$, and to consider function spaces associated with $G$. In practice, the most convenient form of an invariant measure is contained in the following:

Definition: A *Haar measure* on $G$ is a family of non-trivial Radon measures $\{\mu_x;\ x \in X\}$ on $G$ such that:

(1) $\mathrm{supp}(\mu_x) \subseteq G^x$ for each $x \in X$.

(2) The $\mu_x$ are left invariant in the sense that

$$\int_G f d\mu_x = \int_G f \circ L_{\alpha^{-1}} d\mu_y$$

for all $x,y \in X$, $\alpha \in G(x,y)$ and $f \in C_c(G)$.

(3) The map $x \mapsto \mu_x$ is vaguely continuous, i.e. the map



$x \mapsto \int f d\mu_x$ is continuous for each $f \in C_c(G)$.

In this definition, and elsewhere, $C_c(G)$ denotes the space of all continuous scalar functions on $G$ with compact support.

The relationship between two Haar measures on $G$ is, unlike the group case, quite complicated, see [6]. However, any Haar measure on $G$ induces a $*$-algebra structure on $C_c(G)$, as follows. Given $f,g \in C_c(G)$, we define $f*g$ on $G$ by

$$(f*g)(\alpha) = \int_G f(\beta)g(\beta^{-1}\alpha)d\mu_{\pi'(\alpha)}(\beta).$$

We define also an involution $f \mapsto f^*$ by $f^*(\alpha) = \overline{f(\alpha^{-1})}$.

Haar measures and the convolution product above were studied by the author in [6], [7] and by Renault in [5], and one of the main basic results is as follows.

Theorem. $C_c(G)$ is an associative $*$-algebra with these operations and is, moreover, a topological $*$-algebra in the inductive limit topology.

The remainder of this section is concerned with associating a C*-algebra with $G$, and the development is similar to the Effros-Hahn construction of transformation group C*-algebras, see [5].

A *representation* of $C_c(G)$ on a Hilbert space $H$ is a $*$-homomorphism $L : C_c(G) \to \beta(H)$ which is continuous when $C_c(G)$ has the inductive limit topology and $\beta(H)$ the weak operator topology, and is such that the linear span of $\{L(f)\xi;\ f \in C_c(G),\ \xi \in H\}$ is dense in $H$.

For $f \in C_c(G)$, define

$$||f||' = \sup_{x \in X} \int |f| d\mu_x, \qquad ||f||'' = \sup_{x \in X} \int |f| d(\mu_x)^{-1}$$

and finally put $||f||_1 = \max(||f||', ||f||'')$.

Proposition ([5])

(i) $||\ ||_1$ is a norm on $C_c(G)$ defining a topology coarser than the inductive limit topology.

(ii) $||\ ||_1$ is a *-algebra norm on $C_c(G)$, i.e. $||f*g||_1 \leq ||f||_1||g||_1$ and $||f^*||_1 = ||f||_1$ for all $f,g \in C_c(G)$.

Definition ([5]): A representation L of $C_c(G)$ is *bounded* if $||L(f)|| \leq ||f||_1$ for all $f \in C_c(G)$.

Now define, for all $f \in C_c(G)$, $||f|| = \sup ||L(f)||$ where L ranges over all bounded representations of $C_c(G)$.

It is easy to see that $||\ ||$ is a C*-semi norm, and it is shown by exhibiting enough bounded representations (the regular representations in fact) that it is a norm. Finally, we denote by $C^*(G)$ the completion of $C_c(G)$ with respect to $||\ ||$. Then $C^*(G)$ is a C*-algebra, i.e. a Banach algebra with conjugate linear involution $f \mapsto f^*$ such that $||f^*f|| = ||f||^2$ for all f, and is called the *C*-algebra of the groupoid* G.

## 2. Foliations and the Holonomy Groupoid

Foliations have a long history even though the definition and subject matter were not formalised until the 1940s by Ehresmann and Reeb. One encounters foliations in:

(a) Submersions of manifolds (here the leaves are the components of the fibres).

(b) Bundles with discrete structure group.

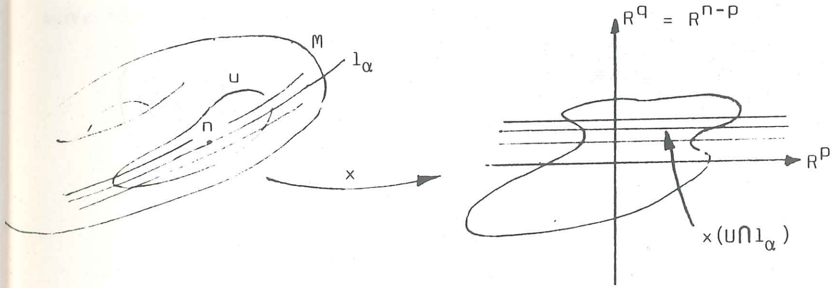
(c) Actions of Lie groups (here the leaves are the orbits).

(d) Differential equations (here the solutions are the leaves).



Definition ([1], [4]): Let M be an n-dimensional manifold and let p,q be natural numbers such that p+q = n. A *p-dimensional class $C^r$ foliation* of M is a decomposition of M into a union of disjoint connected subsets $\{l_\alpha\}_{\alpha \in A}$, called the *leaves* of the foliation, with the following property: every point m of M has a neighbourhood U and a system of local class $C^r$ coordinates $x = (x^1, x^2, \ldots, x^n): U \to R^n$ such that for each $\alpha \in A$ the components of $U \cap l_\alpha$ are described by the equations

$$x^{p+1} = \text{constant}, \ \ldots, \ x^n = \text{constant}.$$




We denote such a foliation by $\mathcal{F} = \{l_\alpha\}_{\alpha \in A}$. p is called the *dimension* and q = n-p the *codimension* of $\mathcal{F}$.

Note that every leaf of $\mathcal{F}$ is a p-dimensional embedded submanifold of M but this embedding need not be proper as the leaves can be dense in M.

Local coordinates with the property mentioned in the definition above are said to be *distinguished* by the foliation. If x,y are two such coordinate systems defined on an open set $U \subset M$, then $yx^{-1}$ is a local $C^r$ diffeomorphism: $R^n \rightarrowtail R^n$ giving the "change of coordinates" and is expressed by the equations

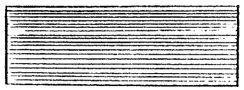
$$y^i = y^i(x^1, x^2, \ldots, x^n), \quad i = 1,2,\ldots,n$$

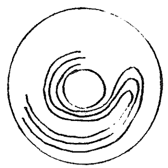
and these must satisfy the differential equations

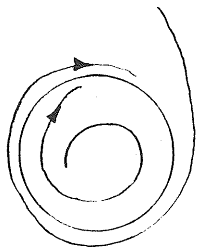
$$\frac{\partial y_i}{\partial x^j} = 0 \quad 1 \leq j \leq p < i \leq n$$

in U. This means that $yx^{-1}$ maps leaves into leaves. Thus, whilst an n-dimensional manifold looks locally like $R^n$, an n-dimensional manifold with p-dimensional foliation looks locally like $R^n = R^{n-p} \times R^p$ trivially foliated by p-dimensional hyperplanes parallel to $R^p$.

Examples







trajectories of a differential equation

The Holonomy Groupoid

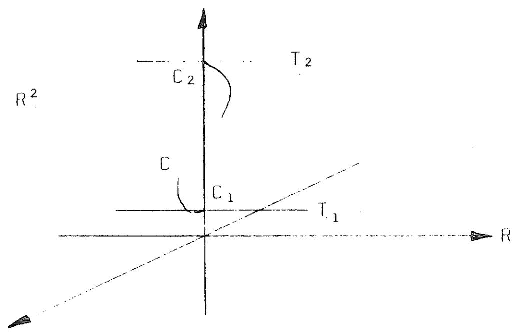
Let $(M,\mathcal{F})$ be a foliated manifold as above, and let $(U,x)$ be a distinguished local coordinate. Then the *plaques* of U are given by the equation $(p_2 x)(m)$ = constant, where $p_2 : R^p \times R^q \to R^q$ is the projection. Give M the "topology of leaves", i.e. the topology on M which has the plaques of distinguished open sets as a basis, and call the resulting space F. A continuous function $U \to R^q$ is called *distinguished* if it is locally of the form $h p_2 o x$, where h is a local homeomorphism of $R^q$. Let D be the sheaf of germs of distinguished functions and $\sigma : D \to F$ the map sending a germ to its source; $\sigma$ is a covering map. [$f \sim g$ if there exists $m \in M$ and a neighbourhood V of m such that $f|_V = g|_V$, then $\sim$ is an equivalence relation and an equivalence class of $\sim$ is called a *germ* at m.] It can



be shown that the fundamental groupoid of F acts on D and we elaborate a little on this below. Finally, on identifying elements of the fundamental groupoid which give the same action we get the *holonomy groupoid* G of $(M,\mathcal{F})$. G is a topological groupoid in a natural way, in fact a locally trivial topological groupoid. It is this construction together with the results of §1 which bring about the sort of application of functional analysis to differential geometry that we have in mind.

