"SHUFFLE THE CARDS, PLEASE"

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Since playing cards was invented, the ranks of card magicians and gamblers have produced quite a deal of mathematics. What follows has been culled from the writings of such people and writers as Martin Gardner, Philosophy graduate, amateur magician and writer of the "Mathematical Games" column in "Scientific American" from 1957 to recent times, and Chicago card magician, Edward Marlo. All this places what follows in the field of Recreational Mathematics so shuffle the cards, please and we'll begin.

Of all card shuffles, that known variously as the "faro", the "weave" or "riffle shuffle" is the most interesting. To execute a faro shuffle, divide the deck in half exactly if the deck contains an even number of cards and as nearly in half as possible with an odd number deck. Now with half a deck in each hand let cards drop alternately from the thumbs thus weaving the two halves together. With odd decks the first card to fall must come from the "half" with the extra card. With even decks if the first card to fall is from the half that was formerly the bottom of the deck the original bottom and top cards retain their respective positions. Magicians call this an out-shuffle because the top and bottom cards remain on the outside. If the first card to fall is from the original top half we get what magicians call an in-shuffle. For odd decks a faro is an out-shuffle if prior to shuffling the deck is cut below the centre card and an in-shuffle if it is cut above the centre card.

Since any shuffle is merely a permutation from the symmetric group S_n, if the deck contains n cards, then the same shuffle repeated a number of times equal to the order of the permutation will restore the deck to its initial order. If n is odd a deck of n cards given x repeated faro shuffles of the same type will return to its original order if $2^x \equiv 1 \pmod{n}$.

For example if we use a full deck with a joker making 53 cards then x = 52 that is 52 in-shuffles (or 52 out-shuffles) are required to restore a 53-card deck.

If the deck is even the number x of out-shuffles is given by

$$2^x \equiv 1 \pmod{(n+1)}.$$

For a normal pack of 52 cards, 52 in-shuffles or 8 out-shuffles restore the original order while 5 out-shuffles and 10 in-shuffles will restore a piquet pack (32 cards).

The reader might like to test the formulae using packets of cards. A perfect faro is difficult to do but this can be circumvented by doing "Reverse faros", i.e. undoing a faro shuffle by stripping out every second card. Obviously if x faro shuffles restore original order then x reverse faros will do the same thing. The easiest way to do the reverse faro is to deal the cards singly into two piles turning the cards face-up as you deal. In putting the piles together if the original top card remains on top it is called an out-sort, if not an in-sort. Do it with ordered packets of cards so that you can see what is happening as you do it.

Alex Elmsley, a British computer programmer and a skilled card magician, discovered a remarkable formula connected with the faro shuffle. Elmsley wished to find the most efficient combination of in-shuffles and out-shuffles - terms incidentally coined by Elmsley - to cause the top card of a deck to go to any desired position from the top. If you want the card for example to go to the 20th position subtract one to give 19 and write this in the binary form 10011. Now let I and O stand for in-shuffle and out-shuffle respectively. The required answer is then the following sequence of faros IOOII. This works regardless of the size of the deck and is done in the least possible time.

We may also use this formula doing reverse faros to bring a card at any chosen position to the top by following the sequence of binary digits backwards. For example to bring the 20th card to the top execute the following sequence of in-sorts and out-sorts IIOOI. A brief analysis shows that in effect what you are doing here is that during each shuffle you are placing the pile containing the card in question on top. Now there is a very old card trick called "Gergonnes Pile Problem" (after Joseph Diez Gergonne, the French Mathematician who in 1813 was first to analyse it experimentally) in which 27 cards are dealt into three piles, face up, and a spectator thinks of any card indicating which pile it is in. The cards are picked up and dealt out again in three piles and the spectator again indicates the pile containing his card. After the procedure is repeated a third time the spectator finds his card at a position in the deck previously specified by him. The trick depends on the order of picking up the piles. Suppose you wish to bring the chosen card to position 22. Find the ternary equivalent of 21, i.e. 210, and reverse the digits to give 012. Now on the first deal place the pile containing the chosen card on top indicated by 0, on the second deal in the center indicated by 1 and on the last deal on the bottom indicated by 2. The card will now be found at position 22. If you want to bring the chosen card to the top then the ternary equivalent of 1-1 = 0 is 000 which tells you to place the pile containing the chosen card on top each time.

Now we find that, if we use two piles instead of three and binary in place of ternary that the above process works for any number of cards although we may need to deal more than twice if the number of cards is greater than 4 depending on the initial position of the chosen card. This all suggests a method for generalising the faro shuffle or rather the inverse faro. (One would need three hands to weave three packets of cards together!) If we deal three piles and pick up in one of the 6! ways, do analogous formulae exist, for example, for finding the number of times such a shuffle must be executed to restore the deck to its original order? I know of no literature on this problem. However, I have found that if we

pick up the three piles so that the top card remains on top and the centre pile is replaced in the centre then the number x of such shuffles is given by

$$3^x \equiv 1 \pmod{(n-1)}, \text{ if } n \text{ is of the form } 3k, \text{ and}$$

$$3^x \equiv 1 \pmod{n}, \text{ if } n \text{ is of the form } 3k-1.$$

I could make no conjecture regarding n of the form 3k+1. The following table gives the number of such shuffles required to restore the original order for decks of 3 through 21 cards.

NO. OF CARDS	3	4	5	6	7	8	9	10	11	12
NO. OF SHUFFLES	1	3	4	4	6	2	2	6	5	5
NO. OF CARDS	13	14	15	16	17	18	19	20	21	
NO. OF SHUFFLES	11	6	6	15	16	16	52	4	4	

At the opposite end of the shuffling spectrum some surprising results occur in the realm of probability. Despite random shuffling the probability will be high that certain properties of a deck are preserved, a fact often made use of by confidence tricksters offering what appear to be good odds to the uninitiated. For example, what are the odds that as two standard decks are dealt simultaneously that at least once the same card will be dealt at the same time. This does not sound like a good bet but in fact is a winner two out of three times. This is of course the classic problem of matching letters with addressed envelopes. The probability, q_{52} say, of failing to get two cards to match is given by

$$q_{52} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{1}{52!} = \frac{1}{3} \text{ approx.}$$

Note that $q_n \to e^{-1}$ as $n \to \infty$.

Harry Blackstone, well-known American stage magician and son of the "Great Blackstone" who rivalled Houdini in Vaudeville in the 1920's, in his book [1] gives the approximate correct answer to this problem but gives an erroneous and simplistic method of getting it.

In the same book he gives the following "betcha": have someone call out any three card values like Ace, Queen and nine. Bet him you can find two of these values together somewhere in the deck. Sounds like a long shot but is far from it. Blackstone himself failed to work out the odds and then as he says "took the simple problem to a probabilities professor at a West Coast University. After two weeks he was still struggling with the problem". Well, I struggled with the problem and eventually, by actually counting the possible combinations found the probability of success to be almost 0.9. The method I used is too long to reproduce here but I would be interested in hearing from anyone who can provide a "nice" solution.

Another good bet is to have someone cut a deck into three piles and turn the top cards. You bet that two will be the same suit. Can you fiqure the odds? There are many more and some may be found in the references below.

References and Further Reading

1. Harry Blackstone Jr., There's One Born Every Minute. Jove, 1978.
2. John Fisher, Never Give a Sucker an Even Break. Sphere, 1978.
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7. Jean Hugard, Encyclopedia of Card Tricks. Faber, 1961.

8. Paul B. Johnson, Congruences and Card Shuffling, Amer. Math. Monthly, Vol. 63 (1956), 718-719.
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