ROBERT CHARLES GEARY

MATHEMATICAL STATISTICIAN, 1896-1983

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Roy Geary, Ireland's greatest statistician, died in Dublin on 8 February 1983, after a long life devoted to mathematical statistics. He was born on 11 April, 1896 and was educated at University College, Dublin, 1913-18, obtaining a B.Sc. and an M.Sc. both with first class honours in 1916 and 1917, respectively. In 1918 he was awarded a Travelling Studentship in Mathematics and attended the Sorbonne in Paris, 1919-21. He was appointed to a lectureship in Mathematics at University College, Southampton in 1922 but returned to Dublin in 1923 as a statistician in the Statistics Branch of the Department of Industry and Commerce and remained there until 1949, apart from a brief period as Senior Research Fellow in the Department of Applied Economics in Cambridge, 1946-7. He was director of the Central Statistics Office in Dublin from 1949 to 1957, when he moved to New York for three years to head the National Accounts Branch of the United Nations Statistical Office. He returned to Dublin in 1960 to the Economic Research Institute (later known as the Economic and Social Research Institute) where he was to spend the rest of his life, as Director until 1966, as consultant thereafter, Among the many honours awarded him were three honorary doctorates from NUI, QUB and TCD and honorary fellowships of the Royal Statistical Society and American Statistical Association. He was President of the International Statistical Institute and a Council member of the International Association for Research in Income and Wealth. He was elected Fellow of the Econometric Society in 1951 and served as Council member from 1962-64.

This appreciation relies heavily on other appreciations by the author published in *The Economic and Social Review*, April 1976 and April 1983, and a forthcoming obituary in *Econometrica*. The first two of these contain lists of Geary's works from which present citations can be identified and all three contain references to other work to aid in the assessment of Geary's achievements. A more rounded assessment of his personality and career can be gained by perusal of all the appreciations and obituaries.

Roy's first paper was published in 1925, a year within what Egon Pearson described as" a period of transition" and as "one of the two great formative periods in the history of mathematical statistics", 1915-1930. The discoveries of Karl Pearson, Edgeworth, Galton and Weldon and the early work of Gosset were to be expanded and unified by the work of Fisher who by 1925 had published the first edition of his enormously influential *Statistical Methods for Research Workers*. His work on likelihood, regression and correlation, his establishing of the distribution of Gosset's t-distribution and of the variance ratio had all just been published. The controversies with E.S. Pearson, Neyman and Wald on fiducial probability, interval estimation and hypothesis testing were still to come and the celebrated collaboration of Neyman and Pearson of 1926 to 1934 had not yet started. A new era in probability theory was also beginning. Liapunov had provided the first satisfactory proof of the central limit theorem under certain conditions in 1901 using the tool of characteristic functions but his work remained fairly unknown for some time and the fundamental refinements of Lindeberg in 1922, Bernstein in 1927 and Feller in 1935 were, in 1925, new or still to come. Indeed it was only by 1925 that Levy published what Cramer has described as "the first systematic treatise of random variables, their probability distributions and their characteristic functions" and the work of Khintchine and Kolmogorov was still in its infancy.

Accordingly Roy was beginning his work at a time which must have seemed full of excitement. Throughout the most productive part of his career, 1930-1956, he was greatly influenced by Fisher although he played no part in the controversies referred to above. Many years later (1976), he confessed in a letter to the author that many of his papers originated in the development of something in Fisher and that it was his great good fortune that his research lifetime coincided with so much of Fisher's. "Everything is in Fisher. One only had to dig it out a bit".

