ON THE DEFINITION OF MEASURABILITY FOR VECTOR VALUED FUNCTIONS

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Various conditions for measurability of a real-valued function are examined in the more general case of a vector valued function.

1. Introduction

In the book "Vector Measures" by Diestel and Uhl, a function $f:\Omega \to X$ (where X is a Banach Space and (Ω,Σ,μ) is a measure space) is called measurable if f is a pointwise almost everywhere limit of measurable simple functions. No mention at all is made of the standard definition in terms of open sets i.e. f is measurable provided $f^{-1}(G)$ is measurable for every open set G.

It is well known that the two definitions are equivalent in the case of real-valued functions. This is established in Section 2 below. Naturally, one asks whether the two conditions are also equivalent in the more general case of vector valued functions. An attempt is made in Section 3 to generalise the standard arguments for the real-valued case to those functions which take their values in a Banach Space.

The generalisation of well-known arguments is not merely a formal exercise. In the process insight is gained into these arguments. For example, the separability of the reals is vital to many of the standard proofs. The standard proof that a measurable real-valued function is a pointwise limit of measurable simple functions depends on the fact that any bounded set can be covered by finitely many translates of a given open interval. This means that the proofs will only generalise to Montel spaces (i.e. locally convex spaces in which bounded sets are totally bounded).

The question of why Diestel and Uhl use the definition in terms of simple functions, rather than the topological definition in terms of open sets, will occur to anyone who reads the book. The answer is presumably known and in the literature somewhere. I have not looked for it. Moreover, I have not answered the question fully here. However, the partial results obtained here may be of some interest and perhaps a member of the Irish Mathematical Society will find the enthusiasm to establish the converse of Theorem 3.10 or, alternatively, to find reasonable necessary and sufficient conditions for the existence of a nonmeasurable function which is topologically measurable.

The notation (Ω,Σ,μ) stands for a finite measure space i.e. a non-empty set Ω, a σ-algebra Σ of subsets of Ω and a finite, positive, countably additive measure μ on Σ. The reader is assumed to be familiar with the terminology in the preceding sentence and should have some acquaintance with elementary measure theory and functional analysis.

2. Real-valued measurable functions.

The various equivalent conditions for measurability of a real-valued function on a measure space (Ω,Σ,μ) are presented here. The proofs are well-known. I give them partly as motivation for section 3 and partly in order to see how they might be generalised to vector-valued functions. There is little point in going for full generality here. It is assumed that the measure μ is finite (i.e. $\mu(\Omega) < \infty$) and that the functions are real-valued. Since the notion of an extended-real-valued function does not readily generalise, I do not consider it at all.

2.1 **Definition.** A function $f:\Omega \to \mathbb{R}$ is called measurable if $f^{-1}((\alpha,\infty)) = \{\omega \in \Omega : f(\omega) > \alpha\}$ is measurable for each real α.

It is hopeless to consider generalising this definition as it stands. If the set $\mathbb{R}$ of real numbers is replaced by an arbitrary topological space, or even a Banach Space, the ordering is lost and the definition makes no sense. One must look for equivalent conditions which involve the topology of $\mathbb{R}$ in some way.

2.2 **Lemma.** If $f:\Omega \to \mathbb{R}$ is measurable then the set $\{\omega \in \Omega : f(\omega) \geq \alpha\}$ is measurable for every real α.

Proof. I can write $\{\omega \in \Omega : f(\omega) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{\omega \in \Omega : f(\omega) > \alpha - 1/n\}$. Since f is measurable, each of the sets $\{\omega \in \Omega : f(\omega) > \alpha - 1/n\}$ is measurable. Since the σ-algebra is closed under countable intersections the set $\{\omega \in \Omega : f(\omega) \geq \alpha\}$ is measurable.

2.3 **Corollary.** If $f:\Omega \to \mathbb{R}$ is measurable then $f^{-1}(I)$ is measurable for every bounded open interval I.

