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Patrick D. Barry: Geometry with Trigonometry, 2nd Edition, Woodhead Publishing, 2015. ISBN:978-1-898563-69-3, USD 84.96, 280pp.

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The second edition of *Geometry with Trigonometry* by Patrick D. Barry is both timely and important. It provides an accurate diagnostic of difficulties encountered in the teaching of geometry at school and undergraduate level; it offers a thoughtful, rigorous and balanced response to this problem; in the context of the new secondary school curriculum Project Maths, it provided solid axiomatic foundations for the development of the geometry component; with its wealth of appealing material, it's a valuable reference for secondary school teachers across Ireland and beyond. Last but not least, this book serves as the basis for a substantial follow-up work – *Some Generalization in Geometry* (see [3]) – which develops a coherent and unitary exposition on conic sections in Projective geometry via computational methods based on sensed-areas and rotors.

The author Patrick D. Barry is Professor Emeritus at University College Cork, where he has started his undergraduate mathematics education by earning first place in the Entrance Scholarship Examination in 1952. After a PhD in Complex Analysis at the Imperial College of Science and Technology, London, and an instructorship at Stanford University, he returned to Ireland in 1964 to become Head of the Mathematics Department at his alma mater. Throughout an extensive practice in research and teaching, Professor Barry has kept in touch with overall developments in mathematics education including at post-primary level. The accumulated experience has led him to provide this rigorous and modern treatment of Euclidean geometry.

Geometry is one of the most ancient areas of mathematical inquiry. Long before the algebraic formulation of equations in the 16-17th centuries, solutions for simple polynomial equations were known to Egyptian, Greek and Chinese mathematicians as early as

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600 BC, based on geometric constructions with ruler and compass. Euclid's elements (ca. 300 BC) record the first attempts at building solid axiomatic foundations, and have determined the general approach to geometry for two millennia. To this day, Euclidean geometry is taught in schools mostly as a means to develop logical deductive thinking, while relying on students' visual observations and an informal understanding of basic facts derived from Euclid's axioms. However, the diminishing role of classical synthetic geometry in school curricula worldwide is undeniable. Inquiring into the causes behind this development is certainly worthwhile as it can help develop comprehensive solutions.

From ancient times, geometry developed under a dual set of methods and motivations: on the one hand, the quantitative geometry of measurements involving lengths, areas and volumes; on the other hand, the qualitative part focusing on structures and the interrelationships between them: incidence, generation, transformations, variations in families and related invariants. For a while, these two aspects have complemented and enriched each other. Nowadays, there is no smooth transition from the qualitative aspects of synthetic geometry introduced in secondary schools to the quantitative content of undergraduate university courses. Indeed, geometry as a subject is missing altogether in the first year of college courses in Ireland, even though geometric problems underpin much of their core content: from calculations of areas and volumes in integral calculus to linear transformations and intersections of linear spaces in linear algebra, as well as applications in classical mechanics. By the time students meet metric spaces and differential geometry in their third/fourth years of study, the break with synthetic methods and motives is complete and profound. No wonder students may question the utility of learning Euclidean geometry in school.

In *Geometry with Trigonometry*, Professor Barry proposes a third level course which revisits Euclidean geometry from a richer perspective: on one hand, setting up solid axiomatic bases, clearly delineating between primary assumptions and their logical consequences; on the other hand, reworking a multitude of classical results quantitatively, by a variety of means: Euclidean coordinates, complex numbers, position vectors, areal coordinates and "mobile coordinates", an ingenious way of moving the reference axes to suit the geometry of the structures studied.

With this book, Professor Barry offers third level students – in particular future teachers of mathematics – an opportunity to ponder more deeply and carefully on the foundations of the subject. Here, the main imperative is rigour and completeness. Although Euclid's Elements attempted to set up logical foundations for geometry, the list of premises provided therein was far from sufficient to support all the theorems derived. On various occasions, Euclid's text relied on observations which seemed "visually obvious", but which had not been spelled out among the original set of postulates. Such "common sense" unstated assumptions included existence of rigid transformations (superposition) on which the treatment of congruences was based; assumptions about relative positions of lines and points e.g. concerning boundaries, sides, "betweenness"; and assumptions on completeness which would insure for example the existence of an expected number of intersection points between circles and lines. At the same time, Euclid attempted to provide definitions for primitive terms like points, lines, planes, which are the basic building blocks of Euclidean geometry, but which cannot be intrinsically described by a list of properties. Hilbert's foundational work in 1890's [10] provided the first logically sound and complete system of axioms; starting from undefined notions like point, line, plane, congruent, and describing the relationships between these in no less than 20 axioms. The long list of axioms seems off-putting for any student aiming to learn about plane geometry. Other equivalent systems have been proposed since, the most notable by Birkhoff ([4], [5]) which uses the theory of real numbers to simplify the exposition, thus making it more amenable to use by teachers. It is this approach that Patrick Barry favours in his book. This choice also facilitates the transition to coordinate geometry in the second part of the book.

