integers and let S_t be the symmetric group on t letters. For $\sigma \in S_t$, let

$$\phi(\sigma) = a_{\sigma(2)}b_{\sigma(1)} + a_{\sigma(3)}(b_{\sigma(1)} + b_{\sigma(2)}) + \dots$$

$$+ a_{\sigma(t)}(b_{\sigma(1)} + b_{\sigma(2)} + \dots + b_{\sigma(t-1)}).$$

For each $k > 1$, let ω be a primitive kth root of one and let

$$u(a_1,\dots,a_t;\ b_1,\dots,b_t) = \sum_{\sigma \in S_t} (\text{sign } \sigma)\omega^{\phi(\sigma)}.$$

Then we have

$u(a_1,\dots,a_t;\ b_1,\dots,b_t) = 0$ for all choices of $a_1,\dots,a_t,\ b_1,\dots,b_t$ if and only if $t \geq 2k$.

It can be shown that this identity is equivalent to the Amitsur-Levitski theorem, so it would be interesting to have an elementary proof of it, e.g. by expressing the right-hand-side as a determinant.

Mathematics in Europe in the 13th and 14th Centuries

F.J. Gaines

1. Fibonacci

The most celebrated mathematician of the Middle Ages in Europe was undoubtedly Leonardo of Pisa (*alias* Leonardo Bigollo, *alias* Fibonacci). His well-known problem on the breeding of rabbits, which leads to the Fibonacci numbers is contained in his *Liber Abaci* which first appeared in 1202. This rather boring book, in its 15 chapters deals with positional numerals and the basic arithmetical operations. It also discusses such matters as factorization into primes, fractions, numerical problems in geometry and problems in commercial arithmetic. The *Liber Abaci* is sometimes credited with introducing the Hindu-Arab system of numerals into Europe, but this is too facile an explanation of a complex historical problem. For example, Gerbert (Pope Sylvester II), who died in 1003, knew symbols for 1 to 9 and is credited with introducing these on markers on the abacus (apices) to help speed up calculations, but he did not know zero.

It may be suggested that the *Liber Abaci* was not a popular work - there is no evidence of its use in any of the Universities and it is perhaps significant that it did not appear in print until the 19th century. Possibly the two most popular works for spreading the new Hindu-Arab arithmetic were the *Carmen de Algorismo* of Alexandre Villedieu (c.1220) and John Sacrobosco's *Algorismus Vulgaris*. Another important 13th century work was the *Arithmeticus* of Jordanus de Nemore (d.1260). In this work he attempted to set up number theory in the axiomatic style of Euclid's geometry. The Arithmeticus was the basis of popular commentaries in the University of Paris up to the 16th century.

Fibonacci wrote five mathematical works that we know of. Part of his Practica Geometriae has been shown to be basically a work of Euclid. In his Flos ... ("The flower of solutions to certain problems concerning numbers and geometry") he shows that the cubic $x^3+2x^2+10x = 20$ cannot have a solution which is either rational or can be obtained by combinations of square or cube roots. He then gives (without explanation) the approximation 1.3688081075 (correct to 9 places) to the only real root. It is possible he may have used Horner's method, which was known to the Chinese at that time. In his Liber Quadratorum (c.1225) Fibonacci finds a rational solution to the pair of equations $x^2+5 = y^2$, $x^2-5 = z^2$. His solution is $x = 3\frac{5}{12}$, $y = 4\frac{1}{12}$, and $z = 2\frac{7}{12}$.

There is no evidence to connect Fibonacci with any of the universities, nor is there any evidence that any of his books were ever used as texts.

2. The 14th Century

Probably the two most capable mathematicians of the 14th century were Thomas Bradwardine at Oxford and Nicole Oresme at Paris.

It is said of Oxford in the 14th century that the principal subjects in the Faculty of Arts were logic and Mathematical Physics. Aristotle had claimed that if a body moved under a force F against a resistance R then its speed v would be proportional to F/R. Bradwardine showed this was incorrect and argued that αv is proportional to $(F/R)^\alpha$, where $\alpha = 1, 2^{\pm 1}, 3^{\pm 1}, \ldots$. This relationship, called "Bradwardine's function", although itself incorrect, was the subject of much study. Oresme at Paris in his discussion of Bradwardine's function was led to the law of exponents $x^m x^n = x^{m+n}$, with m,n rational. At Bradwardine's college, Merton, was produced the "Merton College Rule": the distance s covered by a body travelling in a straight line from rest with uniform acceleration a in time T is given by $s = \frac{1}{2}aT^2$ (modern notation!).

