Irish Math. Soc. Bulletin
Number 79, Summer 2017, 87–91
ISSN 0791-5578

Klaus Truemper: The Construction of Mathematics, Leibniz Company, 2017. ISBN:9-780-9663-5543-7, USD 14.95, 293+viii pp.

REVIEWED BY PETER LYNCH

Is mathematics discovered or created? The Platonic view is that mathematical ideas such as numbers and geometric forms have an a priori existence independent of humanity and gradually come to light as they are discovered through research and investigation. The contrary view is that mathematics is a creation of the human intellect. The question has been debated for centuries. The author of this book, Klaus Truemper, addresses this question and comes to a definite conclusion, strongly in favour of mathematics as a human creation, justifying the subtitle of his book, *The Human Mind's Greatest Achievement.*

Mathematics has emerged over thousands of years, in several civilizations. The first part of the book (Chapters 2 to 7) traces the development of the struggle for insight. Where do mathematical ideas come from? Are they somehow already present in the physical world, hidden and awaiting discovery by inquisitive minds? Or are they the products of human ingenuity? The second half of the book investigates this question from several perspectives, reaching a definite, although hardly definitive, conclusion.

Following the Introduction, Chapter 2 traces the development of numbers from the natural or counting numbers through rational to real and complex numbers. What is the origin of all these numbers? The general thrust is that each successive layer is a result of creation. The question then occurs to this reviewer: if we start with the natural numbers and the additional numbers already exist in some realm awaiting discovery, there seems to be only one way forward. But if we are free to create at will, is there not a multitude of possible extensions, not trivially equivalent and all internally consistent? Are there such alternative number systems, and is it perhaps

Received on 8-4-2017; revised 28-4-2017.

that the standard number system is the one most suited to physical applications? This is not considered in any depth in the book.

Chapter 3 discusses mathematical notation. It is beyond doubt that well-chosen notation can facilitate advances while badly-chosen symbols can severely inhibit it. Truemper discusses the contrast between Newton's awkward fluents and fluxions and Leibniz's elegant notation. The former certainly held up progress in analysis in Britain for more than a century. In Chapter 4 (Infinity) Truemper shows how the application of mathematical arguments in physical contexts can produce nonsensical results. One example is Torricelli's Trumpet, which has finite volume but infinite surface area. Indeed, infinity frequently leads to paradoxical results when we try to apply it to physical systems. The Banach-Tarski Theorem is a particularly sharp example.

In Chapter 5, some classical problems (squaring the circle, etc.) are considered. The key argument here is that all these problems, outstanding for 2000 years, were resolved in the nineteenth century only when mathematics broke free from the natural world. Truemper writes (pg. 77): "mathematics is different from nature, does not need nature and should not be confused with nature."

Chapter 6 examines the role of proof in mathematics from Babylon and Ancient Greece to modern times, when Hilbert's dream of a rock-solid foundation for mathematics was shattered by Gödel's Incompleteness Theorems. The Zermelo-Fraenkel Axioms (ZF), with or without the Axiom of Choice and Continuum Hypothesis, form the basis of most mathematics today. Modern researchers have no real choice but to accept the potential inconsistency of these foundations, hoping — indeed expecting — that if an inconsistency is ever found it will be remedied by suitable modification of the underlying axioms.

The proof by Paul Cohen that the Axiom of Choice may be added to ZF without affecting (in)consistency shows how more than one mathematical system is possible. So, if we consider a single physical universe, at most one of these systems can describe it, implying that the other systems somehow have an existence independent of the physical world.

A chapter on computing machines is interesting but seems inessential to the dominant theme of the book. Still, I cannot resist the temptation to quote Leibniz, inventor of the binary system and of some mechanical calculators: It is beneath the dignity of excellent men to waste their time on calculations when any peasant could do the work just as accurately with the aid of a machine (perhaps mathematicians should avoid quoting this to their colleagues in computer science).

Chapter 8 opens the second part of the book asking in its title "Is Mathematics Created or Discovered?" Mathematical platonism posits that all of mathematics resides in a realm of abstract objects that is separate from the sensible world. This implies that mathematical truths are discovered, not invented. From 1800 onwards many mathematicians departed from this view, starting with Gauss who wrote that "number is purely a product of our mind". Kronecker's famous dictum is that "God made the integers; all else is the work of man". Yet, many twentieth century mathematicians supported the view that mathematical results are discovered.

The concept of "language games", devised by Ludwig Wittgenstein, is introduced in Chapter 9. It is argued that the technique can resolve many thorny philosophical problems. A language game is "a controlled setting of language use that brings a particular facet of a given philosophical problem into focus" and provides insight into the problem. To apply this technique many examples are required and these are drawn from the earlier chapters. In each instance, it is assumed that mathematics is discovered. Then contradictions arising during the course of the game indicate that this assumption must be abandoned.