After a review of the history, basic concepts and pre-requisites in Chapter 1, there follows a systematic presentation of axioms and basic properties. The exposition is thoughtfully divided into different chapters or sections for each axiom and its immediate consequences: Chapter 2 discusses *incidence*, *order*, and *separation* by half-planes. Chapter 3 introduces axioms of *distance* and *angle measurements*, and discusses midpoints, midlines, ratios and perpendicularity. Chapter 4 discusses congruences of triangles, while Chapter 5 deals with *the parallel axiom* and its consequences including similar triangles, Pythagoras' theorem, harmonic ranges and areas. Results are carefully proven by synthetic methods. At this fundamental stage when only basic facts are given, proofs often require elegant solutions based on imaginative additions of auxiliary lines and points to the diagrams. These are presented beautifully and efficiently. While such constructions may hardly seem obvious to beginners in geometry, it is hoped that they are fully appreciated by teachers and 3rd level students looking for an approach that is both complete and coherent.

Chapter 6 signals a change in approach, beginning a coordinate treatment of the main objects developed earlier: Cartesian and parametric equations of a line, criteria for parallelism and perpendicularity, areas of triangles and a thorough discussion of harmonic ranges. In Chapter 7 basic properties of circles are developed, including angles standing on arcs, harmonic ranges and the polar line of a point exterior to a circle; the power of a point with respect to a circle, and the radical axis of two circles (the locus of points having the same "power" with respect to both circles). Synthetic discussions of these topics would necessarily have to distinguish cases according to the relative positions of points and circles. By contrast, a unitary treatment is granted here by the notions of *sensed distance, sensed product* and *sensed ratios* (depending on a chosen order).

Chapter 8 starts a brief discussion of isometries (namely translations and reflections); a treatment of rotations is postponed till the following chapters, when more suitable computational tools have been developed. Chapter 9 consists of a very careful introduction to trigonometric functions and their properties. To deal with angles and rotations, complex coordinates are introduced, as is the homomorphism $\theta \to \text{cis}\theta := e^{i\theta}$ which relates angles to unitary complex numbers. The notion of *sensed angles* distinguishes between angle orientations (clockwise or anticlockwise), and correspondingly, a notion of *sensed area* is defined. These tools are then employed to develop the criterion for 4 concyclic points via a real cross-ratio, and the resulting proof of the famous Ptolemy (c 200 AD) formula involving the lengths of sides and products of diagonals. We are also treated to a special case of Pascal's theorem (1640), involving a cyclic hexagon made of three pairs of parallel sides.

The extended Chapter 11 discusses a variety of coordinate methods, starting with vectors and areal coordinates. We revisit in this context classical results like Menelaus' theorem (c.100 A.D.) about a line cutting a triangle, the dual theorem of Ceva (1678 A.D.), Desargues' perspective theorem (1648 A.D.), and Pappus' theorem (c.300 A.D.). The original concept of *mobile coordinates* employs complex numbers to suitably describe points on perpendicular lines, leading to economical descriptions of special points for a triangle: the centroid, orthocentre, incentre and circumcentre, along with proofs for the existence of Euler's line and the 9 point circle. Other theorems displaying pleasing structural symmetries – a trademark of Euclidean geometry – include the following: *On the sides of a triangle are erected three similar triangles (with the same orientation). Then the circumcentres of these triangles forms a fourth triangle similar to the previous three.* Computational proofs of other well loved theorems, like Feuerbach's, Wallace-Simpson, and Miquel's theorem are provided. The concurrence of isogonal conjugate lines in a triangle (and the particular case of symmedians) complete the delightful collection of classical results.

Chapter 12 develops proofs for the differential formulas of sine and cosine functions, a must-see for every student. Hence the length of an arc of circle can be calculated by integration, and a radian measure can be properly defined.

