such that a_1, b_1 and a_2, b_2 are perspective pairs of elements of $\mathcal{L}(R)$.

The key idea of the proof here is that if A, B are principal right ideals of a bisimple ring R such that $A \cap B = 0$ then A and B are perspective (as before we may assume A = aR and B = bR where Ra = Rb; then c = (a+b)R is a common complement of A and B).

A stronger result is also true: any complemented modular lattice satisfying the condition (*) is easily seen to possess a homogeneous 4-frame (or else be the lattice {0,1}) and so, by von Neumann's result, can be co-ordinatized by a (necessarily bisimple) regular ring.

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TEN COUNTEREXAMPLES IN GROUP THEORY

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Introduction

Many major theorems in the theory of finite groups have been proved by the minimum counterexample technique, which works as follows. We assume that the theorem is false and let G be a counterexample of smallest possible order. The assumption that G exists is then used to force a contradiction and the theorem in question is thereby established. In practice, the contradiction frequently arises from the existence of a counterexample of order less than that of the presumed minimum counterexample (m.c.e.). This technique of course is merely a disguised form of induction or the method of infinite descent used in number theory.

However, even when a conjecture about finite groups turns out to be false, it is often of interest to discover an m.c.e., or "least criminal" as it is often called. Note that an m.c.e. need not be unique. Searching for an m.c.e. is a very good method of becoming familiar with the groups of small order and perhaps the size of an m.c.e. is an indication of how plausible the conjecture was in the first place!

In this article we discuss ten "not implausible" conjectures about finite groups and produce an m.c.e. in each case. We outline the arguments used in establishing that a given group is an m.c.e.

The material in this paper is based on the author's M.A. thesis "Minimum Counterexamples in Group Theory", University College, Cork, 1982, prepared under the supervision of Dr. D. MacHale. I wish to thank Dr. MacHale for suggesting this problem and the Mathematics Department of U.C.C. for their co-operation and facilities.

Preliminaries

In what follows G and H will always denote finite groups and p and q will denote prime numbers. We make use of the following facts from elementary group theory.

1. If $H \triangleleft G$ and $|H| = 2$ then $H \leq Z(G)$.
2. (a) If $|G| = pq$ and $q \not\equiv 1 \pmod p$, then G has a normal Sylow p-subgroup.
   (b) if $|G| = p^2q$ then G has a normal Sylow subgroup, ([1], page 97)
3. For any finite group G, Inn $G \simeq G/Z(G)$.
4. If $N \trianglelefteq G$, then G/N is abelian iff $N \geq G'$. ([1],page 59)
5. If $H, K \leq G$ so that H char $K \trianglelefteq G$, then $H \trianglelefteq G$. ([2], page 73)
6. Let $H \leq G$, $K \trianglelefteq G$, and $\Phi(G)$ denote the Frattini subgroup of G, then (a) $K \leq \Phi(G)$ iff there does not exist $H < G$ so that $HK = G$.
   (b) $K \leq \Phi(H) \Longrightarrow K \leq \Phi(G)$. ([1], page 269)
7. If $H < G$ then (a) G/H_G can be embedded in $S_{|G:H|}$
   (b) $N_G(H)/C_G(H)$ can be embedded in Aut(H). ([1], page 74, 84)
8. If G is a transitive permutation group on a set Ω, then the stabilizers $G_\alpha (\alpha \in \Omega)$ are all conjugate to one another and if $|\Omega| = n$, then $|G:G_\alpha| = n$. ([4], page 15)
9. There exist exactly two non-abelian simple groups of order less than 360 namely A_5 of order 60 and PSL(2:7) of order 168.
10. If G is a non abelian group then G is insoluble iff some subgroup H of G (possibly H = G) contains a non abelian simple factor group (possibly the trivial factor group H/1).
11. The following is a list of all the groups of order less than 16.

    C_n, $1 \leq n \leq 15$; $C_p \times C_p$, $p = 2$ or 3; D_n, $3 \leq n \leq 7$; $C_2 \times C_4$; $C_2 \times C_2 \times C_2$; Q; $C_2 \times C_6$; A_4; Q_6.

Conjectures

Conjecture 1: $G' = G \Longrightarrow Z(G) = 1$.

