SOLUTIONS TO PROBLEMS

We give here outline solutions to some of the problems in Newsletter Number 3.

(3.1) Find all numbers of the form $\frac{1}{2}n(n+1)$ (triangular numbers) which are perfect squares.

Suppose first that n is even. The fact that $\frac{1}{2}n(n+1)$ is a perfect square implies that $n=2a^2$, $n+1=b^2$ where a,b are integers. But then $b^2-2a^2=1$. Conversely for such a,b, setting $n=2a^2$ gives a solution to our problem. The solutions to the equation $b^2-2a^2=\pm 1$ are given by the rational and irrational parts of the binomial expansion of $\pm(1\pm\sqrt{2})^k$ $(k=0,1,\ldots)$ (Hardy and Wright: Introduction to the Theory of Numbers). A similar analysis works in the case n odd.

(J. Kennedy)

(3.2) Let $n>1$ be a given natural number. Find a square matrix A whose entries are zeros and ones such that A^n is a matrix with all its entries equal to one. Show that 2^n is the least possible size for such a matrix A.

Let A be the $2^n \times 2^n$ matrix of the form $\binom{B}{B}$ where row i of B has ones in the $(2i-1)^{st}$ and $(2i)^{th}$ positions and zeros elsewhere. Then $A^n = J(2^n)$ where $J(m)$ is the $m \times m$ matrix all of whose entries are one. On the other hand, if $A^n = J(t)$, the eigenvalues of A^n are $t,0,\ldots,0$, so the eigenvalues of A are $\sqrt[n]{t},0,\ldots,0$. So $\operatorname{trace}(A) = \sqrt[n]{t}$. Since $\operatorname{trace}(A)$ is an integer,

t must be a perfect n^{th} power. So $t \geq 2^n$.

(W. Sullivan)

(3.3) (non-trivial part) Give an example of periodic functions f,g with periods $u > 0$, $v > 0$, respectively, such that (i) u/v is not rational, and (ii) $f+g$ is periodic.

Define a function $f_a:[0,a] \to \mathbb{R}$ as follows: Let $x \in [0,a]$. For each integer m, let $x_m = x+m\pi-ka$ where k is the unique integer so that $x_m \in [0,a)$. For example, when $a=1$, x_m is the decimal part of $x+m\pi$.) Let $V(x) = \{x_m | m \in Z\}$. Note that $V(x) \cap V(y)$ non-empty implies $V(x) = V(y)$. So $\{V(x)\}$ form a partition of $[0,a]$. Choose a distinguished element $y = y(x)$ in each distinct $V(x)$. Then $V(x) = \{y_m | m \in Z\}$. Define $f_a:[0,a] \to \mathbb{R}$ by $f_a(y_m) = m$. Extend f_a to be a map of $\mathbb{R}$ to $\mathbb{R}$ by the rule $f_a(x+a) = f_a(x)$. Then f_a is periodic of period $\leq a$ and f_a satisfies the equation $f_a(x+\pi) = f_a(x)+1$ for all $x \in \mathbb{R}$.

Let $f = f_1$, $g = -f_{\sqrt{2}}$. Then f is periodic of period 1 and g is periodic of period $\sqrt{2}$ (using the fact that π is not algebraic). Also $f(x+\pi) = f(x)+1$ and $g(x+\pi) = g(x)-1$ for all real x. So $f+g$ is periodic of period a (non-zero) rational multiple of π.

(3.4) Let A be a square matrix with entries in a field F. Prove that $A = D+N$ where N is nilpotent and D is diagonalizable <u>over the field</u> F.

We reduce easily to the case where A is a companion matrix, say

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots\cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ & & & & \\ & & & & 1 \\ a_o & a_1 & \cdots\cdots\cdots & & a_{n-1} \end{bmatrix}$$

If $a_{n-1} \neq 0$, take N to be A with the last row replaced by zero and D to have its last row equal to the last row of A and zeros elsewhere. If $a_{n-1} = 0$, let k be the least integer with $a_{n-k-1} \neq 0$. Replace N as above except that the 1 in its $(n-k)^{th}$ row is replaced by $1-a_{n-k-1}$ and let $D = A-N$. In characteristic $\neq 2$, D is diagonalizable with non-zero eigenvalues $\pm a_{n-k-1}$. In characteristic 2, the required result may fail - try $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ over Z_2.

(3.5) This question is answered in the paper by Choi, Laurie and Radjavi in Linear & Multilinear Algebra 8, (1980).

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