Group Theory and Other Abstract Tripe

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After completing the marking of the summer examinations, a disturbing experience for most of us, ones thoughts inevitably become rather philosophical. Why am I here? What is it all for? Especially – what are we teaching our students? Not a lot, usually, but what I want to discuss here is the course design rather than our success or otherwise in teaching these courses (although the two are clearly closely linked.) I will concentrate on the honours courses for the same reasons that we usually over emphasise this aspect of our teaching: we have a free hand here in the course content, we can avoid getting involved in anything unsavoury like questions of applications to other subjects and we all went through honours courses ourselves. (I should explain that the "we" used here refers to the mathematical community at large, and since this article is rather critical of this community's policies the "we" probably doesn't include the reader and definitely does not include the author.)

The first hint honours students have that something may be amiss usually comes with the traditional ε,δ analysis course. Hint here is perhaps understating the case, the experience seems more akin to being hit over the head with an intellectual sledge hammer. This is part of the well known disorientation process first year students undergo, where any connections between this new (learning?) experience and school mathematics are minimized. Moreover great pains are taken to prove results which may appear rather trivial to the uninitiated (as well as Newton and Gauss probably) but which use a definition which takes a year to understand. As a colleague+ put it, "having jumped in the lift and expecting to be shot up to the top floor the student is immediately taken down to the basement." Thankfully this rather austere start is complemented by some light relief in the shape of a course on abstract (what else?) algebra, usually an introduction to group theory. This is also a new experience, so new in fact that it appears hardly to be mathematics at all, more a sort of contrived parlour game (which we call TRIPE). The game involves defining as many new words as possible and relating them with equally many three (but better if you can manage two) line theorems. (Lines here refers to the number of lines of proof; extra marks are awarded if they are exceeded by the number of lines of statement). You know the sort of thing, injective, kernel, subgroup, homomorphism (Patented TRIPE could do more damage to the image of mathematics than the Rubic cube.)

+Why should he remain anonymous – it was Brian Twomey.

The thinking behind the game, its educational value is that it impresses upon the student the importance of structure, a key word. (In retrospect it seems rather amazing that Gauss managed without this experience. Yes, I know there are not many Gauss's(?) about, but if you admit he was good, and structure is important, why didn't he spend more time playing TRIPE?)

The second year is usually the worst in the standard three year course. With the alibi of giving final year options the relevant springboard (and usually with no reference to what is actually taught there) we squeeze in large amounts of good solid (= dreadful?) mathematics. The sort of stuff absolutely essential for the safe release of any self respecting honours student (always assuming there are any such students left by year two). The most important ingredient appears to be the "all you ever wanted to know about linear algebra (and a great deal more besides)" course, and the "advanced (now that you don't understand elementary) calculus" course (on an arbitrary Banach space if you are lucky.) From these huge machines, erected with a great deal of trouble there emerges after a ponderous crank of the handle all sorts of wonderful things! Why, did you know one can classify quadrics up to Euclidean motions? Impressed – well just wait until Lagrange multipliers change your life! Probably the saving grace, universally, is the standard second year course on complex analysis. For my fellow students this was the first thing that looked remotely like school mathematics since they started their university career, and contour integrals were devoured with the same relish with which a starving man takes his first decent meal.

And then we have the ultimate intellectual experience, the final year honours courses, which usually come in a collection of rather standard packages. Group Theory, by now fairly popular with the players of TRIPE, supplies even longer definitions and connecting theorems. To prove that these antics aren't the private property of the algebraists a course on Topology is often given (If you thought connected meant in one piece, wait until you have seen our definition!). Adn then there is Functional Analysis. I remember attending a course which started with measure spaces, integration and went on to L^2 spaces, separable Hilbert spaces, the Riesz representation theorem, and discussed unitary, normal, self adjoint (and other) operators on these Hilbert spaces. Feeling rather like Oliver when he asked for more, I remember asking, at the end of the course, "why?" (or words to that effect.) Our lecturer immediately started talking authoratively about Sturm-Louiville theory and differential equations, and I remember feeling very impressed. The only example of a measure he had introduced was the counting measure on a finite set: solving D.E.'s using that would certainly be a good trick. I could go on (what about the courses on algebraic number theory which don't appear to have anything to do with numbers?) and so could you.