Proof. Set $I = (\alpha,\beta)$ and note that $f^{-1}(I) = \{\omega\in\Omega : f(\omega) > \alpha\} \cap \{\omega\in\Omega : f(\omega) < \beta\}$ $= \{\omega \in \Omega : f(\omega) > \alpha\} \cap [\Omega - \{\omega \in \Omega : f(\omega) \geq \beta\}]$. Lemma 2.2 together with the properties of the σ-algebra now quarantees that $f^{-1}(I)$ is measurable.

The first equivalent condition for measurability can now be established.

2.4 **Theorem.** A function $f:\Omega \to \mathbb{R}$ is measurable if and only if $f^{-1}(G)$ is measurable for every open subset G of $\mathbb{R}$.

Proof. Suppose that f is measurable. The bounded open intervals with rational endpoints form a countable base for the topology on $\mathbb{R}$. Therefore, if G is open, I can find a sequence (I_n) of bounded open intervals such that $G = \bigcup_{n=1}^{\infty} I_n$. Then $f^{-1}(G) = \bigcup_{n=1}^{\infty} f^{-1}(I_n)$. Since each $f^{-1}(I_n)$ is measurable by the above Corollary, $f^{-1}(G)$ is also measurable. The converse is clear since (α,∞) is open.

Notice how the proof depends on the fact that the topology has a countable base. There would be no way of proceeding if we had to write G as an uncountable union of intervals.

2.5 **Definition.** A simple function $S:\Omega \to \mathbb{R}$ is a function with finite range. If $S(\Omega) = \{\alpha_1, \ldots, \alpha_n\}$ then we can write $S = \Sigma_1^n \alpha_i \chi_{E_i}$ where $E_i = \{\omega \in \Omega : S(\omega) = \alpha_i\}$ and, for any subset A of Ω, $\chi_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$

It is easy to see that $S = \Sigma_1^n \alpha_i \chi_{E_i}$ is measurable if and only if each E_i is measurable.

2.6 **Theorem.** If $f:\Omega \to \mathbb{R}$ is measurable, then there exists a sequence (S_n) of mesurable simple functions such that $S_n(\omega) \to f(\omega)\ (n \to \infty)$ for every ω in Ω.

Proof. For each n and for $-n2^n+1 \leq k \leq n2^n$ write $E_{n_k} = \{\omega \in \Omega : (k-1)2^{-n} < f(\omega) \leq k2^{-n}\}$. Then each E_{n_k} is measurable, since it is the inverse image of a half-open interval. The sets E_{n_k} are disjoint. Write $S_n = \Sigma k2^{-n}\chi_{E_{n_k}}$ the sum being over all values of k from $-n2^n+1$ to $n2^n$. Now S_n is a measurable simple function. For any $\omega \in \Omega$, select N such that $|f(\omega)| < N$. If $n > N$ then $|f(\omega)| < n$ and so $-n < f(\omega) < n$. There exists k with $-n2^{-n}+1 \leq k \leq n2^{-n}$ such that $(k-1)2^{-n} < f(\omega) \leq k2^{-n}$. Then $\omega \in E_{n_k}$ and we get $S_n(\omega) = k2^{-n}$. Since $(k-1)2^{-n} < f(\omega) \leq k2^{-n}$ we have $|S_n(\omega) - f(\omega)| < 2^{-n}$. It follows that $S_n(\omega) \to f(\omega)\ (n \to \infty)$.

I mention here that, of all the proofs in this section, the one I have just given is the most difficult to generalise. Indeed, it is because of this that the various equivalent conditions for measurability in this section do not remain equivalent in the more general setting of section 3. The above proof seems to depend ultimately on the fact that bounded sets are totally bounded. This is not true for the norm topology on an infinite-dimensional Banach Space. (It is true for the weak topology, and one should be able to find a generalisation of the above result to that case, with a bit of effort.)

2.7′ Theorem. If there exists a sequence (S_n) of measurable simple functions such that $S_n \to f$ pointwise on Ω, then f is measurable.

