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Donald E. Marshall: Complex Analysis, Cambridge University Press, 2019. ISBN:978-1-107-13482-9, GBP 49.99, 286pp.

REVIEWED BY JAMES E. BRENNAN

Prior to introducing a new subject certain questions always arise, such as what should be emphasized, and where should the presentation begin? In the preface to the book under review the author identifies four points of view each central to the subject, and associated with Cauchy, Weierstrass, Riemann, and Runge, respectively. Each individual point of view offers a possible starting point where the emphasis would be on:

1. Cauchy: functions having a complex derivative, and integral formulas;
2. Weierstrass: functions locally expressible as a power series;
3. Riemann: functions or mappings which preserve angles, a more geometric viewpoint;
4. Runge: functions that can be expressed as the limits of rational functions.

As indicated by the author, the seminal text in this area was written by Ahlfors [1] and stresses Cauchy's viewpoint, while most subsequent texts have followed that lead. Marshall, on the other hand has chosen Weierstrass' point of view and to begin with functions locally expressible as a power series. That approach leads almost immediately to a feature of complex analytic functions not found anywhere in real function theory; namely, to the distinctive property of unique continuation.

Prior to engaging in such such details there is a nice introduction to complex numbers, and the historical events that led to these seemingly imaginary quantities being taken seriously. The initial impetus was the publication of the *Ars Magna* by Cardano in 1545 in which a complete algebraic solution of the depressed cubic

$$x^3 + px + q = 0$$

was presented for the first time, the roots being given by the formula

$$x = \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{1/3} + \left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{1/3}.$$

The first anomaly appeared about 30 years later when Bombelli drew attention to the fact that the equation $x^3 - 15x - 4 = 0$ has threes distinct real roots ($x = 4$ is one), but in terms of Cardano's formula they are expressed as

$$x = (2 + 11i)^{1/3} + (2 - 11i)^{1/3},$$

which clearly involves complex quantities. Here one might have thought that there should be another formula that avoids this difficulty, but almost 270 years elapsed until in 1891 Hölder, making use of Galois theory and the concept of a normal field extension, proved that there can be no formula expressing the roots of the general cubic that does not pass through the complex domain (cf. [2, pp. 450-453]). I have always felt due to its impact on the subsequent development of complex function theory that this is at least as

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important and interesting as the fact that there can be no similar formula expressing the roots of an equation of degree 5 or higher. Other texts mention Bombelli's example as providing evidence that complex numbers must be taken seriously, but to my knowledge Marshall's is the first to acknowledge Hölder's contribution to closing the door on any possibility of avoiding complex numbers. There is another oddity in connection with Cardano's formula that needs to be considered. In particular if the cube roots appearing in the formula are allowed to be specified in all possible ways, then the formula predicts more roots than a cubic can have. This is dealt with in exercise 1.9 on p. 11 in which the student is carefully led in a series of steps to rederive Cardano's formula, where in that process of doing so it becomes clear how the appropriate branch is to be selected. There are a number of such exercises where the student is encouraged with guidance to experience a bit of the joy of discovery.

Before turning to the study of analytic functions proper, Chapter 2 opens with an elegant proof of the fundamental theorem of algebra, a subject usually taken up much later in most texts, and is then based on Lioville's theorem to the effect that a bounded entire function is constant. The proof presented here, however, was first suggested by d'Alembert in 1746 and depends on the fact that a continuous positive function on a compact set attains a minimum, a fact unproven at the time. Since then, of course, the gap in d'Alembert's proof has been filled (cf. [6, p. 266]). At this point an analytic function is formally defined as a function locally expressible as a convergent power series, and the principle of unique continuation is established, along with certain basic properties such as the sum, product and composition of analytic functions are again analytic. Evidentally, certain functions such as e^x, $\sin x$, and $\cos x$ can be extended analytically to the entire complex plain, but the question remains as to how (or whether) other elementary functions such as

$$\log x,\ \sqrt{x},\ x^{4/3},\ldots$$

can be extended analytically from the real line into the plane. That question is first addressed on p. 29 (Ex. 2.10) along with some hints in connection with extending the function $x^{1/n}$ provided that n is a positive integer. Although it is not mentioned in the text, there is at this point sufficient information available to proceed directly to extending $\log x$ from the interval $\{x : |x-1| < 1\}$ to the disc $\{z : |z-1| < 1\}$ by setting

$$\log z = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(z-1)^n.$$

Since it has already been established that the composition of analytic functions is also analytic one can simply set $z^{1/n} = e^{(1/n)\log z}$, and this is the desired unique analytic extension since equality is clearly satisfied on an interval in the real line.