Well what are we teaching our students? In a word, or rather two, formal trips. How can one justify teaching a final year course simply on group theory. No mention of Galois theory or geometric transformations, just dreary theorem after dreary theorem. (If the Jordan Holder theorem does something for you, my condolences.) Yet look at the standard undergraduate textbooks on group theory. Similarly should we really let people who want to teach courses on Hilbert spaces and their operators without mentioning differential equations loose on the streets? What is the point of teaching algebraic number theory when we so often duck out of any course on elementary number theory? Why do we do it? One reason is undoubtedly that this type of course is neat and easily available in nice tidy packages. Another may be in that we are the weirdos who actually liked the stuff; or at least did well at it. Just in case you think the author is an exception, may I shamefacedly admit that I lapped this abstract rubbish up as an undergraduate (but couldn't do a contour integral to save my life.) In common with many others my conversion took place during the first year I spent doing research. Having soaked up courses of the above type I had the vague idea that by writing out a suitable list of axioms all sorts of wonderful theorems would pop out and make me famous; mathematics had become a formal game. It took me a year to grow out of this illusion, to realise that a complete understanding of two good examples is worth (and will probably result in) ten good theorems. That the simple version of the theorem is the important one and that any fool could generalize (and I often did).

The effect of our teaching programme is frequently disastrous, for exactly these reasons. The students have no sense of history of the subject, nor its origins. They see no relationship between mathematics and the world in which we live. They are continually confronted with definitions and theorems completely cut off from their historical and quite valid origins. (Abstraction, structure have their place, but at the right level, and this is usually postgraduate.) But worst of all we kill any enthusiasm our students have for the subject, which we present as a logical and pedestrian development of results from an apparently arbitrary base made up of some axioms. (We know on the other hand from our research experience, that the subject is anything but logical, that anyone sticking with axioms also sticks with three line theorems, that these axioms have their origins in some very important concrete theorems and examples.) Of course we have our occasional successes as well, but the students involved are often so good it is unclear whether they have succeeded because of or despite our efforts. Where we fail, and fail quite dismally, is with

the hard working students of average ability. These people, who constitute the majority of our honours classes frequently leave the universities disillusioned and perplexed, having gained little or nothing from their three year stay. (An extra reason for concern on our part is that they frequently go on to teach in secondary schools; but our bad influence there is another story.)

In overstating my case, I hope I have trodden on as many corns as possible, and look forward to the ensuing criticism. What I would like to see is an approach to the syllabus taught more in line with the historical development of the subject (ε and δ emerged from problems concerned with Fourier series not calculus). The further one is from physics, number theory and geometry, the three main sources of good mathematics, the more careful one must be about the material taught (perhaps even the research one does?). Abstraction for its own sake can quickly degenerate into irrelevant and trivial nonsense. Probably the worst crime of all to commit is not to give many really good examples in courses. Examples first and unifying concepts (much) later. If we must set out on a new theory we should always be able to justify the journey on the grounds that the material developed solves some interesting problem lying outside the course itself. The need for good exercises is of course also well established. Mathematics is not a spectator sport; but too often now examinations are passed on bookwork, and exercises set during the year for the abstract courses involve definition juggling only, so that the problem solving aspect of our students education suffers.

Mathematics is fascinating, vibrant alive - you all believe this. Yet we (and remember what we means!) seem incapable of persuading our students that this is the case. Despite Russell's assertion to the contrary mathematics is not the subject in which we never know what we are talking about, nor whether what we are saying is true. We just teach it as if it was.