Proof. The fact that the S_n are simple functions is not needed at all. Let α be real. I claim that $f^{-1}((\alpha,\infty)) = \bigcup_{\substack{q>\alpha \\ q \text{ rational}}} \bigcup_{N=1}^{\infty} \bigcap_{n>N} S_n^{-1}((q,\infty))$.

This is reasoned out as follows. Suppose that $\omega \in f^{-1}((\alpha,\infty))$ i.e. $f(\omega) \in \alpha$. There exists a rational $q > \alpha$ such that $f(\omega) > q$. Since $S_n(\omega) \to f(\omega)$ $(n \to \infty)$ there exists N such that $S_n(\omega) > q$ for all $n > N$. But this all means that $\omega \in \bigcup_{\substack{q>\alpha \\ q \text{ rational}}} \bigcup_{N=1}^{\infty} \bigcap_{n=N+1}^{\infty} S_n^{-1}((q,\infty))$

The argument is essentially reversible: if ω is in the latter set there exists $q > \alpha$ such that $\omega \in \bigcup_{N=1}^{\infty} \bigcap_{n=N+1}^{\infty} S_n^{-1}((q,\infty))$. Then there exists N such that $\omega \in S_n^{-1}((q,\infty))$ for all $n > N$ i.e. $S_n(\omega) > q$ for all $n > N$. Since $S_n(\omega) \to f(\omega)$ $(n \to \infty)$ we then get $f(\omega) \geq q > \alpha$. Consequently $\omega \in f^{-1}((\alpha,\infty))$. Therefore the set equality holds. Since each S_n is measurable, the sets $S_n^{-1}((q,\infty))$ are all measurable. Since the unions and intersection are countable, the set $\bigcup_{\substack{q>\alpha \\ q \text{ rational}}} \bigcup_{N=1}^{\infty} \bigcap_{n=N+1}^{\infty} S_n^{-1}((q,\infty))$ is measurable i.e. $f^{-1}((\alpha,\infty))$ is measurable, as required.

Observe that the set of rationals plays a fundamental role in the proof. Any generalisation of this type will involve separability.

3 Vector-valued functions.

We can now look at the case where the function f with domain Ω takes its values in a Banach Space X. In view of the results on real-valued functions, there are two reasonable definitions of measurability available in the more general case. The first is purely topological in nature and does not mention the measure μ at all.

3.1 Definition. Let X be a Banach Space and let $f:\Omega \to X$. Say that f is topologically measurable if $f^{-1}(G)$ is measurable for every open subset G of X.

The second definition is based on Theorems 2.6 and 2.7 but the pointwise convergence requirement is relaxed somewhat. This definition involves the measure μ, because sets of measure zero are important here. The theory of vector measures is based on this approach and depends completely on it. A simple function $S:\Omega \to X$ is a function with finite range. If $S(\Omega) = \{x_1 \dots x_n\}$ we can write $S = \Sigma_1^n x_i \chi_{E_i}$ where $E_i = \{\omega \in \Omega : S(\omega) = x_i\}$. Such a simple function is called measurable if each E_i is measurable.

3.2 Definition. Say that a function $f:\Omega \to X$ is measurable provided there is a sequence (S_n) of measurable simple functions such that $S_n \to f$ a.e. on Ω, in the sense that $\|S_n(\omega) - f(\omega)\| \to 0$ $(n \to \infty)$ for almost all ω.

The objective is to investigate the equivalence of measurability and topological measurability so let us first dispose of the case of simple functions.