Before considering such applications we will look a little more closely at the notion of holonomy.

Consider a curve C lying in the plane $R^2$ as shown:




Suppose $C_1$ has coordinates $(0,(0,\alpha_1))$ and $C_2$ has coordinates $(0,(0,\alpha_2))$, and that $T_1$ and $T_2$ are perpendicular, and hence transverse, to $R^2$ and passing through $C_1$ and $C_2$ respectively.

Any neighbourhood U of C in $R^3$ intersects $T_1$ and $T_2$ in neighbourhoods of $C_1$ and $C_2$ in $T_1$ and $T_2$ respectively, and hence induces a $C^r$-diffeomorphism $(x,(0,\alpha_1)) \longmapsto (x,(0,\alpha_2))$ of a neighbourhood of $C_1$ in $T_1$ onto a neighbourhood of $C_2$ in $T_2$. Clearly the same statement is true for general transversals $T_1$ and $T_2$, though the required $C^r$-diffeomorphism is then more complicated to write down.

Now suppose, generally, that $C:[0,1] \to M$ is a path lying in a leaf $l$ of a foliation $\mathcal{F}$ of $M$, and that $T_0$ and $T_1$ are two submanifolds of $M$ transverse to $\mathcal{F}$ and containing $z_0 = C(0)$ and $z_1 = C(1)$ (a submanifold $W$ is transverse to $\mathcal{F}$ if for each $z \in M$, we have $T_zM = T_zW \oplus T_zL$, where $L$ is the leaf passing through $z$ and "$T_z$" denotes the tangent space at $z$). Then to each neighbourhood $U$ of $C$ in $M$ there corresponds a $C^r$-diffeomorphism $\Phi_C$ of a neighbourhood of $z_0$ in $T_0$ onto a neighbourhood of $z_1$ in $T_1$ such that:

(i) If $\Phi_C$ is defined at $z \in T_0$, then $\Phi_C(z)$ belongs to $T_1 \cap$ leaf containing $z$.

(ii) The germ of $\Phi_C$ at $z_0$ does not depend on $U$ nor on the choice of $C$ up to homotopy.

To construct $\Phi_C$ we proceed as follows. Consider a sequence of distinguished functions $f_i$, $i = 0,1,2,\dots,r$ defined on open sets $V_i$ and an ordered set of points $t_i$ of $[0,1]$ such that $t_0 = 0$, $t_r = 1$ and $C([t_k,t_{k+1}]) \subset V_k$ for $k = 0,\dots,r-1$. Let $T^i$, for each $i$, be a submanifold transverse to $\mathcal{F}$ containing the points $C(t_i)$, $i = 0,1,2,\dots,r$, and such that $T^0 = T_0$, $T^r = T_1$. We can suppose that $F_i(C(t_i)) = 0$ and that $f_i$ is of the form $h_i \circ p_2 \circ x_i$, where $x_i$ is a distinguished local coordinate, for all $i$. For each $i < r$, $x_i$ carries the portion of the curve $C$ between $C(t_i)$ and $C(t_{i+1})$, together with $U$, onto a curve in $R^n$ lying in the hyperplane $R^p$ essentially as depicted above, together with a neighbourhood of this curve in $R^n$. Hence, applying $x_i^{-1}$ to the diffeomorphism described there, we see that for each $i < r$ there is a $C^r$-diffeomorphism $\Phi_i$ of a neighbourhood of $C(t_i)$ in $T^i$ onto a neighbourhood of $C(t_{i+1})$ in $T^{i+1}$ such that $\Phi_i(z)$ belongs to the leaf of $V_i \cap U$ passing through $z$ for each $z$ where $\Phi_i(z)$ is defined. Then $\Phi_C$ is simply the composite $\Phi_{r-1} \circ \Phi_{r-2} \circ \dots \circ \Phi_1 \circ \Phi_0$, and it is clear from statements (i) and (ii) that the fundamental groupoid of $F$ does act on $D$, as required.

By means of general results of [6], Haar measures exist on the holonomy groupoid $G$, even though $G$ is not Hausdorff in



general. A natural, geometric construction of a Haar measure on $G$ can be found in [3].

To date, most of the results obtained have concerned the ideal structure of $C^*(G)$ or rather the ideal structure of the reduced $C^*$-algebra $C^*(G)/k$, where $k$ denotes the kernel of the regular representations of $C^*(G)$. It is important to know whether any/all leaves of $\mathcal{F}$ are dense in $M$, and we have the following criteria.

<u>Theorem</u> (Fack and Skandalis [3]). $C^*(G)/k$ is simple (i.e. has no non-trivial closed two sided ideals) if and only if every leaf of $\mathcal{F}$ is dense in $M$.

A $C^*$-algebra $A$ is called *primitive* if it has a faithful irreducible representation on a $C^*$-algebra $\beta(H)$ (i.e. a $*$-homomorphism $A \to \beta(H)$ = bounded linear operators on Hilbert space $H$).

<u>Theorem</u> (Fack and Skandalis [3]). $C^*(G)/k$ is primitive if and only if at least one leaf of $\mathcal{F}$ is dense in $M$.

I have only touched on one small part here of the circle of ideas involved in this subject, a subject which embraces transverse measures on foliations, Connes' generalisation of the Atiyah-Singer index theorem, non-commutative integration in general, to name only a few topics. There is as yet, as far as I know, no general account of this material, and the interested reader will have to consult [2] and subsequent papers/preprints. There is, however, a detailed account of some of the measure theory of [2] to be found in Daniel Kastler's paper "On A. Connes' Non-Commutative Integration Theory", *Commun. Math. Phys.*, *85* (1982) 99-120.

<u>References</u>

1. H. Blaine Lawson, Foliations, *Bull. Amer. Math. Soc.*, *80*

(3) (1974) 369-418.

2. A. Connes, Sur la Théorie Non-Commutative de l'Intégration, Springer Lecture Notes in Math., *725* (1979) 19-143.

3. T. Fack and G. Skandalis, Sur les Représentations et Idéaux de la C*-Algèbre d'un Feuilletage, *J. Operator Theory*, *8* (1982) 95-129.

4. A. Haefliger, Variétés Feuilletées, *Ann. Scuola. Norm. Sup. Pisa*, *16* (1952) 367-397.

5. J. Renault, A Groupoid Approach to C*-Algebras, Springer Lecture Notes in Math., *793* (1980).

6. A.K. Seda, Haar Measures for Groupoids, *Proc. Roy. Irish Acad.*, *76(A)* (1976) 23-36.

7. A.K. Seda, Banach Bundles of Continuous Functions and an Integral Representation Theorem, *Trans. Amer. Math. Soc.*, *270* (1) (1982) 327-332.

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# NON-LINEAR DIFFERENTIAL EQUATIONS IN BIOLOGY*

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## 1. Introduction

In recent years there has been considerable growth in the range of mathematical sciences applied to biology and medicine. For many years the statistics of experimental design had been regarded as the main application in the life sciences, but with the advent of mathematical modelling, both deterministic and stochastic models (see Raymond Flood's lecture to the Easter 1983 Symposium [4]) are gaining widespread acceptance. The introduction of biotechnology courses in Ireland has led to interest in the partial differential equations which arise in biological process engineering, such as the reaction-diffusion equation. Workers in fluid dynamics have linked with medical doctors to consider the equations governing the flow of blood through the heart. Stochastic differential equations arise in population dynamics and interesting problems in branching of solutions of non-linear differential equations have come from transmission in nerve axons and from the study of reversible reactions.

The mathematics involved in biological problems can range from the very recent and sophisticated, such as the sledge-hammer of topological degree theory applied to branching problems, to the ingenious application of the most elementary *ad hoc* methods of classical analysis and geometry, as we shall see in Section 2. But whatever mathematics is used, the final results are only as good as the modelling process employed.

A typical modelling scheme is shown in Figure 1 overleaf. It is rare for this process to flow smoothly from one end to the other. Often the mathematical problem cannot be solved in its original form. A solution may be possible by adding

* Survey Lecture given at the D.I.A.S. Christmas Symposium, 1983.

