PROBLEM SECTION

(3.1) Find all numbers of the form $\frac{1}{2}n(n+1)$ (triangular numbers) which are perfect squares.

(3.2) Let $n > 1$ be a given natural number. Find a square matrix A whose entries are zeros and ones such that A^n is a matrix with all its entries equal to one. Show that 2^n is the least possible size for such a matrix A.

(3.3) Let f be a periodic real function. The least number $u > 0$ (if such exists) such that $f(x+u) = f(x)$ for all real x is called the period of f.

Suppose that f, g are periodic with periods $u \neq 0, v \neq 0$, respectively, and suppose that f is bounded. Prove that $f+g$ is periodic if and only if u/v is rational. Give an example of periodic functions f, g with periods $u>0, v>0$, respectively, such that (i) u/v is not rational and (ii) $f+g$ is periodic.

(3.4) Let A be a square matrix with entries in a field F. Prove that $A = D+N$ where N is nilpotent and D is diagonalizable <u>over the field</u> F.

[It is well known that if F is algebraically closed, $A = D+N$ as above with, in addition, $ND=DN$.]

(3.5) Let A be an $n \times n$ matrix with entries in an algebraically closed field F of characteristic $\neq 2$ and suppose that A has trace zero. Prove that if A has rank greater than one, then A is similar to a symmetric matrix with all its diagonal entries equal to zero.

(3.6) Let $\{a_n\}$ be a sequence of positive terms. Relate the convergence or divergence of $\sum a_n$ to that of $\sum \frac{a_n}{1+a_n^2}$.

(3.7) Let $t = \binom{k}{2}$ be an integer ≥ 10. Show that it is not possible to find an ascending sequence

$$0 = x_1 < x_2 \ . \ . \ . < x_k = t$$

of integers $x_1, \ldots, x_k$ so that the differences $x_j - x_i$ $(1 \leq i < j \leq k)$ are all distinct (and every one of $1,2,\ldots,t$ is such a difference).

(3.8) If A is a complex $n \times n$ matrix, set

$$||A|| = \sup_{||z||=1} ||Az||$$

where $||z|| = [|z_1|^2 + \ldots + |z_n|^2]^{\frac{1}{2}}$, $\quad z = (z_1, \ldots, z_n) \in C^n$.

Let A^* denote the complex conjugate of A transposed. Suppose A,B are $n \times n$ complex matrices over C with

$$\text{trace}[(A^*A)^j A^*B] = 0 \quad \text{for } j = 0,1,\ldots,n-1.$$

Show that $||A|| \leq ||A+B||$.

(3.9) Evaluate

$$\int_0^1 \frac{\tan^{-1}\sqrt{\frac{v}{1+v}}}{\sqrt{1-v^2}}\, dv.$$

(3.10) A man carrying a horizontal 20ft. long pole runs at speed v such that[+] $\gamma(v) = 2$ into a 10ft. long room and closes the door. Explain in the man's frame, in which the room is only 5ft. long, how this is possible. Show also that the minimum length of the room for the performance of this trick is $\frac{20}{\sqrt{3}+2}$ ft.

[+] $\gamma(v) = (1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$, c = velocity of light.

(From W. Rindler, Special Relativity, 1960, Oliver & Boyd.)

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