3.3 Proposition. A simple function $S:\Omega \to X$ is measurable if and only if it is topologically measurable.

Proof. Let $S = \Sigma_{k=1}^n x_i \chi_{E_i}$ be measurable. Let G be an open subset of X. If $S^{-1}(G) = \phi$ then $S^{-1}(G)$ is measurable. Otherwise $S^{-1}(G) = \bigcup_{x_i \in G} E_i$

which is measurable, since $\{x_i : x_i \in G\}$ is finite. Conversely, if $S^{-1}(G)$ is open for every open set G, I can show that each E_i is measurable as follows. I have $E_i = S^{-1}(\{x_i\}) = \Omega - S^{-1}(X - \{x_i\})$. Since $X - \{x_i\}$ is open, $S^{-1}(X - \{x_i\})$ is measurable and therefore so is $E_i = \Omega - S^{-1}(X - \{x_i\})$.

3.4 Proposition. If $f:\Omega \to X$ is measurable then f is "essentially separably valued" in the sense that there is a subset A of Ω such that $\mu(\Omega - A) = 0$ and $f(A)$ is separable.

Proof. Write $A = \{\omega \in \Omega : S_n(\omega) \to f(\omega)\}$ where (S_n) is a sequence of measureable simple functions with $S_n \to f$ a.e. Then $\mu(\Omega - A) = 0$. Now $\bigcup_{n=1}^{\infty} S_n(\Omega)$ is countable since each $S_n(\Omega)$ is finite. Moreover, if $S_n(\omega) \to f(\omega)$ then $f(\omega) \in \overline{\bigcup_{n=1}^{\infty} S_n(\Omega)}$. Consequently $f(A) \subseteq \overline{\bigcup_{n=1}^{\infty} S_n(\Omega)}$ which is separable. This means that $f(A)$ is separable.

3.5 Lemma. Let f be measurable and let $B = B(x,r) = \{y : \|x-y\| < r\}$ be any open ball. Then $f^{-1}(B)$ is measurable.

Proof. There is a sequence (S_n) of measurable simple functions such that $S_n \to f$ a.e. on Ω. Write $A = \{\omega \in \Omega : S_n(\omega) \to f(\omega)\}$ so that $\mu(\Omega - A) = 0$. For any $t > 0$ put $B_t = \{y : \|x-y\| < t\}$. First of all, $f^{-1}(B) = [f^{-1}(B) \cap (\Omega - A)] \cup [f^{-1}(B) \cap A]$ so it suffices to show that the sets $f^{-1}(B) \cap (\Omega - A)$ and $f^{-1}(B) \cap A$ are both measurable. The set $f^{-1}(B) \cap (\Omega - A)$ is certainly measurable since it is a subset of $\Omega - A$ which has measure zero. In order to show that $f^{-1}(B) \cap A$ is measurable we show first that

$$f^{-1}(B) \cap A = \bigcup_{\substack{q < r \\ q \text{ rational}}} \bigcup_{N=1}^{\infty} \bigcap_{n=N+1}^{\infty} [S_n^{-1}(B_q) \cap A] \quad (*)$$

Let $\omega \in f^{-1}(B) \cap A$. Then $\|f(\omega) - x\| < r$. Select a rational $q < r$ such that $\|f(\omega) - x\| < q$. Since $\omega \in A$, I have $S_n(\omega) \to f(\omega)$ $(n \to \infty)$. Therefore there is an N such that $\|S_n(\omega) - x\| < q$ for all $n > N$. All of this shows that ω is in the right hand side of (*). If ω is in the right

hand side of (*) there exists a rational $q < r$ such that for some N, $\omega \in S_n^{-1}(B_q) \cap A$ for all $n > N$. Thus $\omega \in A$ and $\|S_n(\omega) - x\| < q$ for all $n > N$. This gives $\|f(\omega) - x\| \le q < r$ and so $\omega \in f^{-1}(B) \cap A$. The equality (*) is established and the fact that $f^{-1}(B) \cap A$ is measurable now follows from Proposition 3.3 and the properties of the σ-algebra.