His first paper is richly promising as befits the early work of a researcher who is subsequently to gain a high international reputation. The paper shows fine technical ability, a natural flair for stochastic problems and a painstaking care for detail, qualities which were to manifest themselves throughout his later work. This early paper also provides an excellent example of the power of good theory when applied to real problems; here the problem concerned Irish agricultural statistics which had been thrown into some confusion during political troubles of the time. The theoretical result is as follows. Let $m+u_i$, $i=1...N$ be the values of N elements in year one and $m'+u_i'$, $i=1..N$ be the values in year two, where m and m' are the means in the two years. It is desired to measure ratio of true means m/m' by taking a random sample of n elements. The ratio as estimated by

$$\frac{\sum_1^n (m'+u_i')}{\sum_1^n (m+u_i)}$$

is shown to be approximately normally distributed for large n and N with a mean of m'/m. Geary also computes the variance and applies the result successfully to agricultural data.

This interest in finding the density of a ratio continued and five years later the classic (1930b) paper appeared. Let x_1 and x_2 be two jointly distributed normal variables with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2 respectively, with correlation r. The difficult technical problem of finding the exact sampling distribution of $z = x_1/x_2$ was solved on the hypothesis that μ_2 was large relative to σ_2 so that the range of x_2 was effectively positive. On this assumption Geary proved that the ratio $(\mu_2 z-\mu_1)/\sqrt{(\sigma_2^2 z^2 -2r\sigma_1\sigma_2 z+\sigma_1^2)}$ was distributed N(0,1). The result in its original form is still quoted today. An expression for the density of z where x_1 and x_2 are independent

but not necessarily normal variates was discovered in 1937 by Cramer for the case where x_2 is non-negative with finite mean. Writing $\Phi_1(t)$ and $\Phi_2(t)$ for the characteristic functions of x_1 and x_2, the density of z, f(z), if it exists was found to be

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \Phi_1(t)\Phi_2'(-tz)dt,$$

provided the integral converges. Geary (1944b) generalised this result to the case of non-independent x_1 and x_2 with joint characteristic function $\Phi(t_1,t_2)$ and the same condition on x_2. The generalised result is

$$f(z) = \frac{1}{2\pi i}\int_{\infty}^{\infty} \delta\Phi(t_1,t_2)/\delta t_2 dt_1$$

where the partial derivative is evaluated at $t_2=-t_1 z$.

Between these two papers on ratio densities his 1933 paper appeared in which it was established, under normality, that

$$E(m_r/m_2^{\frac{1}{2}r})^p = Em_r^p/Em_2^{\frac{1}{2}rp}$$

where

$$m_r = \sum_1^n (x_i-\bar{x})^r/n.$$

The analysis, from a probabilistic point of view, of ratios constituted one of the three main inter-related themes with which Roy was concerned during his working life. The other two comprise the estimation of relationships between variables whose measured values are subject to error and normality tests and robustness studies of inference formally based on normality. This choice of problems illustrates his deep desire to direct his keen mathematical skill and imaginative flair towards problems of great practical importance and most of his contributions to these three fields are of lasting significance. A possible exception to this can be found in his work in the early 1940's on the estimation of relationships. This work, heavily based on the theory of cumulants, has theoretical importance but has not proved of lasting practical value as the estimators suggested tend to have high variance, in particular if the underlying variables have distributions which are nearly normal. His 1949a paper, however, remains fundamental and is one of his most cited contributions. It outlines an instrumental variable (IV) approach to estimation and allows the citing of Geary, with Reiersöl, as a pioneer of the IV method. The paper begins by considering the problem of estimating $\beta = -\beta_1/\beta_2$ in the model $\beta_1X_1 + \beta_2X_2 = 0$ where the X's are observed with error, the observables x_i being equal to X_i+u_i. Let z be the IV measured without error and suppose X_1, X_2, z have zero means, are joint normal and temporally uncorrelated. Geary writes the IV estimator of β as

$$b = (\Sigma x_2 z)/(\Sigma x_1 z) \quad (\equiv \text{num/den, say})$$

and shows, using Geary (1944b) that its density can be written

$$\phi(b)db \quad = \quad \frac{\{(n-2)/2\}!\,(1+y^2)^{-\frac{1}{2}n}}{\{(n-3)/2\}!\sqrt{\pi}}\, dy$$

where

$$y = \left\{\frac{Ez^2(b^2Ex_1^2 - 2bEx_1x_2 + Ex_2^2)}{(bEx_1z - Ex_2z)^2} - 1\right\}^{-\frac{1}{2}}$$

so that $y\sqrt{(n-1)}$ is distributed as t with n-1 degrees of freedom. By considering confidence intervals, it is shown that the precision of the method (asymptotically) is improved if z is chosen as highly correlated with X_1 and X_2 as possible and the idea of finding an optimal combination of instruments is treated.

