Special values of Legendre’s chi-function and the inverse tangent integral

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ABSTRACT. In our recent publication in this Bulletin [88 Winter (2021), 31–37] a series transform proved via Fourier–Legendre theory and fractional operators in a 2022 article was applied to prove five two-term dilogarithm identities. One such identity gave a closed form for \( \text{Li}_2(\sqrt{2} - 1) - \text{Li}_2(1 - \sqrt{2}) \), and we had attributed this closed form to a 2012 article by Lima. However, as we review in our current article, there had actually been a number of previously published proofs of formulas that are equivalent to the closed-form evaluation for the equivalent expression \( \chi_2(\sqrt{2} - 1) \), letting \( \chi_2 \) denote the Legendre chi-function. We offer a brief survey of the history of special values for \( \chi_2 \) and the inverse tangent integral \( Ti_2 \), in relation to the results given in our previous BIMS publication. Two of the two-term dilogarithm relations proved in this previous publication were actually introduced in 1915 by Ramanujan in an equivalent form in terms of the \( Ti_2 \) function, which adds to the interest in the alternative proofs for these results that we had independently discovered. We also apply special values for \( \chi_2 \) and \( Ti_2 \), together with a Legendre-polynomial based series transform, to obtain evaluations for rational double hypergeometric series with invaluable single sums.

1. INTRODUCTION

In the 2022 article [8], the series transform reproduced as equation (2) in [7] was proved using Fourier–Legendre (FL) theory and fractional calculus, building on an FL-based integration method introduced in the 2019 research article [10]. Using this series transform from [8] together with the generating function for Legendre polynomials, we had proved in [7] five two-term dilogarithm evaluations. These five evaluations are reproduced below. We had incorrectly stated that the first out of the five equations listed below was introduced by Lima in 2012 [18], without our having been aware that an equivalent formulation of this first equation was given in terms of the Legendre chi-function in the 1958 text [15, p. 19]. Lima proved (1) in [18] and one of the main results in [18] follows from (1), but the fact that (1) was previously known, as far back as 1958 [15, p. 19], was not indicated anywhere in [18] or in the zbMATH review [2] of [18] (cf. [11]). Furthermore, while our method for proving the below results using Legendre polynomials is highly original, all of the five formulas below had been known prior to [7], without the author having been aware of this; see [21], [15, p. 19] and [12].

\[
\begin{align*}
\text{Li}_2 (\sqrt{2} - 1) - \text{Li}_2 (1 - \sqrt{2}) &= \frac{\pi^2}{8} - \frac{1}{2} \ln^2 (1 + \sqrt{2}) \\
\text{Li}_2 \left( \frac{1}{\phi^3} \right) - \text{Li}_2 \left( -\frac{1}{\phi^3} \right) &= \frac{\phi^3 (\pi^2 - 18 \ln^2 (\phi))}{3 (\phi^6 - 1)}
\end{align*}
\]
\[
\begin{align*}
\text{Li}_2 \left( i \left( 2 - \sqrt{3} \right) \right) &- \text{Li}_2 \left( -i \left( 2 - \sqrt{3} \right) \right) = \frac{2i \sqrt{7 - 4\sqrt{3}} \left( 8G - \pi \ln \left( 2 + \sqrt{3} \right) \right)}{3 \left( 8 - 4\sqrt{3} \right)} \quad (3) \\
\text{Li}_2 \left( i \left( \sqrt{2} - 1 \right) \right) &- \text{Li}_2 \left( -i \left( \sqrt{2} - 1 \right) \right) \\
&= \frac{1}{32} \left( \sqrt{2} \left( \psi^{(1)} \left( \frac{1}{8} \right) + \psi^{(1)} \left( \frac{3}{8} \right) \right) + 8\pi \ln \left( \sqrt{2} - 1 \right) - 4\sqrt{2}\pi^2 \right) \\
\text{Li}_2 \left( \frac{i}{\sqrt{3}} \right) &- \text{Li}_2 \left( -\frac{i}{\sqrt{3}} \right) = \frac{i \left( 3\psi^{(1)} \left( \frac{1}{8} \right) + 15\psi^{(1)} \left( \frac{1}{3} \right) - 6\sqrt{3}\pi \ln(3) - 16\pi^2 \right)}{36\sqrt{3}}. \quad (5)
\end{align*}
\]