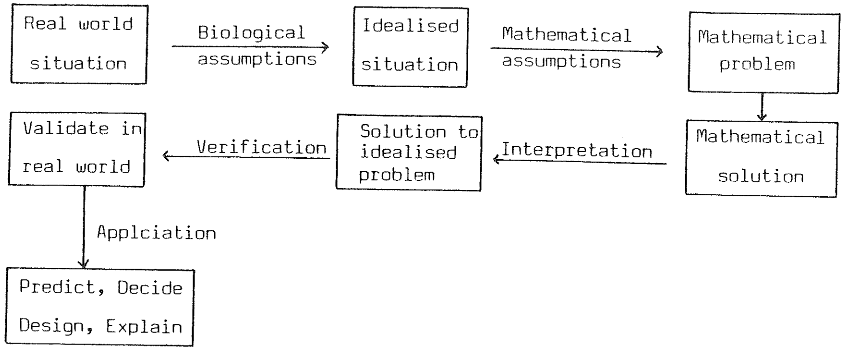


FIGURE 1: The Modelling Process

to the mathematical assumptions, but such assumptions may no longer be in line with the biological reality, with the consequence that the mathematical solution does not make sense when interpreted in the real world. There is an obvious "play-off" between mathematical tractability and biological reality in the model.

There is an enormous temptation for the academic mathematician to pursue only those problems of current, pure mathematical interest, but this should be subordinated to truly interdisciplinary studies, pursued in co-operation with clinical, laboratory and field research workers to provide fresh insight into problems whose solution has important practical consequences. At this time increasing numbers of applied mathematicians throughout the world are finding themselves employed on nuclear, military or defence-related projects. While there is no doubt that an active defence industry is good news for the employment and remuneration of mathematics graduates, many mathematicians in a neutral country would have moral reservations about working on such projects. In biological and medical problems, there exists the opportunity to deploy one's skills, not to add to human suffering, but to alleviate and prevent human misery through the eradication of want and disease.



## 2. The Predator-Prey Interaction

To give the flavour of biological applications, we present some classical work carried out fifty years ago by the Italian mathematician, Vito Volterra, best known for his work on integral equations. This has also been used profitably as a case study for advanced undergraduates or master's degree students and appears in Braun's text on differential equations [1].

We consider an environment where a population of prey, numbering $x(t)$ at time $t$, interacts with a population of predators which numbers $y(t)$. We assume that there are ample resources of food for the prey, but that the prey are the sole source of food for the predators. In the absence of predators the prey population grows at a constant, positive rate $a$. The number of predator-prey contacts will be proportional to the numbers in each population: let $b$ be the "success" rate, from the predator's viewpoint, of each contact, where $b$ is a positive constant. Using dot for differentiation with respect to time, the rate of change in the prey population is thus

$$\dot{x} = ax - bxy \qquad (2.1)$$

For the predators, let $c > 0$ be their natural constant rate of decrease in the absence of prey. But they will increase at a rate proportional to their present number and food supply. Thus

$$\dot{y} = -cy + dxy \qquad (2.2)$$

where $d$ is a positive constant.

Heuristic reasoning leads us to expect that, when prey are plentiful, the predators will multiply to a point at which prey are in short supply and starvation leads to a drop in predator numbers. When these have reached a sustainable number, the cyclic process will start again. Is this substantiated by the mathematics?

The equilibrium states of the system comprising (2.1) and (2.2) are clearly at (0,0) and $(c/d, a/b)$. The state at the origin corresponds to no populations present. We restrict attention to the latter state $(c/d, a/b)$, moving this state to the origin by the transformation $X = dx-c$, $Y = by-a$ to obtain

$$\dot{X} = -(X + c)Y$$
$$\dot{Y} = (Y + a)X. \qquad (2.3)$$

The corresponding linearised system is

$$\dot{X} = -cY$$
$$\dot{Y} = aX \qquad (2.4)$$

which has a pair of purely imaginary characteristic roots $\pm i\sqrt{ac}$. Hence (0,0) is a centre of the linearised system (2.4), and, by the asymptotic perturbation theorem [6, p. 87], will be either a centre or spiral point of the non-linear system (2.3). Fortunately we can find explicit solutions which enable us to distinguish these cases.

An explicit solution of (2.4) is $X(t) = K\cos\sqrt{ac}\,t$, $Y(t) = K\sqrt{ac}\sin\sqrt{ac}\,t$, for any constant K, which describes a system of confocal ellipses in the (X,Y)-plane as shown in Figure 2 below

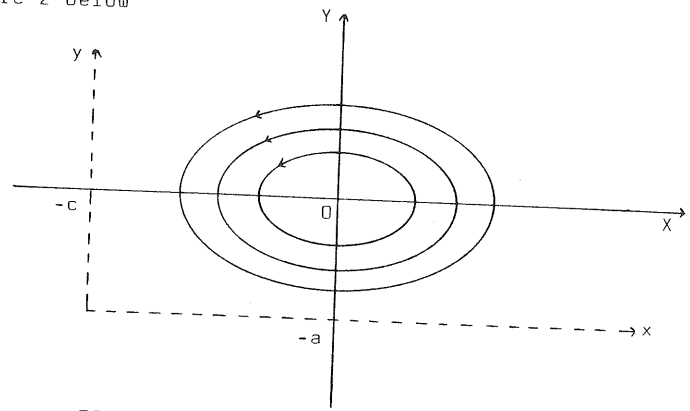


FIGURE 2: Phase-plane of the linear system



There is one for each value of K and the arrows denote the direction of increasing time.

Turning to the non-linear system (2.3), multiplying the equations by $X(Y+a)$ and $Y(X+c)$ respectively and adding yields

$$X(Y + a)\dot{X} + Y(X + c)\dot{Y} = 0.$$

Because both populations are required to be present, both $Y + a$ and $X + c$ are positive and we obtain

$$\frac{X}{X + c}\dot{X} + \frac{Y}{Y + a}\dot{Y} = 0,$$

which may be rearranged as

$$\dot{X} - \frac{c\dot{X}}{X + c} + \dot{Y} - \frac{a\dot{Y}}{Y + a} = 0$$

and integrated directly to give

$$e^{X+Y} = e^{k}(X + c)^{c}(Y + a)^{a}, \qquad (2.5)$$

where again k is an arbitrary constant to be determined by the initial conditions. We shall show that this defines a family of closed curves, not spirals, the other possibility, in the positive quadrant.

Lemma 1. Equation (2.5) defines a family of closed curves in

$$X > -c, \quad Y > -a \quad (\text{that is, } x > 0,\ y > 0)$$

Proof: In original coordinates (2.5) may be written as

$$y^{a}x^{c} = Ke^{by}e^{dx} \qquad (2.6)$$

for some constant K. Define the functions $f(y) = y^{a}/e^{by}$, $g(x) = x^{c}/e^{dx}$. Then f vanishes at 0 and $+\infty$, is positive in between and has a single maximum at $a/b$ with maximum value $M = (a/b)^{a}e^{-a}$. The function g has similar properties with a single maximum of height $m = (c/d)^{c}e^{-c}$ at $c/d$.

It follows at once that (2.6) has no solution with $x, y > 0$ if $K > mM$ and the unique solution $x = c/d$, $y = a/b$ if $K = mM$.

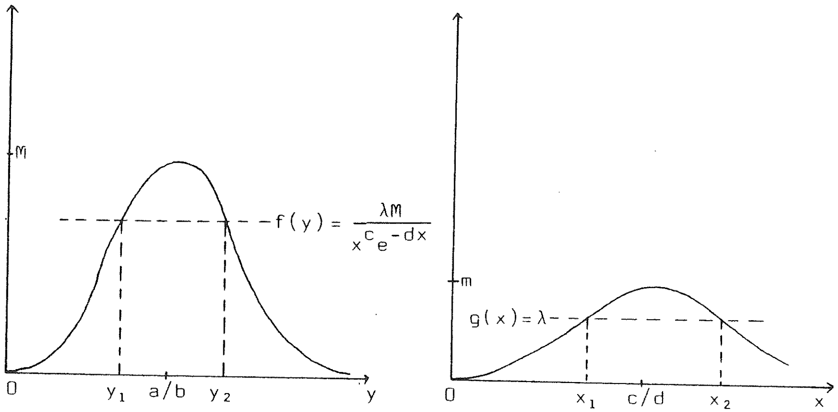


FIGURE 3: The functions f and g

Now let $K = \lambda M$, where $0 < \lambda < m$. We see from Figure 3 that the equation $g(x) = \lambda$ has exactly two solutions $x_1$ and $x_2$ lying on opposite sides of $c/d$. Rewriting the equation $f(y)g(x) = \lambda M$ as

$$f(y) \quad = \quad y^a e^{-by} \quad = \left(\frac{\lambda}{x^c e^{-dx}}\right) M$$

we see that this has: no solution $y$ when $x < x_1$ or $x > x_2$; exactly one solution $y = a/b$ when $x = x_1$ or $x_2$; and two solutions $y_1(x)$, $y_2(x)$ when $x_1 < x < x_2$. The smaller solution $y_1(x)$ is always less than $a/b$ and $y_2(x)$ always greater. Both tend to $a/b$ as $x \to x_1$ or $x_2$. We note also that $y_2(x)$ is increasing for $x_1 < x < c/d$ and decreasing for $c/d < x < x_2$.