The Parisian scholars viewed their Oxford colleagues' work with some envious interest. Walter Burleigh at Oxford said: "The Parisian masters wrap up their doctrines in unskilled discourse and are losing all propriety of logic, except that our English subtleties, which they denounce in public, are the subject of their furtive vigils". Jean Buridan (1300-1360) developed a theory of impetus (= momentum). Oresme, as well as developing Bradwardine's ideas on motion developed a pictorial representation of motion with latitude" = time and "longitude" = speed. These pictures led some people to call Oresme the father of co-ordinate geometry. Oresme wrote on probability, infinite series (he showed the harmonic series diverges), philosophy and theology. He ended his days as Archbishop of Lisieux. Oresme was well-known in his day for his condemnation of astrology (using a probabilistic argument!). He was responsible for bringing some 200 terms into the Old French language.

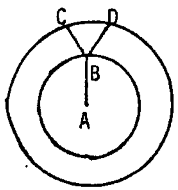
3. Mathematics in Philosophy and Theology

Mathematical arguments were used in the 14th century in discussing the problems of atomism and also of infinity, since Aristotle (whose complete works had not been available in Latin until this time) had shocked the medieval mind with his notion that the world had always existed. Bradwardine in his Tractatus de Continuo wrote (in the style of Euclid) the most complete refutation of 14th century atomism that we have.

A typical argument against the atomists is one due to John Duns Scotus (1266-1308):

If atoms exist they are identified with geometrical points.

Take two circles with centre A. Let C,D be adjacent atoms on the outer circle. Join AC,AD. If these lines always meet the inner circle at different points, then the outer and inner circles have the same number of atoms and hence have the same size. This contradiction implies AC and AD for some C,D meet the inner circle at B. But a tangent to the inner circle at B is perpendicular to both BC and BD. Thus we have two right angles, one greater than the other. Hence a circle is not composed of indivisibles.

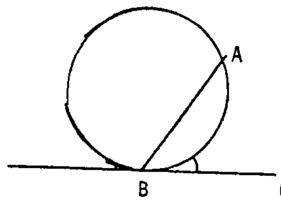


Henry of Harclay (1275-1313) was an atomist who also discussed the problem of infinities. If, as Aristotle says, the world always existed, time up to today can be viewed as a semi-infinite line segment. But this line segment can be superimposed on the line segment representing time up to yesterday. Thus the part equals the whole, contradicting a basic Euclidean

notion. Nicole Oresme and Albert of Saxony argued that the rules which apply to finite quantities do not necessarily apply to infinite ones. We note that arguments on parts, wholes and infinities also occur at times in Greek, Roman and Arab writings.

4. The Horn Angle

The authoritative medieval translation of Euclid's Elements into Latin was made from the Arabic by Johannes Campanus of Novara (c.1260). This was in fact the first edition of Euclid to be printed in 1482. In his edition of Euclid Campanus (and other writers elsewhere) discussed the concept of the horn angle or angle of contact, i.e. the angle between the tangent to a circle and the curve of the circle. No matter how finely the rectlinear angle ABC is subdivided, we never obtain an angle smaller than the horn angle.



This contradicts Euclid X, Prop.1, but Campanus realised that Euclid's proposition must apply only to magnitudes of the same type. Of course, the horn angle can be dealt with in the context of curvature.

5. Conclusion

It is hoped that in this note we have shown that in the 13th and 14th centuries, mathematics was not a complete wasteland, but that a number of significant ideas were discussed, even if their complete explanation had to wait for later times.

Bibliography

1. T. Bradwardine, *Tractatus de proportionibus*, translated by H.L. Crosby, Jr. (Madison, 1955).

2. E. Grant (Ed.), *A Source Book in Medieval Science* (Harvard University Press, 1974).

3. N. Oresme, *De proportionibus proportionum* and *Ad pauca respicientes*, translated by E. Grant (Madison, 1966).

PROBLEM SECTION

It is hoped to include in each issue of the Newsletter a set of problems of general mathematical interest. Readers are invited to submit problems for inclusion in this section. It is envisaged that the problems posed should be intelligible to (though not necessary soluble by) people with a degree in Mathematics. It is hoped that most of the problems posed should be soluble and it is intended to publish solutions to those in subsequent issues of the Newsletter. Readers are invited to submit solutions to the problems posed for consideration for inclusion in the Newsletter. Correct solutions will be acknowledged in the Newsletter.