The theory is then extended to the time series case where the true values of the observables are taken as non-stochastic with the errors normally distributed, independent of each other and of the other variables. The estimator b in this case is

the ratio of two independent normal variables so that Geary (1930b) applies by which $\{bE(\text{den})-E(\text{num})\}/\sqrt{\{b^2\text{var}(\text{den})+\text{var}(\text{num})\}}$ is a N(0,1) variate. Of course, the expression involves unknown error variances while the previous expression only involves expectations of functions of observables which can easily be consistently estimated. Although the first model would seem to be inapplicable to the time series "sequences of n" case it is found that the operative first theory can be used with confidence in such a case for moderately sized samples with error variances not too large.

Roy's interest in testing for normality and appreciation of the need for robustness studies developed early. By 1930 large sample approximations to the densities of the statistics $\sqrt{b_1}$ and b_2, the sample analogues of the $\sqrt{\beta_1}$ and β_2, the classical measures of skewness and kurtosis, had been derived. By 1935, several of the lower order moments for normal $\sqrt{b_1}$ and b_2, useful in approximating their distributions, had been derived by various writers notably Craig, Wishart and Fisher. Roy was to contribute to several aspects of this broad range of topics. In *Biometrika* 1935 he criticised tests of kurtosis based on b_2 due to the slowness with which b_2 approached normality and suggested tests based on the ratio of the mean deviation to the standard deviation. In *Biometrika* 1936 he found moments of his ratio under normality and provided a table ofcritical values. Other papers dealing with small sample properties of $\sqrt{b_1}$ and b_2 culminated in his outstanding 1947 *Biometrika* paper (1947c) in which he derived approximations to the first four normal moments and asymptotic distribution of a generalization of b_2 and showed that while b_2 was asymptotically the efficient test for kurtosis in the class considered against a wide field of alternatives, there appeared to be little superiority over his own ratio for moderate sample sizes. Similar analysis was carried out on skewness measures. Recent reconsideration of his test, both theoretical and Monte Carlo, has been interestingly favourable. Geary's work on robustness is mostly contained in his 1936 JRSS paper and the

1947 paper just described. In the latter he considers the effect of nonnormality on Fisher's z ($=\frac{1}{2}\log F$) and one and two sample t-tests. Regarding the one sample t-test, for example, he approximated the first six moments in terms of the cumulants of the parent population and approximated the density of the non-normal t with a Charlier differential series with the normal t as generating function using his results for the first few cumulants of non-normal t, assuming a Pearson system population, and collecting terms according to their orders of magnitude. The resultant formula is used to approximate the true left hand tail probabilities. The main trouble is found to be asymmetry, with positive skewness leading to left tail rejection of the null hypothesis too often. This contrasts with the variance ratio test, where kurtosis is the key problem.

Further, as follows from general formulae in Geary (1947c), the approximation to order n^{-1} of var z ($z=\frac{1}{2}\log F$) is

$$\left(\frac{\beta_2-1}{4}\right)\left(\frac{1}{n_1}+\frac{1}{n_2}\right)$$

for two independent samples drawn from the same population, an expression which generalizes Fisher's normal approximation for $\beta_2=3$, and suggests that positive kurtosis would lead to too frequent rejection of variance equality on uncritical use of normality assumptions. Much of this is now standard knowledge.

In the course of these investigations he presented a method for calculating the exact small sample frequency of normal $\sqrt{b_1}$ to any required degree of accuracy (1947a) and calculated, with Worlledge, the seventh moment of normal b_2 (1947). The method of the former paper was an elaboration of that which he used in 1935b and 1936b in finding the distribution of his ratio and consisted of establishing a relation in integral form between the frequency ordinate for n (the sample size) with that for n-1, with a certain actual frequency being shown to be close to its Gram-Charlier representation.