3.6 Theorem. If f is measurable then f is topologically measurable.

Proof. Let (S_n) and A be as in the proof of Lemma 3.5. Let G be any open subset of X. Since $f^{-1}(G) = [f^{-1}(G) \cap (\Omega - A)] \cup [f^{-1}(G) \cap A]$ it is enough to show that the sets $f^{-1}(G) \cap (\Omega - A)$ and $f^{-1}(G) \cap A$ are measureable. The first of these is measurable since it is a subset of the set $\Omega - A$ which has measure zero. Observe that $f^{-1}(G) \cap A = f^{-1}(G \cap f(A)) \cap A$. By Proposition 3.4 f(A) is separable. This means that the relative topology on f(A) has a countable base of open balls. Consequently I can find a sequence (B_n) of open balls in X with $G \cap f(A) = \bigcup_{n=1}^{\infty} B_n \cap f(A)$

Now we get

$$f^{-1}(G) \cap A = f^{-1}(G \cap f(A)) \cap A = f^{-1}\left(\bigcup_{n=1}^{\infty} B_n \cap f(A)\right) \cap A$$

$$= \bigcup_{n=1}^{\infty} f^{-1}(B_n \cap f(A)) \cap A =$$

$$= \bigcup_{n=1}^{\infty} f^{-1}(B_n) \cap f^{-1}(f(A)) \cap A = \bigcup_{n=1}^{\infty} f^{-1}(B_n) \cap A,$$

since $f^{-1}(f(A)) \cap A = A$.

But now each of the sets $f^{-1}(B_n)$ is measurable by Lemma 3.5. It follows that $f^{-1}(B_n) \cap A$ is measurable for each n and therefore so is $f^{-1}(G) \cap A$. This completes the proof.

At this point the investigation is half over. It is established that measurability implies topological measurability. It would be nice if the converse could be established, but we run into problems. However, a partial converse can be found, and it will be convenient to use the following known result from the theory of vector-valued measurable functions. (see Diestel and Uhl).

3.7 Pettis Measurability Theorem. Let $f:\Omega \to X$ where X is a Banach Space. Then f is measurable if and only if f is essentially separably valued and weakly measurable (in the sense that the real-valued function $\varphi o f$ is measurable for every φ in the dual X^* of X)

3.8 Theorem. If f is topologically measurable and essentially separably valued then f is measurable.

Proof. By the Pettis Measurability Theorem, it is enough to show that if $\varphi \in X^*$ then $\varphi o f$ is measurable (as a real-valued function on Ω). By section 2, it suffices to show that $\varphi o f$ is topologically measurable. Let G be an open subset of $\mathbb{R}$. Then $(\varphi o f)^{-1}(G) = f^{-1}(\varphi^{-1}(G))$. Now $\varphi^{-1}(G)$ is open since φ is continuous. Then $f^{-1}(\varphi^{-1}(G))$ is measurable since f is topologically measurable. It follows that $\phi o f$ is measurable.

The only remaining question is whether the condition that f is essentially separably valued can be dropped from Theorem 3.8. In order to obtain more insight, we can adopt the following device which simplifies the problem conceptually.

3.9 Proposition. The following are equivalent

(a) There exists a measure space (Ω,Σ,μ) and a topologically measurable function $f:\Omega \to X$ which is not measurable.

(b) There exists a Borel measure λ on X such that the identity on X is not λ-measurable.

(c) There exists a Borel measure λ on X which is not concentrated on any separable subset of X i.e. such that there is no separable subset P of X with $\lambda(X-P) = 0$.