Also, a different formulation of the main transform from our recent article [7] was included in an unpublished online note [23] from 2000, but was proved differently; also, a different formulation of this same result was given by Bradley in [3], and proved in much the same way as in [23]. The above identities for the dilogarithmic expressions in (3) and (4) had been given by Ramanujan in 1915 [1, 21] in an equivalent form in terms of the special function known as the inverse tangent integral Ti_2. Ramanujan’s approach toward evaluating (3) and (4) was very different compared to our Legendre polynomial-based proofs for equivalent evaluations [7], which further motivates the application of our methods from [7]. As indicated in Section 2.2 below, there have actually been a number of previously published proofs of identities equivalent to (1) [4, 5, 22].

The corrections to our publication [7] covered above motivate the brief survey offered in Section 2 on past literature concerning the above evaluations for the two-term dilogarithm combinations in (1), (2), (3), and (4), relative to the methods and results from [7].

**Remark 1.1.** Subsequent to the publication of [7], the five dilogarithmic identities indicated in (1)–(5) were reproduced in the Wolfram MathWorld encyclopedia entry on the dilogarithm function [25], with [7] cited as a Reference for these identities. This same MathWorld entry [25] contains links to the corresponding encyclopedia entries on the inverse tangent integral [26] and Legendre’s chi-function [14], and this led the author to discover that equivalent formulas for the values in (1)–(4) had been previously recorded in mathematical literature prior to both [7] and [18]; this, in turn, had inspired the author to explore the history of special values for \( \chi_2 \) and Ti_2 in relation to the material in [7] and [18], culminating in the survey offered in Section 2 below.

### 2. Survey

#### 2.1. The Legendre chi-function.

The special function known as Legendre’s chi-function is defined as follows [14]:

\[
\chi_\nu(z) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)^\nu}.
\]

From the above definition, it is immediate that

\[
\chi_\nu(z) = \frac{1}{2} \left( \text{Li}_\nu(z) - \text{Li}_\nu(-z) \right).
\]

So, we see that the left-hand sides of (1) and (2) may be naturally expressed with the \( \chi \)-function. As it turns out, the identities

\[
\chi_2 \left( \sqrt{2} - 1 \right) = \frac{1}{16} \pi^2 - \frac{1}{4} \ln^2 \left( \sqrt{2} + 1 \right) \quad (6)
\]

and

\[
\chi_2 \left( \sqrt{5} - 2 \right) = \frac{1}{24} \pi^2 - \frac{3}{4} \ln^2 \left( \frac{\sqrt{5} + 1}{2} \right), \quad (7)
\]
which are easily seen to be equivalent to (1) and (2), respectively, were previously known [14] [15, p. 19], prior to the publication of [7]. New identities involving the Legendre chi-function were recently given in [24], in which the classical identity
\[ \chi_2 \left( \frac{1-x}{1+x} \right) + \chi_2(x) = \frac{3\zeta(2)}{4} + \frac{1}{2} \ln(x) \ln \left( \frac{1+x}{1-x} \right) \]
is reproduced from the classic text [16]. We see that (6) follows directly from the identity for \( \chi_2 \left( \frac{1-x}{1+x} \right) + \chi_2(x) \) given above, and this same identity may be used in a direct way to prove (7). The foregoing considerations add to the interest in the new and Legendre polynomial-based alternate proofs of (6) and (7) given in [7]. The evaluations in (6) and (7) are also reproduced in [23], again with reference to Lewin’s text [16]. The formulas in (6) and (7) are well-known and were recently noted [20] in the context of applications related to the special function known as the Barnes G-function.