We now conclude that the curves defined by (2.6) are closed in $x > 0$, $y < 0$ and have the form shown overleaf in Figure 4.



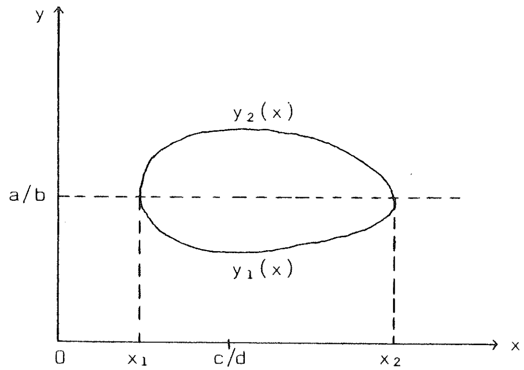


FIGURE 4: A trajectory of the non-linear system

Hence the solution curves of (2.5) are closed for $X > -c$, $Y > -a$ as required.

Corollary: All solutions of (2.1), (2.2) with positive initial conditions are periodic functions of time.

Lemma 2. Let $x(t)$, $y(t)$ be a solution of (2.1), (2.2) with period $T > 0$. Define the mean values by

$$\bar{x} \quad = \quad \frac{1}{T}\int_0^T x(t)dt, \quad \bar{y} \quad = \quad \frac{1}{T}\int_0^T y(t)dt.$$

Then $\bar{x} = c/d$, $\bar{y} = a/b$, that is, the mean and equilibrium values coincide.

Proof: Dividing both sides of (2.2) by $y > 0$ and integrating yields

$$\frac{1}{T}\int_0^T \frac{\dot{y}}{y}\,dt \quad = \quad \frac{1}{T}\int_0^T (-c + dx)dt.$$

Since the left-hand side is zero by periodicity of $y$, the result $\bar{x} = c/d$ follows on evaluating the right-hand side.

The other result follows similarly by dividing (2.1) by x.

Thus the interpretation of the mathematical model is that the growth of each population can be described as regular increase and decrease around a mean level. From (2.1), if x ever vanishes, then it is zero for all future time. Since we know that for given initial conditions there is exactly one solution of (2.1), (2.2), we conclude that if x is ever positive then it will always remain positive. This means, under the assumptions preceding (2.1), that the prey population can never be wiped out by the predators.

Our next stage is to validate this model against sets of field observations. It may be that we have to modify the model to include more realistic interaction terms p(x,y), q(x,y) where p,q are polynomial functions, giving

$$\dot{x} = ax + p(x,y)$$
$$\dot{y} = -cy + q(x,y)$$

The explicit phase-plane analysis given above holds only for the particularly simple forms of p, q in (2.1), (2.2). In general, it is possible to obtain spiral points, where the sizes of both populations oscillate about longterm equilibrium values.

## 3. The Effect of Harvesting

Volterra's contribution was to explain the effect of a reduction in fishing levels on fish stocks in the Adriatic, observed by the Italian biologist D'Ancona. He studied the interaction between the predatory selachians (sharks, skates and rays) and the food fish which formed their prey. The table below shows the percentage of selachians in the total catch recorded at the port of Fiume in the years 1914-1923.

| 1914 | 1915 | 1916 | 1917 | 1918 | 1919 | 1920 | 1921 | 1922 | 1923 |
|---|---|---|---|---|---|---|---|---|---|
| 11.9 | 21.4 | 22.1 | 21.2 | 36.4 | 27.3 | 16.0 | 15.9 | 14.8 | 10.7 |



Was the rise in percentage of selachians due to the decreased level of fishing during the First World War or simply part of the predator-prey cycle observed in Section 2?. As the selachians were not in demand for human consumption, there were important implications for the fishing industry.

D'Ancona's theory was that, when fishing was reduced, there were more prey available to the selachians, who flourished and multiplied. Unfortunately, it was found that the absolute numbers of food fish also increased in this period. The theory did not explain why a reduced level of fishing was more beneficial to the predators than their prey.

In [7], Volterra formulated predator-prey equations like (2.1), (2.2) with an extra term to describe the effects of fishing. Assume that fishing decreases the food fish population at rate $\varepsilon x(t)$ and the selachian population at rate $\varepsilon y(t)$, where $\varepsilon$ describes the intensity of harvesting. This can be measured by the number of boats at sea or nets in the water: see Clark [2]. We then have

$$\begin{aligned} \dot{x} &= (a - \varepsilon)x - bxy \\ \dot{y} &= -(c + \varepsilon)y + dxy \end{aligned} \qquad (3.1)$$

Provided $a > \varepsilon$, the system (3.1) is identical to that of Section 2, with a replaced by $a-\varepsilon$ and c by $c+\varepsilon$. The mean values of x and y are given by Lemma 2 as

$$\bar{x} = \frac{c + \varepsilon}{d}, \quad \bar{y} = \frac{a - \varepsilon}{d}. \qquad (3.2)$$

The ratio of selachians to food fish is seen to be

$$\frac{a - \varepsilon}{c + \varepsilon} \cdot \frac{d}{b}$$

which increases as $\varepsilon$ is reduced, accounting for the observed effect. The increase in percentage of selachians is due to a shift in the equilibrium values and not to cyclical variations. We observe also from (3.2) that a moderate amount of harvesting

($\varepsilon < a$) actually increased the number of food fish and reduces the number of selachians. An excessive amount of fishing ($\varepsilon > a$) leads to the eradication of both populations, with obvious implications for EEC fishing policies in Irish waters.

This result is known in biology as Volterra's principle. It is interesting to note that another distinguished analyst of the same period, G.H. Hardy, also better known to mathematicians in other fields, has his name enshrined in biology through the Hardy-Weinberg ratio in genetics.

## 4. Other Applications of the Model

The use of insecticides, which destroy both the insect predators and their prey, may have the undesired effect of increasing the population of insect pests kept under control by natural insect predators. The cottony cushion scale insect was accidently carried from Australia to the U.S.A. in 1868 and spread to such proportions that it threatened the Californian citrus industry. To combat this, the ladybird beetle, a natural predator, was introduced from Australia and succeeded in keeping the scale insect in check. When the insecticide, DDT, was discovered, farmers applied it in an attempt to eradicate the scale insects. Instead they found that, as predicted by Volterra's principle, the scale insect population increased.

Similar effects have been observed in the spraying of lakes to kill off mosquito larvae, which also had the effect of reducing the population of natural predators on the larvae. Spraying of DDT had damaging longterm effects on the environment, while spraying with oil; to reduce the surface tension, causing the eggs to sink, led to pollution of water suppplies. Mathematical work on alternative methods for mosquito control has been carried out by a former student, F.M. Dube [3], in 1982. Let $x(t)$, $z(t)$ and $p(t)$ denote the populations of adult female mosquitoes, immature mosquitoes (water-borne larvae) and aquatic predators respectively. The interaction of the two mosquito



populations, in the absence of predators, is shown in Figure 5 below.

