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PROBLEMS

IAN SHORT

Problems

The first problem this issue is due to Yagub Aliyev of ADA University, Azerbaijan. A similar problem attracted popular interest in Azerbaijan during 2020.

Problem 86.1. Find the nearest integer to

$$10^{2021} - \sqrt{(10^{2021})^2 - 10^{2021}}.$$

The second problem was proposed by Seán Stewart of Bomaderry, Australia.

Problem 86.2. Evaluate

$$\int_0^1 \frac{1}{x} \arctan\left(\frac{2rx}{1+x^2}\right) dx,$$

where r is a real constant.

The third problem comes from Finbarr Holland of University College Cork.

Problem 86.3. Prove that

$$\sum_{n=0}^{\infty} \frac{9n+5}{9n^3+18n^2+11n+2} = 3\log 3.$$

Solutions

Here are solutions to the problems from *Bulletin* Number 84.

The first problem in Issue 84 was a corrected version of Problem 82.1, which was missing some hypotheses. The problem uses the usual notation $x_1, x_2, \ldots, x_n$ for the components of a vector x in $\mathbb{R}^n$. It was solved by the North Kildare Mathematics Problem Club and the proposer, Finbarr Holland. We present a version of Finbarr's solution with modifications from the Problem Club.

Problem 84.1. Suppose that u and v are linearly independent vectors in $\mathbb{R}^n$ with

$$0 < u_1 \leqslant u_2 \leqslant \cdots \leqslant u_n \quad \text{and} \quad v_1 > v_2 > \cdots > v_n > 0.$$

Given $x \in \mathbb{R}^n$, let y be the orthogonal projection of x onto the subspace spanned by u and v; thus $y = \lambda u + \mu v$, for uniquely determined real numbers λ and μ. Prove that if

$$x_1 > x_2 > \cdots > x_n > 0,$$

then μ is positive.

Solution 84.1. A straightforward calculation shows that

$$\mu = \frac{x \cdot w}{|w|^2}, \quad \text{where} \quad w = v - \frac{u \cdot v}{|u|^2} u.$$

Thus it suffices to show that

$$|u|^2 (v \cdot x) > (u \cdot x)(u \cdot v).$$

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Let us now define

$$a_i = \frac{x_i}{u_i}, \quad b_i = \frac{v_i}{u_i}, \quad c_i = u_i^2, \quad \text{for } i = 1, 2, \dots, n,$$

the coordinates of vectors a, b and c. Then

$$\begin{aligned}
|u|^2(v \cdot x) - (u \cdot x)(u \cdot v) &= \sum_{i=1}^{n} c_i \sum_{j=1}^{n} a_j b_j c_j - \sum_{i=1}^{n} a_i c_i \sum_{j=1}^{n} b_j c_j \\
&= \sum_{i,j=1}^{n} c_i c_j (a_j b_j - a_i b_j) \\
&= \tfrac{1}{2} \sum_{i,j=1}^{n} c_i c_j (a_j b_j - a_i b_j + a_i b_i - a_j b_i) \\
&= \tfrac{1}{2} \sum_{i,j=1}^{n} c_i c_j (a_i - a_j)(b_i - b_j).
\end{aligned}$$

Observe that $a_1 > a_2 > \cdots > a_n > 0$ and $b_1 > b_2 > \cdots > b_n > 0$. Hence

$$c_i c_j (a_i - a_j)(b_i - b_j) \geqslant 0, \quad \text{for } i, j = 1, 2, \dots, n,$$

with equality if and only if $i = j$. Thus $|u|^2(v \cdot x) - (u \cdot x)(u \cdot v) > 0$, as required. □

The next problem was solved by JP McCarthy of the Cork Institute of Technology, the North Kildare Mathematics Problem Club and the proposer, Finbarr Holland. We present JP's solution.

Problem 84.2. Given any finite collection $L_1, L_2, \dots, L_n$ of infinite straight lines in the complex plane, find a formula in terms of data specifying $L_1, L_2, \dots, L_n$ for a differentiable function $f\colon \mathbb{R} \longrightarrow \mathbb{C}$ with the property that each line L_i is tangent to the curve $f(\mathbb{R})$.