2.2. Landen’s identity and the Rogers L-function. One of the main results in [18], as highlighted in the title of [18] and in the corresponding zbMATH review [2], is as given below:
\[ \text{Li}_2 \left( \sqrt{2} - 1 \right) + \text{Li}_2 \left( 1 - \frac{1}{\sqrt{2}} \right) = \frac{\pi^2}{8} - \frac{\ln(1 + \sqrt{2})}{2} - \frac{1}{8} \ln^2 2. \tag{8} \]
However, this follows in a direct way from (1) together with the famous Landen identity
\[ \text{Li}_2(z) = -\text{Li}_2 \left( \frac{z}{z-1} \right) - \frac{1}{2} \ln^2(1 - z), \]
but it is not indicated in [18] or its reviews [2, 11] that (1) was previously known in an equivalent way via the Legendre chi-function, as far back as Lewin’s classic 1958 text [15, p. 19]. The article [18] was the main inspiration behind our publication in [7], but it is suggested in [18] that (1) was introduced in Lima’s 2012 article in [18]. Part of the reason as to the confusion concerning the origins of identities as in (1) is due to a number of different special functions and notational conventions that have been used to express such identities, with reference to the \( \chi_\nu \)-function defined above, along with the \( \text{Ti}_2 \)-function defined below and the different definitions/notations for the Rogers dilogarithm function indicated below. Again, our published proof of (1) [7], which relied on a fractional calculus-derived transform from the 2022 article [8], is original, as is the case with our proofs in [7] of the above symbolic forms for (2), (3), (4), and (5).

The fact that the formula in (8) that was highlighted as a main result in [18] and presented as being new in Lima’s paper [18] follows directly from Landen’s identity together with the classically known evaluation in (1) recorded in the 1958 text [15, p. 19] has not been noted in any past literature citing [18], including [13, 17, 19]. Letting
\[ L(x) = \frac{6}{\pi^2} \left( \sum_{n=1}^{\infty} \frac{x^n}{n^2} + \frac{1}{2} \ln x \ln(1 - x) \right) \]
denote the normalized Rogers dilogarithm function, in the 1999 article [5], it was noted that an equivalent formulation of the above equation for \( \text{Li}_2 \left( \sqrt{2} - 1 \right) + \text{Li}_2 \left( 1 - \frac{1}{\sqrt{2}} \right) \) follows in a direct way from the identity
\[ L(x) + L(1 - x) = 1 \tag{9} \]
together with Abel’s duplication formula, which follows from Abel’s functional equation
\[ L(x) + L(y) = L(xy) + L \left( \frac{x(1-y)}{1-xy} \right) + L \left( \frac{y(1-x)}{1-xy} \right). \]
This is also noted in [18]. So, we find that the formula in (1), which traces back to the 1958 text [15, p. 19], may also be proved using the functional relations for the Rogers dilogarithm given in (9) together with Abel’s duplication formula and Landen’s identity. This provides a remarkably different proof compared to our Legendre polynomial-based proof of (1) that we had introduced in [7].

Using the alternative notation/definition

\[ L_R(x) = \text{Li}_2(x) + \frac{1}{2} \ln x \ln(1 - x) \]

for the Rogers \( L \)-function indicated in [27], the formula

\[ L_R\left(2 - \sqrt{2}\right) - L_R\left(\frac{2 - \sqrt{2}}{2}\right) = \frac{\pi^2}{24} \]

was proved in 1981 in [22] through the use of the Rogers–Ramanujan and the Andrews–Gordon identities. Using the functional relation in (9), this can be used to produce yet another proof of (1).

Bytsko [4] proved the identity

\[ L_R\left(1 - \frac{1}{\sqrt{2}}\right) + L_R\left(\sqrt{2} - 1\right) = \frac{\pi^2}{8} \]  \hspace{1cm} (10)

as the \( k = 2\) case of the formula

\[ \sum_{i=1}^{k-1} L_R\left(\frac{\sin^2\frac{\pi}{3k+2}}{\sin^2\frac{(i+1)\pi}{3k+2}}\right) + L\left(\frac{\sin\frac{\pi}{3k+2}}{\sin\frac{(k+1)\pi}{3k+2}}\right) = \frac{\pi^2}{6} \frac{3k}{3k + 2} \]

given in [4]; we see that (10) is equivalent to (8), which, as indicated above, is equivalent to (1).

2.3. **Ramanujan’s inverse tangent integral.** Integrals of the form

\[ \text{Ti}_2(x) = \int_0^x \frac{\arctan t}{t} \, dt \]

were of interest to Ramanujan, and remarkable results on the special function \( \text{Ti}_2 \) defined above were given in his 1915 article [21] (cf. [1, §17], [26]). From the series expansion

\[ \text{Ti}_2(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{(2k - 1)^2}, \]

we obtain that

\[ \text{Ti}_2(x) = \frac{1}{2i} \left(\text{Li}_2(ix) - \text{Li}_2(-ix)\right). \]