I warmly recommend this book for those who wish to deepen their understanding of Euclidean geometry. The first part sets up solid comprehensive bases, through carefully formulated axioms and elegant efficient proofs. The second part involves considerable computational sophistication and ingenuity in the choice of coordinates.

Geometry with Trigonometry is a perfect resource for post-primary teaching. The axiomatic part is covered at a level of rigour and sophistication not required (or indeed productive) when working with school children, but useful for teachers' own understanding. The major classical theorems present in this book have a perennial aesthetic appeal, and I hope that teachers will feel tempted to include a good part of them in their teaching, either during class time or for maths clubs and maths circles. When planning their own teaching methodologies though, it is left to the teachers to compare and select between synthetic and coordinate strategies, on a case by case basis. When applying synthetic methods, the main difficulties spring from a number of factors:

- The need for casework – depending on the relative positions of points, lines and circles. This is more a case of awkwardness than difficulty, as whenever casework is required, closely similar strategies tend to work for all cases. The problem is mostly pedagogical, as systematic working of cases may try the patience of young minds; this can be mitigated by applying discretion in the level of detail and rigour in school geometry proofs.
- The need for an ability to navigate complex diagrams, focusing on their useful configurations, ignoring elements irrelevant to the problem at hand, and sometimes drawing new lines and points to bring a new perspective. Modern graphic design tools like Geogebra [11] are quite helpful, as they produce crisp, exact diagrams which can be zoomed in or varied continuously while preserving - and thus highlighting - the relevant features in each configuration.
- The need to combine a variety of measurement methods within one proof, e.g. making transitions from angle-chasing to computations of lengths and areas and vice-versa via congruences, similarities, or trigonometric functions. At first sight, the coordinate method does away with this difficulty. However, as seen in this book, one needs flair in choosing the most efficient coordinate setting. In particular, simple angle-chasing arguments can become quite complicated in Cartesian coordinates; only the use of complex numbers can offer comparable simplicity, but these are only taught in more advanced classes. An alternative way to deal with this difficulty would be to organize the teaching of school synthetic geometry into distinct stages:
  (1) The *foundational stage*, involving primary objects, notations, definitions, axioms and immediate consequences.
  (2) the *isometry stage*, including the use of congruent triangles in deriving properties of parallelograms and other special quadrilaterals; as well as properties of rotations and reflections;
  (3) the *angle-chasing stage*, including the theorems on sums of angles in triangles and other convex polygons; cyclic quadrilaterals; the concurrency of altitudes; orthocentre

as incentre of the orthic triangle; and many other beautiful theorems like Miquel's point, Miquel and Steiner's quadrilateral theorem, Miquel's pentagon and six circles theorems;

(4) the *similarity stage*, making the transition from angle-chasing to more complex computations involving lenghts and ratios: including Pythagoras' theorem, and topics like Ceva's and Menelaus' theorems, power of the point, Simson's line, Ptolemy's theorem, harmonic ranges;

(5) the *geometric loci stage*, including all important lines in the triangle and their *universal properties*, as well as radical axes for circles; leading to the existence of incentre, circumcentre, centroid, orthocentre, radical centre, Euler's line, the 9 point circle, the Newton-Gauss line.

This book inspires the reader to think more deeply about ways to develop a coherent approach to teaching geometry from school to university level: including seamless transitions in the development of topics, methods and themes from classical plane geometry to advances undergraduate and graduate courses. Here are a few of reviewer's thoughts on this issue:

- Offer 3rd level courses that lead to a deeper perspective on Euclidean geometry, for example based on professor Barry's book.
- Study conic sections by both synthetic and coordinate approaches (e.g. defining focal points as tangency points of the plane section with certain spheres, and using synthetic methods to deduce the reflection properties applied in optics).
- Transition from plane Euclidean geometry to projective geometry over division rings or fields following Hilbert's ideas (defining operations on the abstract line using translations and similarities; rehashing Desargues' and Pappus' configurations as associativity and commutativity of the base field). Thus setting basic premises for Algebraic Geometry.
- Use the "power of the point" to study inversion in a circle and build Poincare's model of Hyperbolic Geometry.
- Transition from study of triangles and triangulations to simplices and simplicial complexes in early courses on Algebraic Topology.

- Show how the theme of geometric loci from Euclidean geometry is continued with the study of parameter or moduli spaces and universal families in Algebraic, Complex and Differential Geometry.
- Transition from the study of invariants in transformation geometry to Representation Theory.

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