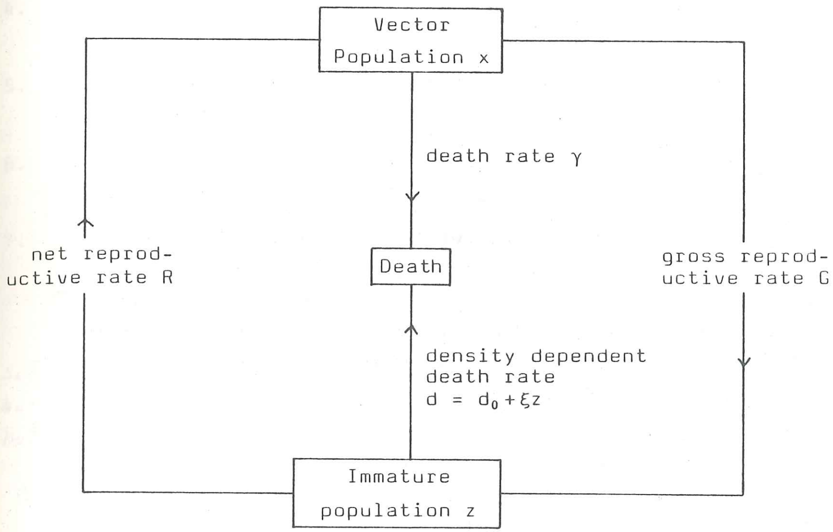


FIGURE 5: Interaction of mosquito population

This is described by the system of equations

$$\begin{aligned} \dot{x} &= -\gamma x + Rz \\ \dot{z} &= Gx - (d_0+\xi z)z. \end{aligned} \qquad (4.1)$$

A key parameter is the reproductive threshold $R_0 = GR/\gamma d_0$ which is interpreted as the average reproductive contribution of one female mosquito to the next generation, roughly the number of viable progeny.

We now introduce an aquatic predator population with carrying capacity (maximum sustainable population) $\kappa$, intrinsic growth rate $\alpha$ and rate of kill $\beta$. The system (4.1) is modified to:

$$\dot{x} = -\gamma x + Rz$$
$$\dot{z} = Gx - (d_0 + \beta p)z - \xi z^2 \qquad (4.2)$$
$$\dot{p} = \alpha p(1 - p/\kappa) + \beta pz.$$

This system has four equilibrium states

$E_0$ : (0,0,0) ; no populations present
$E_1$ : (0,0,$\kappa$) ; eradication of mosquitoes
$E_2$ : ($\hat{x},\hat{z}$,0) ; predator fails to thrive
$E_3$ : ($\bar{x},\bar{z},\bar{p}$) ; control of mosquito population

Let $\beta^* = d_0(R - 1)/\kappa$. It can be shown that, for $R_0 > 1$, the equilibrium states $E_0$ and $E_2$ are always unstable. If, in addition, $\beta > \beta^*$, then the equilibrium state $E_1$ is asymptotically stable, while $E_3$ (which is of no biological significance in this case) is unstable. Conversely, if $\beta < \beta^*$, then $E_3$ is asymptotically stable and $E_1$ is unstable. As $\beta \to \beta^*$, the states $E_1$ and $E_3$ coalesce. We conclude that, for a model with a density dependent death rate, $\beta^*$ represents the threshold between control and eradication of the mosquito population.

This result has practical implications for the choice of natural predator introduced. For habitats with a low carrying capacity $\kappa$, the predatory worm of the genus *Mesostama* has been found to reduce mosquito emergence by 70 to 90% [5]. In habitats with large $\kappa$, the introduction of predatory species of fish, with higher rate of kill $\beta$, will prove more effective.

## References

1. Braun, M., Differential Equations and their Applications, Springer-Verlag, 1975.

2. Clark, C.W., The Optimal Management of Renewable Resources, Wiley-Interscience, 1976.



3. Dube, F.M., Some Theoretical Models for Mosquito Populations, Ph.D. Thesis, Cranfield Institute of Technology, 1982.

4. Flood, R.G., "Population Biology of Infectious Diseases", *Irish Mathematical Society Newsletter*, 9, 1983.

5. Mead, A.P., "A Dreadful Way to Die - A New Mosquito Killer", *Overseas Universities*, 27, 1980.

6. Sanchez, D.A., Ordinary Differential Equations and Stability Theory, Freeman, San Francisco, 1968.

7. Volterra, V., Lecons sur la Theorie Mathematique de la Lutte pour la Vie, Gauthier-Villiers, Paris, 1931.

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## BOOK REVIEWS

"CALCULUS AND ANALYTIC GEOMETRY" (Sixth Edition)

By *G.B. Thomas and R.L. Finney*

Published by *Addison-Wesley,* Reading, Mass, 1984, paper.
Stg £15.95, pp. xxiii + 1146, ISBN 0-201-16309-9.

From the authors' preface:

"In preparing the *Sixth Edition* from the *Fifth Edition*, we have tried to make calculus easier for people to learn. The number of exercises has nearly doubled ...

"This edition ... contains references at the end of some of the problem sets to a series of twenty-seven Apple II™ microcomputer programs appropriate for students enrolled in calculus courses as well as for engineers, physicists, and others who use calculus as a tool in their work. The programs and accompanying manual are available from Addison-Wesley Publishing Company under the title *A Calculus Student's Microcomputer Toolkit*.... It is to be emphasized ... that ... progress through the text is in no way dependent upon having access to these or any other computer programs.

"The level of rigor of the text is about the same as in earlier editions. ...

"Chapter 1, on the rate of change of a function, has new sections on continuity and on infinity as a limit.

"Chapter 2, on derivatives, now has sections on linear (tangent line) approximations, and on inverse functions and the Picard method for finding roots.

"Chapter 3, on applications of derivatives, begins with new sections on curve sketching, concavity, and asymptotes.



Maxima and minima now precede related rates. The chapter concludes with a section that extends the Mean Value Theorem and develops error estimates for the standard linear and quadratic approximations of functions.

"Chapter 4, on indefinite and definite integrals, begins as it has in the past with differential equations of the form $y' = f(x)$, solved by separation of variables. The chapter contains a new development of the two fundamental theorems of integral calculus, however, and devotes a separate section to the technique of integration by substitution (for both definite and indefinite integrals). It is in this section that differentials are first introduced.

"Chapter 5, on applications of definite integrals, has more art, problems, and worked examples than before, and frequent formula summaries.

"Chapter 6, which introduces the logarithmic, exponential, and inverse trigonometric fucntions, also discusses relative rates of growth of functions. It concludes with a section on applications of exponential and logarithmic functions to cooling, exponential growth, radioactive decay, and electric circuits, and a section on compound interest and Benjamin Franklin's will.

"In Chapter 7, on techniques of integration, the section on improper integrals has been expanded to include comparison tests for convergence. There is also a new section on using integral tables, and the treatment of integration by parts has been moved to the beginning of the chapter.

"Chapters 8 (plane analytic geometry) and 9 (hyperbolic functions) have been shortened somewhat and contain additional art and problems.

"Chapter 10, on polar coordinates, is shorter than before and contains a new technique for graphing polar equations of the form $r = f(\theta)$.

"The presentation of infinite sequences and series has been moved forward in the book and divided into Chapters 11 and 12. Chapter 11 is devoted to sequences and infinite series of constants, Chapter 12 to Taylor's theorem (as an extended mean value theorem) and power series. Series of complex numbers are mentioned briefly. (An introduction to complex number arithmetic and Argand diagrams appears in Appendix 8.)

"Chapter 13, on vectors, begins with motion in the plane and moves from there to the study of vector algebra and geometry in space.

"Chapter 14, on vector functions and their derivatives, has a new treatment of tangent vectors, velocity and acceleration, and concludes with a section on Kepler's laws of planetary motion.

"Chapter 15, on partial derivatives, has new treatments of limits of functions of two variables, continuity, surfaces, partial derivatives, chain rules, directional derivatives, linear approximation and increment estimation, maxima and minima (both constrained and free), Lagrange multipliers, exact differentials, and least squares. Computer graphics have made it possible to visualize and discuss a number of surfaces that could not have been shown in earlier editions of this book. The chapter also looks briefly at solutions of some of the important partial differential equations of physics (in connection with higher order derivatives) and has a short section on how to apply chain rules when a function's variables are not independent.

"Chapter 16, on multiple integrals, contains a new introduction to the subject, along with more examples, problems, and frequent formula summaries. It concludes with a presentation of surface area based on the notion of gradient.



"Chapter 17, on vector analysis, begins with vector fields, surface integrals, line integrals, and work, and concludes with Green's theorem, the divergence theorem, and Stokes's theorem. In addition to many new examples and problems, the chapter now contains a brief derivation of the continuity equation of hydrodynamics.

"In Chapter 18, on ordinary differential equations, the treatment of linear second order equations with constant coefficients has been expanded to include the method of undetermined coefficients in addition to the method of variation of parameters. The chapter concludes with short sections on power series solutions, direction fields and Picard's theorem, and Euler and Runge-Kutta methods.

"The appendixes include expanded sections on determinants and Cramer's rule, and on matrices and linear equations, as well as new sections on mathematical induction and number systems.