So, we find that the expressions in (3), (4), and (5) are naturally expressible as specific values of \( \text{Ti}_2 \). Ramanujan introduced the identity whereby

\[ \sum_{n=0}^{\infty} \frac{\sin(4n + 2)x}{(2n + 1)^2} = \text{Ti}_2(\tan x) - x \ln \tan x \]  \hspace{1cm} (11)

for \( 0 < x < \frac{1}{4} \pi \), and noted that this may be proved by applying term-by-term differentiation to the above series [21] (cf. [1, §17]). A corrected version [1, p. 365] of Ramanujan’s formula for \( \text{Ti}_2(\sqrt{2} - 1) \) is such that:

\[ \sum_{n=0}^{\infty} \frac{(-1)^{n(n-1)/2}}{(2n + 1)^2} = \sqrt{2} \text{Ti}_2\left(\sqrt{2} - 1\right) + \frac{\pi}{4\sqrt{2}} \ln(1 + \sqrt{2}). \]  \hspace{1cm} (12)
Also, from Ramanujan’s identity in (11), we obtain that
\[ Ti_2(1) = \frac{3}{2} Ti_2(2 - \sqrt{3}) + \frac{1}{8} \pi \ln(2 + \sqrt{3}), \] (13)
and we find that the above equalities due to Ramanujan in 1915 [21] (cf. [1, §17]) are equivalent to our formulas for (3) and (4), which we had proved in a completely different way in [7]. Ramanujan’s formulas in (12) and (13) were recently noted in [20], again in the context of applications pertaining to the Barnes G-function. Our discovery presented in [7] given by the equality in (5) may be rewritten so that
\[ Ti_2 \left( \frac{1}{\sqrt{3}} \right) = 3 \psi^{(1)} \left( \frac{1}{6} \right) + 15 \psi^{(1)} \left( \frac{1}{3} \right) - 6 \sqrt{3} \pi \ln(3) - 16 \pi^2. \] (14)
This can also be proved using Ramanujan’s identity
\[ \sum_{n=0}^{\infty} \frac{\left( \frac{1}{2} \right)_n \cos^{2n+1} x + \sin^{2n+1} x}{n! \left( 2n + 1 \right)^2} = Ti_2(\tan x) + \frac{1}{2} \pi \ln(2 \cos x) \]
for \( 0 < x < \frac{1}{2} \pi [21] \), but this is nontrivial in the sense that plugging \( x = \frac{\pi}{6} \) into the above series produces a linear combination of the hypergeometric series
\[ _3F_2 \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \mid \frac{1}{4}, \frac{1}{4} \right] \quad \text{and} \quad _3F_2 \left[ \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \mid \frac{3}{2}, \frac{3}{2} \right]. \]
which computer algebra systems such as Maple 2020 are not able to evaluate.

2.4. Sherman’s and Bradley’s formulas. The main transform from [7], our proof of which relied on results from our 2022 article [8], is such that
\[ \frac{1}{1 + z} \sum_{n=0}^{\infty} \frac{\left( \frac{16z}{(1 + z)^2} \right)_n}{\left( 2n + 1 \right)^2 \left( 2n \right)_n} = \operatorname{sgn}(z) \frac{i \left[ \text{Li}_2(-\sqrt{-z}) - \text{Li}_2(\sqrt{-z}) \right]}{2\sqrt{z}} \] (15)
holds if both sides converge for real \( z \). Our proof of this in [7] relied on the generating function for Legendre polynomials together with a fractional calculus-derived series transform from the 2022 article [8]. A different formulation of this result was given in an unpublished note by Sherman in 2000 [23]. In [23], by integrating the Maclaurin series expansion
\[ \sum_{n=0}^{\infty} \frac{1}{(2n)_n} \frac{(2x)^n}{2n + 1} = \frac{\arcsin \sqrt{x}}{\sqrt{x(1 - x)}}, \]
it was shown that
\[ \sum_{n=0}^{\infty} \frac{1}{(2n)_n} \frac{(2x)^n}{(2n + 1)^2} \]
is expressible as a linear combination of
\[ \chi_2 \left( e^{i \arcsin \sqrt{x}} \right) \]
and elementary expressions, in contrast to our identity in (15) [7]. It appears that our dilogarithm transform identity indicated in [7, p. 36] had not been considered previously. With regard to our formula in (15) and its derivation in [7], the following closely related formula was proved in a different way in [3]:
\[ \int_0^x \ln(\tan \theta) \, d\theta = x \ln x - \frac{1}{4} \sum_{k=0}^{\infty} \frac{(2 \sin 2x)^{2k+1}}{(2k + 1)^2 \binom{2k}{k}}, \] (16)
Bradley [3] also showed that