While these three themes account for his most systematic work, other important and attractive results appear throughout his work. For example, he showed in the 1936 paper on robustness that independence of mean and variance imply normality, not just the well-known reverse. In a 1942 JRSS paper, he showed, under regularity conditions, that maximum likelihood estimators minimise the generalized variance for large samples. His 1954 paper in *The Incorporated Statistician* introduced a contiguity ratio of spatial autocorrelation, a generalization of the von Neumann ratio, together with sampling theory which has been used by geographers, sociologists and others and has been included in a book of readings in Statistical Geography. In a 1944 *Biometrika* paper he compared Pitman's 'closeness' with efficiency showing a certain equivalence result in the bivariate normal case and providing an early analysis of "the racing car problem", the estimation of b in the uniform [0,b] distribution. He was also a pioneer in national income accounting, a topic in which he consistently retained interest and influence.

In his later work through the 1960's and 1970's he concentrated more on economic and social issues showing great concern for problems of population movement, poverty, inequality and inflation. He was never an avid reader of other people's work (except Fisher) preferring to work things out for himself. He had a deep love, almost reverence for the arts and mathematics which he thought of as the sublime art. In no sense did he conceive of the application of mathematics as its justification. In fact, he was often sceptical and at times deeply antagonistic towards the use of mathematics in social problems, especially if little mathematical manipulation was involved, which he conceived to be the essence of mathematics. The essence of statistics, he thought, lay in the efficient manipulation of measures so as to derive inference from them. In partial response to a question from the author and earlier, apparently from Fisher as to the essential distinction between mathematics and statistics, he emphasised that mathematics had no place in statistics unless clearly relevant to a statistical problem. He frequently emphasised that a problem generally involved much more than its relevant statistics and resented any charge of materialism on statisticians as much as he resented the use of mathematics for its own sake in the ostensible address of a statistical problem.

He was undoubtedly himself a powerful and energetic mathematician and a magnificently creative statistician, with an unusual emphasis on applicability throughout his work at all times. He had an amazing all-round talent and was a man whose contributions to statistics will not be forgotten.

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PUTTING COORDINATES ON LATTICES

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In this article I shall show how the problem of putting coordinates on certain types of lattices leads to the class of von Neumann regular rings, and discuss briefly the resulting connexion between ring and lattice properties. The article is based on a talk I gave at the Group Theory Conference in Galway on May 13th, 1983, and I would like to thank the organizers both for their invitation to speak and for tolerating the presence of a ring theorist.

Recall that a lattice is a partially ordered set in which any pair of elements a,b have a greatest lower bound $a \wedge b$ and a least upper bound $a \vee b$. We shall be considering complemented modular lattices in what follows. A lattice L is said to be ***complemented*** if it has a least element (denoted by 0) and a greatest element (denoted by 1) and if every element $a \in L$ has a complement $a' \in L$; that is, $a \wedge a' = 0$ and $a \vee a' = 1$. Such complements are not usually unique. We say that L is ***modular*** if whenever $a,b,c, \in L$ with $a \leq c$ then $(a \vee b) \wedge c = a \vee (b \wedge c)$.

Example 1: Let V be any vector space (possibly infinite dimensional) and let L be the set of all subspaces of V ordered by inclusion, so that $a \wedge b = a \cap b$ and $a \vee b = a + b$. Then L is a complemented modular lattice.

2. : Von Neumann, studying rings of operators on Hilbert spaces, came across rings whose sets of projections (that is, self-adjoint idempotent operators p, so that $p = p^* = p^2$) formed lattices if $p \leq q$ was taken to mean that $p = qp$ (so that q is a left and right identity for p). Although there was no simple algebraic formula for $p \wedge q$ and $p \vee q$ in this case, von Neumann was able to show that this lattice was complemented and modular.