"The text is available in one complete volume ... or as two separate parts ... Both parts contain answers to odd-numbered problems."

## PROBLEM PAGE

First of all, here's a proof of the inequality

$$(1 + x^2)^p \leq x^{2p} + (2^p - 2)x^p + 1, \qquad x \geq 0,\ 1 \leq p \leq 2, \qquad (1)$$

which was posed last time.

For $x > 0$ we write

$$(x^{2p} + (2^p - 2)x^p + 1) - (1 + x^2)^p = x^p(x^p + (2^p - 2) + \frac{1}{x^p} - (x + \frac{1}{x})^p) = x^p\phi(x),$$

say. By symmetry, it is enough to prove that $\phi(x) \geq 0$ for $0 < x \leq 1$, and since $\phi(1) = 0$, we need only show that $\phi$ is decreasing in $(0,1)$. But

$$\phi'(x) = px^{p-1} - \frac{p}{x^{p+1}} - p(x + \frac{1}{x})^{p-1}(1 - \frac{1}{x^2})$$

so that

$$x^{p+1}\phi'(x) = p(x^{2p} - 1 + (1 - x^2)(1 + x^2)^{p-1}).$$

For $0 < x \leq 1$ we have

$$x^{2p} + (1 - x^2)(1 + x^2)^{p-1} \leq 1, \qquad 1 \leq p \leq 2.$$

since the left-hand side is convex as a function of $p$, and there is equality at $p = 1,2$. Thus $\phi'(x) \leq 0$ for $0 < x \leq 1$, and the proof is complete.

The idea of keeping $x$ fixed and varying $p$ has been used by Harold Shapiro to give a proof of (1) based on Descarte's rule of signs! Another proof of (1) depends on the expansion

$$\alpha^p = (1 - (1 - \alpha))^p = 1 - p(1 - \alpha) + \frac{p(p - 1)}{2!}(1 - \alpha)^2 - \ldots,$$

where $0 < \alpha < 1$.



The more general inequality, of which (1) is a very special case, arises in the following way. For any polynomial

$$P(z) = a_0 + a_1 z + \ldots + a_n z^n,$$

let us write

$$|P|_p = |a_0|^p + |a_1|^p + \ldots + |a_n|^p.$$

If $P,Q$ are both polynomials of degree $n$ with non-negative coefficients is it then true that

$$|PQ|_p^2 \leq |P^2|_p\,|Q^2|_p, \qquad 1 \leq p \leq 2? \qquad (2)$$

For $p = 1$ both sides are equal to $(P(1)Q(1))^2$ and, for $p = 2$, (2) is a form of the Schwarz inequality. All computer calculations (now done by several people) point to the truth of (2) but there are only a few positive results.

For example, if $P(z) = 1 + az$, $Q(z) = a + z$, where $a \geq 0$, then (2) reduces to (1). From (1) we can also deduce the general linear case, $P(z) = 1 + az$, $Q(z) = 1 + bz$, where $a,b \geq 0$. However the next case to consider

$$P(z) = 1 + az + bz^2,\ Q(z) = b + az + z^2,$$

where $a,b \geq 0$, has only been verified (using Shapiro's method) in certain special cases.

I hope to survey known results and the computer evidence in a future article in the *Newsletter*. For now, here are two more problems.

1. (Suggested by Finbarr Holland) Prove that

$$\sum_{k=1}^{m} \cot^2 \frac{k\pi}{2m+1} = \frac{m(2m-1)}{3},$$

and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

2. (Posed by the topologist Morton Brown) Suppose that $a_1$, $a_2$ are real and not both zero, and

$$a_{n+2} = |a_{n+1}| - a_n, \quad n = 1,2,\dots .$$

Prove that the sequence $\{a_n\}$ always has period 9.

*Phil Rippon,*
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# CONFERENCE REPORTS

## FOURTH CONFERENCE ON APPLIED STATISTICS IN IRELAND

The Fourth Conference on Applied Statistics in Ireland was held in the Kilkea Castle Hotel, Castledermot, Co. Kildare, on 29-30 March 1984. This conference was the fourth in a series which brings together individuals of diverse statistical interests from industry, government and education. Fifty-three participants (including two from overseas) attended the conference and helped create an atmosphere conducive to the exchange of statistical ideas. An added bonus to this year's conference was the book displays provided by both Chapman and Hall Ltd and John Wiley and Sons Ltd. C.O.P.S. Ltd displayed IBM personal computers and some relevant statistical software.

The conference programme was divided into five sessions of contributed papers as well as two principal invited addresses. The first invited address was given by Mr Thomas P. Lenihan, Director of the Central Statistics Office. Mr Lenihan gave an overview of the C.S.O. and its activities, and one could not but be impressed by the diversity and scope of this important information collecting agency. Mr Charles Smith, chief statistician at Guinness Ireland Ltd, gave the second principal address in which he described the role of statistics at Guinness. It was quite interesting to note how diversely talented a large company like Guinness expects its statisticians to be. Although the role of the statistician in industry seems to be well appreciated (for historical and other reasons) at Guinness, it was perhaps a bit discouraging to learn that the number of statisticians employed at Guinness has decreased markedly in recent years.

The first session of contributed papers was led off by Adrian Dunne (UCD) who demonstrated the potential of an objective design strategy for pharmacokinetic model discrimination. Graham Horgan (TCD) then described some of the practical problems in the statistics of image processing, particularly with regard to the case of satellite photography for environmental purposes. John Haslett (TCD) discussed the utilization of spatial information in performing discriminant analysis on multivariate data (e.g. with LANDSAT). Arnold Horner (UCD) and James Walsh (Carysfort) completed the initial session when they described a project they have undertaken to make information on the geography of Irish agricultural statistics more readily intelligible and more widely and rapidly diffused. Their recently published *Agriculture in Ireland - A Census Atlas* is the result of this project in which the mapping done is computer assisted.

Aidan Moran (UCC) began the second session by discussing some of the problems which arise in ranking students on the basis of an entrance scholarship examination at UCC which involves the selection of several different subjects. He demonstrated that as certain subjects seem to dominate the scholarship awards, some form of standardization is needed. Owen Egan (Educational Research Centre) discussed in the context of a regression model the performance of primary teachers' assessments of their pupils' abilities versus assessments based on test scores, and concluded that the teachers' assessments are as defensible as any that might be made under the prevailing error factors. Eamonn McEntee (Ulster Polytechnic) concluded the session by presenting some useful ideas on how the microcomputer might be used in the teaching of statistics.

Stephen Gardiner (Department of Agriculture for Northern Ireland) initiated the third (early morning) session in discussing recent research at the Plant Testing Station, Department of Agriculture (NI) involving the use of electrophoresis in studying competitive ability of perennial ryegrass cultivars in conventional swards. David McSherry (QUB) described ISIS (Interactive Statistical Information System), a programme he has developed designed for patient record management and statistical analysis in a study of femoral neck fracture in the elderly. Peter Whalley (Open University) presented a paper on the applicability of psychological data to multivariate



analysis. Eddie Gillespie (Ulster Polytechnic) concluded the session with a paper showing how in multiple linear regression the assumption of measurement error in the explanatory variables does not always cause regression parameters to be biased towards zero (as in simple linear regression), and that the direction of the bias depends on the degree of orthogonality between the explanatory variables.

In the fourth session Adrian Raffery (TCD) presented a new multivariate exponential distribution and compared it favourably with several of the more traditional multivariate exponential distributions. Mohammed Khan (Kent State University, Kent, Ohio) presented a sequential design scheme for estimating the optimal inspection time of a system of N independent and identical components.

Antony Unwin (TCD) in the fifth session described models for estimating the probability that another sample of votes would give a different election result and discussed the importance of the sampling effect in recent Irish elections. Don Bary (UCC) presented an interesting paper highlighting the need in many situations for a nonparametric approach to regression theory. Gabrielle Kelly (UCC) concluded the conference with a paper demonstrating the usefulness of the bootstrap method in estimating standard errors of regression coefficients.

This year's conference was organized by Dr P. Boland, with the assistance of Dr F. Murtagh and Dr D. Williams. It is anticipated that the tradition of these successful conferences will be continued in 1985 under the organization of Cork statisticians.

*P.J. Boland*

## MATRIX THEORY CONFERENCE

A conference on Matrix Theory and its Applications was held in University College Dublin on March 22-24. The conference was organized by F.J. Gaines and T.J. Laffey, and was sponsored by the Department of Mathematics, U.C.D., the Irish Mathematical Society and the Symposium Fund of the Royal Irish Academy. It was well-attended, with about 35 participants. The lectures and discussion covered the many aspects of matrix theory - algebraic, analytic, combinatorial and computational - as well as its applications in applied mathematics. There was also a stimulating problem session, with several challenging questions.