The motivation for this conjecture is of course the fact that if G is abelian then G' is trivial and also G' is in some way a measure of the commutativity of G. Now we know that if G is soluble then G' is properly contained in G and so a counter-example can only be found within the class of insoluble groups. $G \simeq A_5$ satisfies the conjecture. Hence all groups of order less than 120 are ruled out as counter-examples (9, 10). We show $G \simeq SL(2:5)$ of order 120 is an m.c.e. The map det : $GL(2:5) \rightarrow \{1,2,3,4\}$ is an onto homomorphism whose kernel is SL(2:5). Hence $|G| = 480/4 = 120$. $G/Z(G)$ is simple non abelian ([1], page 66) $\Rightarrow$ Z(G) is maximal normal in G (Isomorphism Theorem) $\Rightarrow$ Z(G) is the only proper normal subgroup in G. [For suppose there exists $H \neq Z(G)$ such that $H \triangleleft G$. Then $HZ(G) \trianglelefteq G$ and HZ(G) contains Z(G) properly. Hence $HZ(G) = G$. Now

$$Z(G) = \left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}4 & 0\\0 & 4\end{pmatrix}\right\} < \left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}2 & 0\\0 & 3\end{pmatrix}, \begin{pmatrix}4 & 0\\0 & 4\end{pmatrix}, \begin{pmatrix}3 & 0\\0 & 2\end{pmatrix}\right\} = K,$$

and $K \simeq C_4$. Hence by (6) $Z(G) \subsetneq \Phi(G)$ and $H = G$]. G non abelian $\Rightarrow G' \neq 1$. $G/Z(G)$ non abelian $\Rightarrow G' \nsubseteq Z(G)$ (4). Since $G' \trianglelefteq G$ we must have $G' = G$.

Conjecture 2: Conformal groups are isomorphic, i.e. if G and H have exactly the same number of elements of each order then $G \simeq H$.

This conjecture is true for abelian groups but unfortunately non abelian groups do not fit this pattern. In fact within the groups of order 16 we can find 3 non isomorphic groups all of which are conformal. We also see that an abelian and a non abelian group may be conformal. For a counter-example it is clear that we need two non isomorphic, non cyclic groups of the same order and this condition rules out all the groups of order less than 16 except the groups of order 8 and 12, (11), and these groups are eliminated by counting elements of each order. $G \cong C_4 \times C_4$ and $H \cong Q \times C_2$ supply us with an m.c.e. as both groups each contain three elements of order 2 and twelve elements of order 4. G is abelian. H is non abelian. The group $K = \langle x,y : x^4 = 1 = y^4,\ xy = yx^{-1} \rangle$ is also conformal with G and H. We also note that this m.c.e. is not unique as among the groups of order 16 there exist two other groups which are conformal, namely:

$$\langle a,b,c : a^2 = 1 = b^2 = c^2, \quad abc = bca = cab \rangle$$

and

$$\langle a,b,c : a^4 = 1 = b^2 = c^2,\ ab = ba,\ ac = ca\,b,\ bc=cb \rangle$$

<u>Conjecture 3</u>: Given a nilpotent group G, there exists a finite group H such that $G \cong \Phi(H)$.

For any group G, $\Phi(G)$ is nilpotent so this conjecture poses the interesting question "is every nilpotent group the Frattini subgroup of some group?" Since $\Phi(C_{p^2}) \cong C_p$, $\Phi(C_8) \cong C_4$, $\Phi(C_4 \times C_4) \cong V_4$, $\Phi(C_9 \times C_4) \cong C_6$ and S_3 is not nilpotent all groups of order less than 8 are ruled out as counter-examples (11). $G \cong Q$ is an m.c.e. For suppose there exists H such that $\Phi(H) \cong Q$ and let $C = C_H(\Phi(H))$. Then $C \trianglelefteq H$. Now M/C is max. in H/C $\Longrightarrow$ M max. in H(Isomorphism Theorem) $\Longrightarrow$ $M \supseteq \Phi(H)$. Hence $C\Phi(H)/C \subseteq \Phi(H/C)$ ......(*). Also $C\Phi(H)/C \cong \Phi(H)/(C \cap \Phi(H))$ (Isomorphism Theorem). Now $Z(\Phi(H))$ char $\Phi(H) \trianglelefteq H$ ($Z(G)$ and $\Phi(G)$ are char in G) $\Longrightarrow$ $Z(\Phi(H)) \trianglelefteq H$ (5). $|Z(\Phi(H)| = 2$ ($\Phi(H) \cong Q$). So $Z(\Phi(H) \leq Z(H)$ (1) and clearly $C \cap \Phi(H) = Z(\Phi(H))$ Hence $C\Phi(H)/C \cong V_4$. Hence $\Phi(H/C)$ contains a subgroup isomorphic to V_4 (*). On the other hand $H = N_H(\Phi(H))$ and $C = C_H(\Phi(H))$ so H/C can be embedded in $Aut(Q) \cong S_4$ (7). But all subgroups of S_4 have Frattini subgroup of order 1 or 2 which gives a contradiction.