Having begun from the point of view of power series, within 30 pages there is already a rich collection of analytic functions which can serve as examples from which to infer what might be true in more general situations. From here the discussion moves quickly into the heart of complex function theory to such topics as: the maximum principle, the local behavior of analytic functions, contour integration, Cauchy's theorem, Runge's theorem on rational approximation, the argument principle, and so forth. Along the way it is shown that if a function is continuously differentiable in an open set Ω, then it is in fact analytic. Goursat's theorem to the effect that continuity can be dropped and that the mere existence of a derivative throughout an open set is sufficient for analyticity is another of those instances where the student is encouraged in an exercise, along with hints, to fill in an important gap (cf. p. 62, Ex. 4.12). As a kind of sequel on p. 121 a function f is defined to be *weakly-analytic* in a region Ω if

$$\int_{\Omega} f \frac{\partial \varphi}{\partial \bar{z}}\, dA = 0$$

for all continuously differentiable functions φ defined on Ω, where dA denotes two-dimensional Lebesgue (or area) measure. The problem for the student is to verify *Weyl's lemma*, which states that a function is weakly-analytic if, and only if, it is analytic, and to state a similar result for harmonic functions replacing $\partial/\partial\bar{z}$ by the Laplacian.

Another example of this particular teaching device occurs earlier on p. 61, where the student is asked to show that the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges and is analytic in the half-plane $\{s : \operatorname{Re} s > 1\}$, to prove that whenever $\operatorname{Re} s > 1$

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \int_{n}^{n+1} \left(n^{-s} - t^{-s}\right) dt$$

and to conclude from this that $\zeta(s)$ can be continued analytically to $\{\operatorname{Re} s > 0\} \setminus \{1\}$. Although this is not as strong as Riemann's proof that the zeta function can be continued analytically to $\mathbb{C} \setminus \{1\}$, it is sufficient, however, to establish the prime number theorem which is presented much later on p. 191 as an exercise with hints.

These are marvelous problems, but I do not want to leave the impression that all exercises are of the same difficulty as those I have chosen to highlight. Throughout the text exercises are arranged in groups designated A, B and C. Those in group A are meant to be routine and intended to be solved as the student is reading the text. Problems in the other two groups are more challenging, the groups being listed in the order of increasing difficulty. In some cases, as in the preceding two paragraphs, a problem assigned at one stage will reappear later in a more challenging context.

Finally, there are certain features of this book that distinguish it from other texts currently available. Here are two examples:

First, normal families are treated in the context of the chordal metric, a concept introduced on p. 10. Based on Marty's theorem from 1931 characterizing normal families of meromorphic functions on plane domains (cf. [1, p. 226]) together with a lemma of Zalcman (cf. [7, p. 216]), both the great and little theorems of Picard are obtained in a short efficient manner (cf. Marshall pp.162-166). Marty's work is mentioned in Ahlfors, but not in Stein and Shakarchi, while Zalcman's lemma appears in neither.

Second, there is a strong emphasis on conformal mapping beyond what one usually encounters in an introductory text. There is an extensive discussion concerning the actual construction conformal maps. Moreover, Marshall presents two different proofs of the Riemann mapping theorem. One is somewhat constructive and based on what the author refers to as the *zipper algorithm*. The other is based on normal families and is usually associated with Koebe. And, in the end there is a beautiful exposition of the *uniformization theorem* for simply connected Riemann surfaces, not found in either of the texts mentioned above.

When I entered Brown University in 1961 as a graduate in mathematics I had the great pleasure of being introduced to complex analysis through a masterful series of lectures delivered by John Wermer, and starting from the point of view of power series. Whenever I have had the opportunity to teach the subject I have always taken that point of view, and begun in the same way. Although Cartan [3] also begins from the point of view of power series, I never felt that there was a suitable text to assign that students could follow in connection with the course lectures. THERE IS NOW!

References

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