The lecturers in order of appearance were as follows:

Professor G.N. De Oliveira (Coimbra) *Matrices over finite fields*

Professor T.T. West (T.C.D.) *Left/Right Symmetry in semi-simple algebras*

Professor T.J. Laffey (U.C.D.) *Factorization of matrices as products of Skew-symmetrics*

Professor R. Grone (Auburn, Alabama) *Computation of an immanant*

Dr R. Timoney (T.C.D.) *Reinhardt decompositions of operator matrix spaces*

Dr D.W. Lewis (U.C.D.) *Hermitian forms and von Neumann regular matrices*

Dr E.P. O'Reilly (N.I.H.E. Dublin) *The recursion method - a matrix technique in solid state physics*

Dr N.B. Blackhouse (Liverpool) *Grassman matrices*

Professor F. Holland (U.C.C.) *Counterparts of Hankel and Toeplitz operators on $C^n$*

Dr R. Gow (U.C.D.) *Some properties of unitary matrices*

Dr D. O'Connor (U.C.D.) *Sparse matrices*

Professor H. Wimmer (Wurzburg) *The algebraic Riccati equation*

T. Laffey



## REPORT OF THE NASECODE III CONFERENCE

(Communicated by J.J.H. Miller of the Numerical Analysis Group, Dublin)

The third international conference on the Numerical Analysis of Semiconductor Devices and Integrated Circuits, NASECODE III, was held in Galway, Ireland, from June 15th to 17th, 1983, under the auspices of the Numerical Analysis Group. It was attended by over 120 delegates from 18 countries. The aim of this series of conferences is the fostering of a fruitful exchange of ideas between electronic engineers and numerical analysts, who are using existing and developing new computer codes for semiconductor process, device and integrated circuit modelling.

As on previous occasions the industrial sector was strongly represented and it is our policy to ensure that the topics discussed at these conferences are relevant to the needs of industry. This ensures that the scientific and technical material presented at the conference is not only intellectually challenging, but also of great practical importance.

The application of numerical methods to semiconductor device modelling began about 17 years ago, and since then it has developed and broadened in scope very rapidly. To date relatively few professional numerical analysts have worked in this area, and consequently it is still a fertile source of stimulating unsolved problems of widely varying degrees of difficulty.

The models of technological importance are mainly in two space dimensions and they may also be time dependent. Typically, two or three nonlinear differential equations have to be solved on complicated domains with a variety of boundary conditions. Computational experience indicates that the systems are often very stiff.

For the numerical analyst there is a wealth of problems. Frequently, underflow and overflow occur and special tricks have to be used to allow the computation to proceed. Convergence of the iterative method for solving the discrete nonlinear system is usually a problem. The very fine meshes generally used in certain parts of the domain give rise to large discrete systems, and consequently the systems to be solved after linearisation are large. Many standard linear equation solvers, both direct and iterative, are impractical or simply fail for these problems. The development of practical and efficient techniques for solving extensions of these problems to three space dimensions and to the non-stationary case are also needed.

For a representative collection of papers on the subject the reader may consult the five publications [1], [2], [4], [9] and [10] associated with the NASECODE conferences. The first two monographs on the subject are Kurata [3] and Mock [5]. The main journals covering engineering aspects are [6] and [7], while the more computational and mathematical aspects are discussed in journal [8]. The fourth conference in the series, NASECODE IV, will be held in Dublin, Ireland, from June 9th to 21st, 1985.

Bibliography

[1] B.T. Browne and J.J.H. Miller (eds), Numerical Analysis of Semiconductor Devices, Proceedings of the NASECODE I Conference, Boole Press, Dublin (1979).

[2] B.T. Browne and J.J.H. Miller (eds), Numerical Analysis of Semiconductor Devices and Integrated Circuits, Proceedings of the NASECODE II Conference, Boole Press, Dublin (1981).

[3] M. Kurata, Numerical Analysis for Semiconductor Devices, Lexington Books, Massachusetts (1981).

[4] J.J.H. Miller (ed.), An Introduction to the Numerical Analysis of Semiconductor Devices and Integrated Circuits,



Lecture Notes of a Short Course held in association with the NASECODE II Conference, Boole Press, Dublin (1981).

[5] M.S. Mock, Mathematical Analysis of Semiconductor Devices, Boole Press, Dublin (1981).

[6] I.E.E.E. Transactions on Electron Devices, The Institute of Electrical and Electronics Engineers, New York.

[7] Solid-State Electronics - An International Journal, Pergamon Press, Oxford.

[8] COMPEL - The International Journal for Computation and Mathematics in Electrical and Electronic Engineering, Boole Press, Dublin.

[9] J.J.H. Miller (ed.), NASECODE III, Proceedings of the Third International Conference on the Numerical Analysis of Semiconductor Devices and Integrated Circuits, Boole Press, Dublin (1983).

[10] J.J.H. Miller (ed.), Finite Element Programming with Special Emphasis on Semiconductor Device and Process Modelling, Lecture Notes of a Short Course held in association with the NASECODE III Conference, Boole Press, Dublin (1983).

IRISH MECHANICS GROUP

(Conference of the Irish Mechanics Group held at the Dublin Institute for Advanced Studies on 17th April, 1984)

The opening lecture was given by Dr R.K. Li of Trinity College, Dublin, who spoke on "Scalar polynomial linear flow potentials". He was followed by Dr D.W. Reynolds of N.I.H.E. whose topic was "The buckling of viscoelastic rods" and a lecture by Prof. J.N. Flavin of U.C.G. on "Some asymptotic bounds for end-bonded elastic cylinders" brought the first session to a close.

The second session consisted of three lectures. The first one on "Slow perturbations of fast plane shear flow of a simple fluid" was by Dr J.N. Dunwoody of Queen's University, Belfast, and the second on "Inhomogeneous plane waves" by Prof. M.A. Hayes of U.C.D. Prof. P.M. Quinlan of U.C.C. deputised for Prof. M.M. Carroll of the University of California, who was unable to attend. Prof. Quinlan spoke on "The complex displacement method in elasticity".

The conference was attended by Prof. J. Ericksen of the University of Minnesota who was recently the recipient of an honorary degree from N.U.I.

*G. Kelly*

## GROUPS IN GALWAY 11-12 MAY 1984

A conference on Group Theory sponsored by the Irish Mathematical Society, Royal Irish Academy and University College, Galway, was again held at University College Galway on Friday-Saturday 11-12 May 1984. The main speakers were David Lewis (U.C.D.), Charles Leedham-Greene (Q.M.C. London), Pat Fitzpatrick (U.C.C.), Marty Isaacs (Wisconsin), Ted Hurley (U.C.G.) with further contributions from Rex Dark (U.C.G.), Mark Cartwright (Christ Church, Oxford) and Martin Newell (U.C.G.). A very successful addition to the Conference this year was a Problem Session in which many of the participants contributed a number of unsolved problems for discussion.

David Lewis spoke on the Merkuryev-Suslin Theorem (see article in this issue). This Theorem, concerning the algebraic K-theory and the Brauer group of a field, has only recently (1982) appeared, but has already answered many hard problems in simple algebras, quadratic forms and in algebraic geometry. It is destined to become a classic which people in many areas will find useful. David very eloquently set the scene and led us through an outline of this famous result.

Charles Leedham-Greene spoke on "Space groups and p-groups"



and reported on work by S. McKay, W. Plesken, M.F. Newman and himself. The idea is to classify p-groups according to co-class (if $|G| = p^n$ and G has nilpotency class c then co-class G = n-c). There are 5 co-class conjectures and one of these involving a tremendous amount of hard mathematics has recently been settled by Charles and others.

Pat Fitzpatrick gave an excellent survey of problems, questions and some answers concerning boundedness of conjugacy classes of a finite group in his talk "Some questions on conjugacy". If $m_i = |C(g_i)|$ and $|G| = m_1 \geq m_2 \geq \ldots \geq m_k$, then the idea is to look at k and the $m_i$ and determine properties of G from these (e.g. $m_1 > m_2 > \ldots > m_k$ and G supersoluble $\Longrightarrow$ $G = S_3$).

Marty Isaacs spoke on "Characters of soluble groups". If $\Pi$ is a set of primes, $G^*$ = set of $\Pi$-elements in G, the idea is to find a good basis for the vector space of "class functions" of $G^*$. His results can be applied to $\Pi$-separable groups. This is a unique approach to this vast area and is certain to lead to new developments.