<u>Conjecture 4</u>: If G is not simple then G has a normal Sylow subgroup.

A p-group cannot furnish us with a counter-example and groups of order pq and p^2q are ruled out by (2). Hence all groups of order less than 24 are ruled out as counter-examples. We show S_4 of order 24 is an m.c.e. S_4 prime $\cong A$ of order 12. Hence no Sylow 2-subgroup is normal (4). A_4 and hence S_4 contains eight elements of order 3 $\Longrightarrow$ a Sylow 3-subgroup is not unique and so not normal.

<u>Conjecture 5</u>: If G is not simple then G has a non-trivial endomorphism.

The motivation for this conjecture comes from the fact that if G is not simple, G certainly has a non-trivial homomorphic image as there is a 1:1 correspondence between the normal subgroups of G and the homomorphic images of G. Now we know that if G is soluble then G contains a normal subgroup of prime index, H say, and by Cauchy's Theorem, G/H can be embedded in G. Hence G has a non-trivial endomorphism with kernel H. So this rules out as counter-examples all groups of order less than 120 (9, 10). We show that $G \cong SL(2:5)$ of order 120 is an m.c.e. By conjecture 1, $Z(G)$ is the unique normal proper subgroup of G and $G/Z(G) \cong A_5$. Hence the only possibility for a non-trivial endomorphism α is $\ker \alpha = Z(G)$. But G contains no subgroup, H, isomorphic to A_5, otherwise $H < G$, contradicting the uniqueness of $Z(G)$. Hence $SL(2:5)$ has no non-trivial endomorphism.

<u>Conjecture 6</u>: If an automorphism α of G sends every conjugacy class of G onto itself then α must be inner.

Let A denote the set of all automorphisms of G which send each conjugacy of G onto itself. Then $A < Aut(G)$ by checking closure. Clearly $Inn(G) \leq A$. Hence in eliminating groups as counter-examples we merely show $|A| < 2|Inn(G)| = 2|G/Z(G)|$, (3). All groups of order less than 32 can be eliminated as counter-examples. Much of the detail is just routine so we merely outline the arguments one may use. We note here that if $x \in G$, then $|G:G'| \le |C_G(x)| \ge 2|Z(G)|$ and $K(x)$, the conjugacy class of x, has $|G:C_G(x)|$ elements. If G is a non-abelian 2-generator group of order less than 32 then x and y can always be chosen such that $G = \langle x,y \rangle$ and $|K(x)||K(y)| < 2|G/Z(G)|$ and so $|A| < 2|G/(Z(G)|$. If G is a 3-generator group of order less than 32, then $z,x,y \in G$ can nearly always be chosen such that $G = \langle z,x,y \rangle$ where $z \in Z(G)$ and $|K(x)|\,|K(y)| < 2|G/Z(G)|$. There is only one exception, namely if G is a non abelian 3-generator group of order 18 with trivial centre. Clearly in this case we have $G = \langle a,b,x \rangle$ where $|a| = |b| = 3$ and $|x| = 2$. $\langle a,x \rangle$ is a proper subgroup of G and clearly $\langle a,x \rangle \simeq S_3$. Hence $a = g^{-1}a^2g$ for some $g \in \langle a,x \rangle$, We conclude that if $k_1, k_2 \in \langle a,b \rangle$ and $k_1 = g^{-1}k_2g$ $(g \in G)$, then $k_1k_2 = 1$ .....(*) (Otherwise G is a 2-generator group). Now $G = \langle a,b,x \rangle$ where $|K(a)| = |K(b)| = 2$ and $|K(x)| = 9$. So $|A| \le 36$. But α defined by $a \mapsto a^2$, $b \mapsto b$, $x \mapsto x$ is one of the 36 possible maps in A and $\alpha(ab) = a^2b$ but $ab\,a^2b = b^2 = 1$ contradicting (*). Hence $\alpha \notin A$ and $|A| < 36 = 2|G/Z(G)|$.