\[ L(2, \chi_6) = \pi \sqrt{3} \ln 3 + \frac{1}{2} \sum_{k=0}^{\infty} \frac{3^k}{(2k+1)^2 \binom{2k}{k}^2}, \]

which, together with (16), can be used to give an equivalent formulation of (14), where the expression \( \chi_6 \) denotes the non-principal Dirichlet character modulo 6. This is shown using an equivalent formulation of Ramanujan’s 1915 identity in (11) together with (16), in contrast to our methods from [7].

An evaluation for \( T_i \left( \sqrt{3} \frac{2}{3} \right) \) was also given in 1984 in [12], using a previously known relation [16, p. 106] involving \( T_i \) and the special function known as the Clausen integral.

3. Double series

We conclude by briefly considering how the special values for \( \chi_2 \) and \( T_i \) considered in this article may be applied using our previous work on double series [6, 9]. As a special case of a hypergeometric transform introduced in [6] using the FL-based evaluation technique from [10], it was shown that: For a suitably bounded parameter \( p \),

\[ \frac{\pi}{2} \sum_{m,n \geq 0} \left( \frac{1}{16} \right)^m \left( \frac{1}{20} \right)^n \sum_{k=0}^{\infty} \frac{\left( \frac{2m}{m} \right)^2 \left( \frac{2n}{n} \right)^2}{m+n+1} = \sum_{m,n \geq 0} \left( \frac{1}{16} \right)^m \left( \frac{1}{20} \right)^n \sum_{k=0}^{\infty} \frac{\left( \frac{2m}{m} \right)^2 \left( \frac{2n}{n} \right)^2}{m+n+1} \]

(17)
equals

\[ -1 \times \left( \mathrm{Li}_2 \left( \frac{1}{2} \sqrt{1 - 4p} \right) \right) - \mathrm{Li}_2 \left( \frac{1}{2} \sqrt{1 - 4p + 1} \right). \]

In [9], we had applied this identity for (17) together with the known closed form for \( \mathrm{Li}_2(\sqrt{2} - 1) - \mathrm{Li}_2(1 - \sqrt{2}) \) to obtain new bivariate hypergeometric series evaluations. Setting \( p = \frac{1}{12} \) in (17) and using the closed form in (2), we obtain the remarkable formula

\[ \sum_{m,n \geq 0} \left( \frac{1}{16} \right)^m \left( \frac{1}{20} \right)^n \sum_{k=0}^{\infty} \frac{\left( \frac{2m}{m} \right)^2 \left( \frac{2n}{n} \right)^2}{m+n+1} = \frac{5\pi}{3} - 6 \ln^2(\phi) \]

Summing over \( n \in \mathbb{N}_0 \), we obtain an inevaluable \( 2F_1 \left( \frac{1}{2} \right) \)-series; summing over \( m \in \mathbb{N}_0 \), we obtain a \( 3F_2(1) \)-series with no closed form. Similarly, by setting \( p = -\frac{1}{12} \) in (17) and using Ramanujan’s formula in (13), we may obtain that

\[ \sum_{m,n \geq 0} \left( \frac{1}{16} \right)^m \left( \frac{1}{12} \right)^n \sum_{k=0}^{\infty} \frac{\left( \frac{2m}{m} \right)^2 \left( \frac{2n}{n} \right)^2}{m+n+1} = \frac{2G}{\sqrt{3} \pi} - \frac{2 \ln(2 + \sqrt{3})}{\sqrt{3}} \]

Summing over \( n \in \mathbb{N}_0 \), we obtain an inevaluable \( 2F_1 \left( -\frac{1}{2} \right) \)-series; summing over \( m \in \mathbb{N}_0 \), we again obtain a \( 3F_2(1) \)-series that does not admit any closed form. We leave it to a separate project to pursue a full exploration of the application of the techniques from [6, 9] together with the special values for \( \chi_2 \) and \( T_i \) considered in this article.

References


Special values of $\chi_2$ and $T_{12}$


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