FOR YOUR DIARY

L.M.S. Research Symposium on Finite Simple Groups. Durham July 31 - August 10. (For details contact M. Collins, Math. Inst., St. Giles, Oxford OX1 3LE).

A.S.I. Conference on Rings with Polynomial Identities. Antwerp. August 1-13. (For details, contact Professor Dr. F. van Oystaeyen, University of Antwerp, U.I.A., Dept. of Mathematics, Universiteitsplein 1, 2610 Wilrijk, Belgium).

International Congress of Mathematicians. Helsinki. August 15-23. (For details of group travel from Ireland to the congress, contact T.C. Hurley, U.C.D.).

I.M.S. Conference on the history of Mathematics, U.C.C., late September. (Full details of this conference wi-l be issued later).

Dublin Institute for Advanced Studies Christmas Symposium, December 20-21, 1978.

Polynomial identities and central identities for matrices

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Let A,B,C be 2×2 matrices. It is easy to verify that $(AB-BA)^2$ is a scalar matrix and hence

$$(AB-BA)^2C - C(AB-BA)^2 = 0$$

(This is called Wagner's identity). This may be expressed by saying that $(xy-yx)^2z - z(xy-yx)^2$ is a polynomial identity for 2×2 matrices. In general, a nonzero polynomial $f(x_1,\dots,x_m)$ in the noncommuting indeterminates (or symbols) $x_1,\dots,x_m$ is called a polynomial identity for $n \times n$ matrices if $f(A_1,A_2,\dots,A_m) = 0$ for all $n \times n$ matrices $A_1,A_2,\dots,A_m$.

For example, it is easy to check that Wagner's identity is not a polynomial identity for 3×3 matrices. Another example of a polynomial identity for 2×2 matrices is $(x_1^2x_2-x_2x_1^2)(x_1x_2-x_2x_1)-(x_1x_2-x_2x_1)(x_1^2x_2-x_2x_1^2$

We now give an example of a polynomial identity for $n \times n$ matrices. First, we need a definition.

Definition The standard polynomial $s_m(x_1,\dots,x_m)$ of degree m is defined by

$$s_m(x_1,\ldots,x_m) = \sum_{\sigma\in S_m} (\text{sign } \sigma) x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(m)}$$

where S_m denotes the symmetric group of degree m and order $m!$, and sign $(\sigma) = +1$ or -1 depending on whether the permutation σ is even or odd.

Thus, for example, $s_2(x_1,x_2) = x_1x_2 - x_2x_1$,
$s_3(x_1,x_2,x_3) = x_1x_2x_3 - x_1x_3x_2 - x_3x_2x_1 - x_2x_1x_3 + x_2x_3x_1 + x_3x_1x_2$,
etc. $s_m(x_1,\ldots,x_m)$ is a homogeneous polynomial of degree m.

We now state the famous theorem of Amitsur-Levitski (Proc. Amer. Math. Soc. 1, 449-463, (1950)).

Theorem $s_{2n}(x_1,\ldots,x_{2n})$ is a polynomial identity for $n \times n$ matrices. There is no polynomial identity of degree $<2n$ for $n \times n$ matrices.

Up till recently, no simple proof was known for this result. However, a simple proof was discovered a few years ago by S. Rosset (Israel J. Math. 23, (1976), 187-188). Rosset's proof also appears in Cohn's book Algebra II (Wiley 1977), p.457.

Remark The polynomial $s_m(x_1,\ldots,x_m)$ is multi-linear i.e.

$$s_m(x_1,\ldots,x_{i-1}, x_i + x_i', x_{i+1},\ldots,x_m) = s_m(x_1,\ldots,x_{i-1}, x_i, x_{i+1},\ldots,x_m) + s_m(x_1,\ldots,x_{i-1}, x_i', x_{i+1},\ldots,x_m)$$

and vanishes if two of the x's are equal, i.e. $s_m(x_1,\ldots,x_i,\ldots,x_i,\ldots,x_m) = 0$.