In analysing the groups of order 24, it may be useful to divide them into the following four categories: (1) both Sylow subgroups normal; (2) only the Sylow 3-subgroup normal; (3) only the Sylow 2-subgroup normal, and (4) no Sylow subgroup is normal. In (1) $G \simeq D_4 \times C_3$ or $G \simeq Q \times C_3$ and G is a 2-generator group. In (2) $G \simeq H_\phi \times K$, where $|H| = 8$ and $|K| = 3$. G may be a 3-generator group but one generating element can always be chosen from the centre. In (3) $G \simeq H_\phi \times K$, where $|H| = 3$ and $|K| = 8$ and G is a 2-generator group. In (4) $G \simeq S$ and $Aut(S_4) = Inn(S_4)$. An m.c.e. of order 32 can be found in [4], page 24.

Conjecture 7: If every maximal subgroup of G has prime power index, then G is soluble.

The converse to this conjecture is true and in fact the conjecture itself is "very nearly true" as $PSL(2:7)$ seems to be the only non abelian simple group with the property that

every maximal subgroup has prime power index. Now G insoluble of order less than $168 \Rightarrow G \simeq A_5$ or $|G| = 120$ (9, 10). Let P be a Sylow 5-subgroup of A_5. Then $N_{A_5}(P)$ is a maximal subgroup of A_5 of index 6. Hence by (9), (10) and the Isomorphism Theorem all insoluble groups of order 120 also have maximal subgroups of composite index. The subgroup of A_7 generated by $\{(1234567), (26), (34)\}$ is simple of order 168 (See [4], page 18). Let P be a Sylow 7-subgroup. $|G:N_G(P)| = 8$ is the number of Sylow 7-subgroups (Sylow Theorem). Hence the normalizers of the eight Sylow 7-subgroups form a single conjugacy class of 8 maximal subgroups each having index 8 in G. Also G is a transitive permutation group of degree 7, so $\{G_i : 1 \le i \le 7\}$ (where G_i is the stabilizer of i in G) is a conjugacy class of 7 subgroups in G each having index 7 in G (8). Now every proper subgroup of G of composite index is contained in one of the above maximal subgroups of prime power index. But G is insoluble as it is simple.

Conjecture 8: Every finite group G has a maximal subgroup of prime index.

Here we use the minimum counter-example technique. Let G be an m.c.e and N a maximal normal subgroup of G. Suppose $|N| > 1$. Then by hypothesis G/N contains a maximal subgroup of prime index, H/N say, $\Longrightarrow$ H has prime index in G. (Isomorphism Theorem). Hence $|N| = 1$ and G is simple. The two simple groups of order less than 360 (9) are eliminated by (8) and conjecture 7. $G \simeq A_6$ of order 360 is an m.c.e. for if there exists $H < G$, such that $|G:H|$ is prime then by (7) and the simplicity of G, $G \le S_5$ which is impossible.

Conjecture 9: Let $K \trianglelefteq G$. Then if both K and G/K are supersoluble then G is supersoluble.

This conjecture is true for solubility. Groups of order pq are ruled out by (2) and finite p-groups are nilpotent and hence supersoluble. This eliminates all groups of order less than 12. In A_4 of order 12, $1 < V_4 < A_4$ is the only possible normal chain and V_4 is not cyclic. But V_4 and A_4/V_4 are supersoluble.

Conjecture 10: If H is a subnormal subgroup of G, then the sequence from H of subgroups of G formed by taking successive normalizers in G reaches G.

If H is subnormal in G, then by definition there exists a normal chain from H to G. However the conjecture is not true. One of the characteristics of nilpotent groups is that every proper subgroup is a proper subgroup of its normalizer and so all finite p-groups are eliminated as counter-examples. In groups of order pq or p^2q a non-trivial subnormal subgroup is either normal or its normalizer is normal. This rules out all groups of order less than 24. $G \simeq S_4$ of order 24 is an m.c.e. since in S_4.

$$\{1\} \triangleleft \{(1), (12)(34)\} \triangleleft \{(1), (12)(34), (13)(24), (14)(23)\} \triangleleft A_4 \triangleleft S_4$$

is a composition series. So H = {(1), (12)(34)} is subnormal in G but $N_G(H)$ is a Sylow 2-subgroup of G, has index 3 in G and is not normal in G. Hence $N_G(H)$ is self normalizing.

We conclude with three "partially solved" problems and the author invites comments or solutions.

(1) The kernel, K, of a Frobenius group G is abelian.

If G is an m.c.e. then $120 \leq |G| \leq 256$.

(2) G non-abelian $\Rightarrow$ Aut(G) non-abelian.

If G is an m.c.e. then G is a p-group and $|G| \geq 32$.

(3) Every group G has a p-complement where $p \mid |G|$.

If G is an m.c.e., $120 \leq |G| \leq 360$.

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