Chapter 10 looks at several stages in the historical emergence of mathematics, using the language games framework. Before the concept of numbers, came one-one correspondences or bijections, for example between pebbles and sheep or fingers of the hand and children. Soon names were made up for groups of pebbles or fingers, leading to the counting numbers. All this could be regarded as creative. More species of numbers negatives, fractions, square roots followed as the need for them arose. Again, all could be described as invention rather than discovery. Other areas considered include logarithms, calculus, function theory, Lebesgue integration and the hierarchy of infinities. In each case, the assumption of discovery leads the author to bizarre and untenable consequences.

Analogies between mathematics and art are considered. Truemper gives a strange argument constructing a mathematical function that precisely specifies Michaelangelo's David: the function is defined in 3-dimensional Cartesian space and takes the value 1 for points within the statue and 0 for points outside. He then argues that, if the function existed before the statue was made, Michaelangelo must have discovered rather than created David. But the same argument holds if we replace *David* (created) by *Mount Everest* (discovered). I did not find this example illuminating. A comparison of the development of music and mathematics is more enlightening. Musical compositions are universally held to be creations, not discoveries. Why then should mathematical results be regarded as discoveries?

Truemper next addresses the proposition "Mathematical concepts are created, whereas the consequences provable from these concepts are discovered". I might paraphrase this: "Definitions are created, theorems/proofs are discovered". By analogies with sculpture, music and literature, the author shows that such a proposition leads to unreasonable conclusions. But I cannot easily accept such analogies as valid, or as vitiating the proposition. Indeed, this idea (definitions created, theorems discovered) has arguments and evidence in its favour (See "Invention or Discovery?" at `https://thatsmaths.com`).

The "unreasonable effectiveness" of mathematics in the physical sciences is examined in Chapter 11. This concept is often advanced in support of the idea that mathematics is discovered. Evidence is amassed in the book that, contrary to a widespread view, mathematics is actually quite ineffective in providing solutions to many problems in the modern world. There is selection bias: failed mathematical models tend to be ignored in favour of successful ones.

There are many natural processes for which we have not been able to construct useful mathematical models. Truemper considers these as evidence of the "reasonable ineffectiveness" of mathematics. However, our inability to solve the non-linear Navier-Stokes equations in closed form does not diminish the remarkable power of these equations to describe accurately a huge range of fluid phenomena. Truemper then compares the limitations of weather forecasting and economic prediction. This is to miss a crucial distinction: there are no Navier-Stokes equations for the economy!

Mathematics appears to be essential to civilization and is often considered to be an inherent part of nature. However, this view is disputed in Chapter 12, which gives the absence of any mathematics in an Amazonian tribe as an agument against the discovery of mathematics. I found this unconvincing and feel that the entire chapter is irrelevant and superfluous. In the last substantive chapter, recent advances in brain science are used to account for divergent opinions amongst experts concerning creation/discovery. In the past, Gauss and Cantor argued for creation, while Frege and Gödel supported discovery. It is claimed that differences arise from "embodiment of different learning experiences". Modern neuroscience is hardly needed to see that scholars with different backgrounds, knowledge and experience may reach different conclusions, and appeal to recent research does not really provide any additional insight into the creation/discovery dilemma. As with the previous chapter, I feel that this one could have been omitted without loss.

Braoadly speaking, mathematics involves the study of quantity (number), structure (algebra), space (geometry) and change (analysis). This book concentrates mostly on the first and last categories. The concept of symmetry is not mentioned. It would be interesting to examine the concept of symmetry in the context of creation/discovery.

The main text of the book covers 207 pages and is supplemented by 67 pages of endnotes containing much fascinating background material. A good bibliography follows this.

In summary, I found the book well-written with generally clear and convincing arguments (despite the counterexamples mentioned above). If there is a general criticism it is that the author has tried too hard to support his main conclusion, giving more weight to arguments supporting it and less to those that might refute it. Notwithstanding this, the book is an interesting, enjoyable and thought-provoking read. Of course, it cannot provide the final word on the central question, which I feel has the characteristics of a Gödelian problem, irresolvable with our current tools of thought.

Peter Lynch is emeritus professor at UCD. His interests include all areas of mathematics and its history. He writes an occasional mathematics column in *The Irish Times* and has recently published a book of articles entitled *That's Maths*. His blog is at `http://thatsmaths.com`.

School of Mathematics & Statistics, University College Dublin
E-mail address: `Peter.Lynch@ucd.ie`