Hence, to prove the Amitsur-Levitski theorem, it suffices to show that $s_{2n}(B_1,\ldots,B_{2n}) = 0$ for all distinct $B_1,\ldots,B_{2n}$ chosen from a fixed vector space basis of the algebra of $n \times n$ matrices, for example, from the set E_{ij} $i,j = 1,2,\ldots,n$ where E_{ij} is the matrix with 1 in the (i,j) position, zeros elsewhere. This enables one to verify the theorem easily for $n = 2$ since these reduce the theorem to verifying that $s_4(E_{11},E_{12},E_{21},E_{22}) = 0$ in that case. Several proofs of the Amitsur-Levitski theorem are based on this idea, but Rosset's one depends on a particularly ingenious application of the Hamilton-Cayley theorem.

As mentioned at the outset, if A,B are 2×2 matrices, then $(AB-BA)^2$ is scalar, (though not necessarily zero), and thus lies in the centre of the algebra of 2×2 matrices. This is expressed by saying that $(x_1x_2 - x_2x_1)^2$ is a central identity for 2×2 matrices. In general, a nonzero polynomial $f(x_1,\ldots,x_m)$ (with constant term zero) in the non-commuting indeterminates $x_1,\ldots,x_m$ is called a central identity for $n \times n$ matrices if $f(B_1,\ldots,B_m)$ is a scalar matrix for all $n \times n$ matrices $B_1,\ldots,B_m$ and $f(B_1,\ldots,B_m) \neq 0$ for some choice of $n \times n$ matrices $B_1,\ldots,B_m$. Kaplansky posed the problem of deciding whether central polynomials exist for $n \times n$ matrices, $n > 2$. This problem was solved in 1972 by Formanek and independently by Razmyslov, both of whom explicitly constructed (different) central polynomials for every $n > 2$. Razmyslov's construction is described in Cohn's book Algebra II (Wiley 1977), p.462. We now describe Formanek's construction.

Let $z_1,\ldots,z_{n+1}$ be distinct commuting indeterminates and let

$$g(z_1,\dots,z_{n+1}) = \left[\prod_{i=2}^{n}(z_1-z_i)(z_{n+1}-z_i)\right]\left[\prod_{\substack{i\ j=2\\ i\neq j}}^{n}(z_i-z_j)^2\right]$$

Then

$$g(z_1,\dots,z_{n+1}) = \sum_{(\nu)} a_{(\nu)} z_1^{\nu_1} z_2^{\nu_2} \dots z_{n+1}^{\nu_{n+1}}$$

for some integers $a_{(\nu)}$.

Let $x, y_1, \dots, y_n$ be distinct non-commuting indeterminates.
Put

$$f(x;\, y_1,\dots,y_n) = \sum_{(\nu)} a_{(\nu)} x^{\nu_1} y_1 x^{\nu_2} y_2 \dots x^{\nu_n} y_n\, x^{n+1}$$

(Thus f is constructed from g by first sticking in y's between the distinct z's and then replacing all the z's by x's).

Finally put

$$F(x;y_1,\dots,y_n) = f(x;y_1,\dots,y_n) + f(x;y_2,y_3,\dots,y_n,y_1)$$

$$+ f(x;y_3,y_4,\dots,y_n,y_1,y_2) + \dots$$

$$+ f(x;y_n,y_1,\dots,y_{n-1}).$$

Then F is a homogeneous polynomial of degree n^2 and Formanek (J. Algebra 23 (1972) 129-133) proved that F is a central polynomial for $n \times n$ matrices. It is clear that F is multilinear in $y_1,\dots,y_n$ and thus in proving the result one can assume that the y_i are replaced by matrices of the form $E_{s_i t_i}$. It can also be assumed that x is

replaced by diagonal matrix. Now the "discriminant-like" form of g makes the proof succeed.