It is a pleasure to be able to state that the reporter understood every little detail of Ted Hurley's talk "What can you do with a set of variables?"! This surveyed the connections between various objects, groups, Lie Algebras, Polynomial Rings, Power Series Rings, group algebras, varieties and some of the associated problems, e.g. Burnside's, Dimension Subgroup problem, isomorphic group rings problem. Modesty forbids further comment!

Further contributions included Rex Dark "Isotropic tensors and symmetric group algebras" (see I.M.S. Newsletter, December 1983, No. 9); Mark Cartwright "Bounded conjugacy conditions"; and Martin Newell "2-generator groups of exponent $\leq p^3$".

We would like to thank all our speakers, contributors and participants for their continued support and we hope to continue with "If it's May, it must be Galway (Groups)".

We are also happy to report that the famous group theory program CAYLEY has now been implemented at U.C.G. This is a tremendously powerful program (about 4500 blocks - $\frac{3}{4}$ million lines of FORTRAN) which has taken over 10 years to develop. It can deal with computations in, e.g. finite presented groups, permutation groups, matrix groups, low index subgroups, character tables and has over 200 algorithms. It is used in some universities for undergraduate teaching of group theory - it has its own mathematical language and no knowledge of programming is required. We also hope to implement MATRIX soon on a trial basis. This is an undergraduate teaching aid developed by John Cannon and a group at Sydney (who are also responsible for CAYLEY). It is best described as a *laboratory tool* (and so is not a "package" as such) for mathematics and I understand that this particular program will include among others, Gaussian Elimination, eigenvalues-vectors, linear (in)dependence, simplex algorithm. Others being developed are NEWTON (calculus!), KOENIC (graph theory).

*Ted Hurley*



# CONFERENCE ANNOUNCEMENTS

## PROTEXT I

The First International Conference, Exhibition and Workshop on Text Processing Systems

Gresham Hotel, Dublin, Ireland
22 - 26 October, 1984

*Organised by Professor John Miller, Trinity College, University of Dublin*

### Aims and Scope

These events aim to bring together a cross-section of people from business, industry and academia who share an interest in computer-aided text processing systems. Particular emphasis will be placed on the following areas:

* computer-aided generation of generalised copy (e.g. graphics, mathematical, non-English language)
* computer generated book-quality masters for print production
* interactive editing systems
* computer-aided typography
* human factors (e.g. the handicapped, the unions).

Both software and hardware aspects are included.

### Conference (24 - 26 October 1984)

This will consider future developments and current research in both the hardware and software areas. Keynote speakers at the conference include:

Brian Kernighan (Bell Laboratories)
Pierre MacKay (University of Washington)
Brian Reid (Stanford University)

Vincent Quint (University of Grenoble and INRIA)
A.N. Other (Hewlett-Packard)

Several formal discussions may be held on controversial topics of current interest by protagonists of international repute (e.g. Whither typesetting?).

Workshop (22- 23 October 1984)

This will be concerned with state-of-the-art computer-aided text processing systems. Live demonstrations of actual text processing systems (including SCRIBE, TEX, TROFF and EDIMATH) will be presented and tutorials given on their use.

Exhibition (22 - 26 October 1984)

Companies marketing software and hardware related to computer-aided text processing systems will exhibit their products during the Workshop and Conference. It is expected that several new products will be announced at this exhibition.

Registration Fees

These cover entry to all technical sessions, a copy of the Workshop Lecture Notes/Conference Proceedings (as applicable), morning coffee, lunch and afternoon tea. There is no charge for associates of delegates, and they are welcome at all events in the Social Programme. They may not, however, attend any of the technical sessions or appear as joint authors on papers.

| | Early US$ | Late US$ |
|---|---|---|
| Workshop only | 250 | 295 |
| Conference only | 250 | 295 |
| Workshop and Conference | 350 | 395 |

The early rate applies to all fees received by 1st June 1984.



All full-time *bona fide* research students with a letter of introduction from their supervisor may claim a US$100 discount.

All correspondence should be addressed to:

PROTEXT I Organising Committee/Boole Press Ltd,
P.O. Box 5, 51 Sandycove Road,
Dun Laoghaire, Co. Dublin,
Ireland.

Announcing NASECODE IV

The Fourth International Conference
on the Numerical Analysis of Semiconductor Devices
and Integrated Circuits

19th to 21st June, 1985, in Dublin, Ireland
under the auspices of
the Numerical Analysis Group
and co-sponsored by the

Commission of the European Communities
Electron Devices Society of the IEEE
Institute for Numerical Computation and Analysis
*Technical Group on Semiconductor and Semiconductor Devices of the IECE
Irish Mathematical Society

* applied for

Contributed papers are solicited from engineers, physicists and mathematicians on any topic relevant to the numerical analysis, modelling and optimisation of electronic, opto-electronic and quantum electronic semiconductor devices and integrated circuits.

A special feature of the conference will be a number of public debates led by distinguished personalities holding different views on key technical issues.

## Contributed Papers

THE DEADLINE FOR THE RECEIPT OF ABSTRACTS AND PRELIMINARY VERSIONS OF 20-MINUTE CONTRIBUTED PAPERS IS 1ST FEBRUARY, 1985.

## Short Course

A Short Course of relevance to the Conference will be held in association with NASECODE IV on 17th and 18th June, 1985.

All publications associated with NASECODE I, NASECODE II and NASECODE III Conferences held in 1979, 1981 and 1983 respectively and the Lecture Notes of the NASECODE II and NASECODE III Short Courses are available from Boole Press Limited.

All correspondence concerning the Conference and/or Short Course should be addressed to:

NASECODE Organising Committee, c/o Boole Press Limited, P.O. Box 5, 51 Sandycove Road, Dun Laoghaire, Co. Dublin. Ireland.



## WORDS

TRILLION. On 7 June the *Standard*, reporting on 'America's ballooning budget deficit,' wrote that Federal government spending last year was 'running at $1.5 trillion a year... (A trillion has 12 noughts)'. Twelve? Surely a trillion is a million times a million: 18 noughts. Then I remembered how in 1974 Mr Callaghan, then Prime Minister, had given his blessing in a parliamentary answer to the American billion (nine noughts) against ours (12 noughts).

The struggle has been going on for some time. According to the OED, two Frenchmen of the late 1400s and early 1500s, N. Chuquet and Etienne de la Roche, explained billion, trillion etc as 'successive powers of a million [i.e. six noughts for each jump], the trillion being the third power of a million ... as always used in England.' Then, in the mid-1600s, the 'erroneous custom' was established in France of 'calling a thousand millions a billion and a million millions a trillion, an entire perversion of the nomenclature of Chuquet and de la Roche, an error unfortunately followed by some in the US.'

Unfortunately or not, the Americans seem to be winning. Trillion with 12 noughts, says the forthcoming Vol. IV of A Supplement to the OED, 'is increasingly common in British usage.' (Incidentally, a centillion, a million to the power of 100, has - English style - 600 noughts, which would fill at least 12 of these lines.)

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From *The Observer*, Sunday 17 June, 1984. (Compare with "Word Conservation", *I.M.S. Newsletter*, No. 7 (March 1983), page 88.

# THE IRISH MATHEMATICAL SOCIETY

## Instructions to Authors

The Irish Mathematical Society seeks articles of mathematical interest for inclusion in the *Newsletter*. All parts of mathematics are welcome, pure and applied, old and new.

In order to facilitate the editorial staff in the compilation of the *Newsletter*, authors are requested to comply with the following instructions when preparing their manuscripts.

1. Manuscripts should be typed on A4 paper and double-spaced.
2. Pages of the manuscript should be numbered.
3. Commencement of paragraphs should be clearly indicated, preferably by indenting the first line.
4. Words or phrases to be printed in capitals should be doubly underlined, e.g.

   Print <u>these words</u> in capitals → Print THESE WORDS in capitals
5. Words or phrases to be italicized should be singly underlined, e.g.

   Print <u>these words</u> in italics → Print *these words* in italics.
6. Words or phrases to be scripted should be indicated by a wavy underline, e.g.

   Print these words in script → Print *these words* in script.
7. Diagrams should be prepared on separate sheets of paper (A4 size) in black ink. Two copies of all diagrams should be submitted: the original without lettering, and a copy with lettering.
8. Authors should send two copies of their manuscript and keep a third copy as protection against possible loss.

If the above instructions are not adhered to, correct reproduction of a manuscript cannot be guaranteed.

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