The following solution assumes that none of the lines L_j are vertical; it can easily be adjusted to deal with the omitted special cases.

Solution 84.2. Each line L_j has a parametrization

$$\ell_j(t) = t + i(m_j t + c_j), \quad \text{for } t \in \mathbb{R}.$$

For $j = 2, 3, \dots, n$ and $t \in \mathbb{R}$, we define

$$\begin{aligned}
\phi_j(t) = (1-t)^3(1 + i(m_{j-1} + c_{j-1})) &+ 3(1-t)^2 t(2 + i(2m_{j-1} + c_{j-1})) \\
&+ 3(1-t)t^2(-1 + i(-m_j + c_j)) + t^3 i c_j.
\end{aligned}$$

This is the cubic Bézier curve from the point on L_{j-1} with real part 1 to the point on L_j with real part 0. It uses the point on L_{j-1} with real part 2 and the point on L_j with real part -1 to match the slopes of L_{j-1} and L_j at $t = 0$ and $t = 1$.

By construction, the function $f\colon \mathbb{R} \to \mathbb{C}$ that, for $j = 2, 3, \dots, n$, satisfies

$$f(x) = \begin{cases} \ell_1(x) & \text{if } x < 1, \\ \phi_j(x - (2j-3)) & \text{if } 2j-3 \leqslant x < 2j-2, \\ \ell_j(x - (2j-2)) & \text{if } 2j-2 \leqslant x < 2j-1, \\ \ell_n(x - (2n-2)) & \text{if } x \geqslant 2n-1, \end{cases}$$

has the desired properties. □

The third problem was solved by the North Kildare Mathematics Problem Club, and it is their solution presented here.

Problem 84.3. Suppose that each edge of a finite directed graph G is coloured in one of some finite collection of different colours, with the property that for each colour c and vertex v, there is precisely one directed edge with colour c and target vertex v. Prove that for any infinite sequence of colours $c_1, c_2, \dots$ there is an infinite walk $e_1, e_2, \dots$ of directed edges of G such that, for each index i, e_i has colour c_i and the target vertex of e_i equals the source vertex of e_{i+1}.

Solution 84.3. (We assume G has a nonempty vertex set V and there is at least one colour.) The finite edge set E is nonempty. We give it the discrete topology, give $E^{\mathbb{N}}$ the product topology, and give the set $W \subset E^{\mathbb{N}}$ of infinite walks the relative topology. Observe that W is compact, since it is a closed subset of the compact Hausdorff space $E^{\mathbb{N}}$. Note also that $E^{\mathbb{N}}$ is metrizable, and a sequence (w_n) of walks converges if and only if for each $i \in \mathbb{N}$ the sequence formed by taking the ith edge $w_n(i)$ of w_n, $n = 1, 2, \dots$, is eventually constant.

Let $c_1, c_2, \dots$ be a given sequence of colours. For any $n \in \mathbb{N}$ and each vertex $v \in V$ we see by working backwards that there is a finite walk $e \in E^{\{1,2,\dots,n\}}$ ending at v such that e_i has colour c_i for $1 \leqslant i \leqslant n$. By taking n greater than the order $|V|$ of V we see that G contains a cycle.

If we remove terminal vertices and the edges to those vertices from G, we are left with a nonempty graph having the same property – nonempty because it will contain each loop. Repeating the process at most $|V|$ times, we obtain a graph without terminal vertices having the same property. Thus we may assume without loss in generality that G has no terminal vertices. Then each finite walk may be continued to some infinite walk. In particular, each set

$$K_n = \{w \in W : w(i) \text{ has colour } c_i,\ \forall i \leq n\}$$

is nonempty. Moreover $K_{n+1} \subset K_n$ for each $n \in \mathbb{N}$, and each K_n is closed and hence compact. Thus

$$K = \bigcap_{n=1}^{\infty} K_n \neq \varnothing,$$

and any element $w \in K$ is an infinite walk having colour sequence $c_1, c_2, \dots$. □

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer Latex). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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