Proof (a) $\Rightarrow$ (b). Suppose that (a) holds. The collection $\mathcal{M} = \{E \subseteq X : f^{-1}(E) \text{ is measurable}\}$ is a σ-algebra on X, as is easily verified. Since f is topologically measurable $\mathcal{M}$ contains the open sets and therefore also contains the Borel sets of X. Define $\lambda(E) = \mu(f^{-1}(E))$ for every Borel set E. Then λ is a Borel measure on X. Suppose that the identity is λ-measurable. There exists a sequence $S_n : X \to X$ of λ-measurable simple functions with $S_n(x) \to x$ a.e. on X. Let $E_o = \{x : S_n(x) \to x\}$ so that $\lambda(X-E_o) = 0$. Define $t_n : \Omega \to X$ by $t_n = S_n o f$. Then t_n has finite range. Moreover, if $G \subseteq X$ is open then $t_n^{-1}(G) = f^{-1}(S_n^{-1}(G))$. But $S_n^{-1}(G)$ is λ-measurable and so $f^{-1}(S_n^{-1}(G))$ is (μ-)measurable. This means that t_n is topologically measurable and therefore measurable by Proposition 3.3. If $\omega \in f^{-1}(E\ )$ then $f(\omega) \in E_o$ and so $S_n(f(\omega)) \to f(\omega)$ $(n \to \infty)$. This gives $t_n \to f$ on $f^{-1}(E_o)$. But $\Omega - f^{-1}(E_o) = f^{-1}(X - E_o)$ and $\mu(f^{-1}(X-E_o)) = \lambda(X-E_o) = 0$. Hence $t_n \to f$ a.e. on Ω and f is measureable. This contradicts the assumption that (a) holds and therefore the identity on X cannot be λ-measurable.

(b) $\Rightarrow$ (c). If (b) holds then the identity on X is topologically measurable but not measurable. By Theorem 3.8 the identity cannot be essentially separably valued. This establishes (c) at once.

(c) $\Rightarrow$ (a). Let $\Omega = X$. Let Σ be the Borel algebra on X and take f as the identity on X. Then f is topologically measurable. However, for the given measure λ there is no sequence (S_n) of measurable simple functions converging almost everywhere to f. If there were such a sequence then f would be essentially separately valued, which is impossible, since λ is not concentrated on any separable subset of X.

The problem now reduces to finding an answer to the following question: is it possible to have a Banach Space X and a finite Borel measure λ on X which is not concentrated on any separable subset of X?

The question is related to Ulam's Measure Problem: does there exist a set D and measure ν such that every subset of D is measurable, $0 < \nu(D) < \infty$ and every countable subset of D has measure zero?

The answer to this question is somewhere in the foundations.

3.10 Theorem. If the answer to Ulam's Measure Problem is 'Yes' then there exists a topologically measurable function which is not measurable.

Proof. Let D be a set with measure ν satisfying Ulam's criteria. Let X be the Hilbert space $\ell_2(D)$ consisting of all functions $\phi: D \to \mathbb{R}$ for which $\{\delta \in D : \varphi(\delta) \neq 0\}$ is countable and $\Sigma_{\delta \in D} |f(\delta)|^2$ is finite. The inner product on $\ell_2(D)$ is given by $< \varphi\ \psi > = \Sigma_{\delta \in D} \varphi(\delta) \psi(\delta)$. The measure space is $(D, 2^D, \nu)$ and we define $f: D \to X$ by $f(\delta) = \ell_\delta$ where $\ell_\delta(y) = \begin{cases} 0 & y \neq \delta \\ 1 & y = \delta \end{cases}$.

As is well-known, the set $\{\ell_\delta : \delta \in D\}$ is an orthonormal basis for $\ell_2(D)$ and so $||\ell_\delta - \ell_y|| = \sqrt{2}$ for $\delta \neq y$. The set f(D) therefore has no limit points and so every subset of f(D) is closed. This means that a subset of f(D) is separable if and only if it is countable.

If f is measurable then it is essentially separably valued and so there is a subset A of D with $\nu(D-A) = 0$ and f(A) separable. But then f(A) is countable and hence so is A which gives $\nu(A) = 0$. Now we have $\nu(D) = \nu(A) + \nu(D-A) = 0$, a contradiction. This means that f is not measurable.

It would be nice to complete the work by proving the converse of Theorem 3.10. However, I have not succeeded in establishing it. My numerous attempts have left me with the impression that the converse holds. On the other hand, the fact that those attempts failed leaves room for doubt. It is left to the reader to pursue this question.

Reference.

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