If $X, Y_1, \dots, Y_n$ are $n \times n$ matrices of the form $\begin{pmatrix} U & \begin{matrix} 0 \\ \vdots \end{matrix} \\ 0 \;\cdots\; 0 & 0 \end{pmatrix}$ (i.e. the last row and column are zero), $F(X;Y_1,\dots,Y_n)$ being scalar can only be the zero-matrix. Thus $F(x;y_1,\dots,y_n)$ is a polynomial identity for $(n-1)\times(n-1)$ matrices but not for $n \times n$ matrices.

Several other examples are known now of polynomial identities and central identities for $n \times n$ matrices. Accounts can be found in Procesi's book "Rings with polynomial identities", Marcel-Dekker, 1973, Jacobson's notes "P.I. algebras" (Springer Lecture Notes in Mathematics # 441), and the recent papers of Amitsur and Rowen.

The existence of polynomial identities for matrix algebras has led to the development of a major theory of algebras satisfying a polynomial identity. In recent years the existence of central identities has led to very considerable simplifications and improvements in this theory and also a theory of forming central quotients (for details, see for example the books of Procesi and Jacobson above and the papers of Formanek and Rowen). In addition, they play a role in elucidating the structure of the generic division algebras constructed by Amitsur (Israel J. Math. 12 (1972), 408-420) which he has used to solve many outstanding problems - in particular, he has constructed finite dimensional division algebras which are not crossed products (i.e. they have no maximal subfields normal over their centres).

We conclude with an amusing identity. Let $a_1,\dots,a_t$ $b_1,\dots,b_t$ be integers and let S_t be the symmetric group on t letters. For $\sigma \in S_t$, let

$$\phi(\sigma) = a_{\sigma(2)}b_{\sigma(1)} + a_{\sigma(3)}(b_{\sigma(1)} + b_{\sigma(2)}) + \cdots$$

$$+ a_{\sigma(t)}(b_{\sigma(1)} + b_{\sigma(2)} + \cdots + b_{\sigma(t-1)}).$$

For each $k > 1$, let ω be a primitive kth root of one and let

$$u(a_1,\ldots,a_t;\ b_1,\ldots,b_t) = \sum_{\sigma\in S_t} (\text{sign } \sigma)\omega^{\phi(\sigma)}.$$

Then we have

$u(a_1,\ldots,a_t;\ b_1,\ldots,b_t) = 0$ for all choices of

$a_1,\ldots,a_t,\ b_1,\ldots,b_t$ if and only if $t \geq 2k$.

It can be shown that this identity is equivalent to the Amitsur-Levitski theorem, so it would be interesting to have an elementary proof of it, e.g. by expressing the right-hand-side as a determinant.

Mathematics in Europe in the 13th and 14th Centuries

F.J. Gaines

1. Fibonacci

The most celebrated mathematician of the Middle Ages in Europe was undoubtedly Leonardo of Pisa (alias Leonardo Bigollo, alias Fibonacci). His well-known problem on the breeding of rabbits, which leads to the Fibonacci numbers is contained in his Liber Abaci which first appeared in 1202. This rather boring book, in its 15 chapters deals with positional numerals and the basic arithmetical operations. It also discusses such matters as factorization into primes, fractions, numerical problems in geometry and problems in commercial arithmetic. The Liber Abaci is sometimes credited with introducing the Hindu-Arab system of numerals into Europe, but this is too facile an explanation of a complex historical problem. For example, Gerbert (Pope Sylvester II), who died in 1003, knew symbols for 1 to 9 and is credited with introducing these on markers on the abacus (apices) to help speed up calculations, but he did not know zero.

It may be suggested that the Liber Abaci was not a popular work - there is no evidence of its use in any of the Universities and it is perhaps significant that it did not appear in print until the 19th century. Possibly the two most popular works for spreading the new Hindu-Arab arithmetic were the Carmen de Algorismo of Alexandre Villedieu (c.1220) and John Sacrobosco's Algorismus Vulgaris. Another important 13th century work was the Arithmeticus