PROBLEMS

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The first problem this issue was contributed by Peter Danchev of Plovdiv University, Bulgaria.

Problem 79.1. Suppose that k and n are positive integers with $1 \leqslant k \leqslant n$. Find the largest integer m such that the binomial coefficient $\dbinom{2^n}{k}$ is divisible by 2^m.

The next problem was suggested by Prithwijit De of the Homi Bhabha Centre for Science Education, Mumbai, India.

Problem 79.2. Let f be a function that is continuous on the interval $[0, \pi/2]$ and satisfies $f(x) + f(\pi/2 - x) = 1$ for each x in $[0, \pi/2]$. Evaluate the integral

$$\int_0^{\pi/2} \frac{f(x)}{(\sin^3 x + \cos^3 x)^2} \, dx.$$

We finish with an elegant identity involving sums of powers of integers. It would be pleasing to see a simple geometric proof of this classic identity, but perhaps that is asking too much.

Problem 79.3. Prove that, for any positive integer n,

$$(1^5 + \cdots + n^5) + (1^7 + \cdots + n^7) = 2(1 + \cdots + n)^4.$$

SOLUTIONS

Here are solutions to the problems from *Bulletin* Number 77. The first problem was solved by the North Kildare Mathematics Problem Club as well as the proposer, Finbarr Holland of University College Cork. The two solutions were similar in spirit; we give the solution of the problem club.

Problem 77.1. Suppose that $f : [0,1] \to \mathbb{R}$ is a convex function and $\int_0^1 f(t)\,dt = 0$. Prove that

$$\int_0^1 t(1-t)f(t)\,dt \leqslant 0,$$

with equality if and only if $f(t) = a(2t-1)$ for some real number a.

Solution 77.1. Let

$$\alpha = \tfrac{1}{2}(f(1) - f(0)), \quad g(t) = \alpha(1 - 2t), \quad \text{and} \quad h(t) = f(t) + g(t).$$

Then h is convex, $h(0) = h(1)$, and

$$\int_0^1 h(t)\,dt = 0.$$

Let $k(t) = h(t) + h(1-t)$. Then k is convex, $k(t) = k(1-t)$, and

$$\int_0^{1/2} k(t)\,dt = \int_0^1 k(t)\,dt = 0. \tag{1}$$

Also,

$$\int_0^{1/2} t(1-t)k(t)\,dt = \int_0^1 t(1-t)f(t)\,dt.$$

It cannot be that $k(0) < 0$, because if that were so then $k(1) < 0$, and hence (by convexity) $k(t) < 0$ on $[0,1]$, which contradicts (1). Reasoning similarly, we see that if $k(0) = 0$, then $k(t) = 0$ for all t, and

$$\int_0^{1/2} t(1-t)k(t)\,dt = 0.$$

The remaining possibility is that $k(0) > 0$. In this case, since k is convex with integral zero, there must be exactly two zeros of k between 0 and 1, and by symmetry they are at points β and $1-\beta$ for some $\beta \in (0, \tfrac{1}{2})$. Moreover, k is strictly decreasing on the interval $(0, \tfrac{1}{2})$ and $t(1-t)$ is positive and increasing on $(0, \tfrac{1}{2})$. Thus one can check that the inequality $t(1-t)k(t) < \beta(1-\beta)k(t)$ is satisfied on both intervals $(0, \beta)$ and $(\beta, 1/2)$. Therefore

$$\int_0^{1/2} t(1-t)k(t)\,dt < \beta(1-\beta)\int_0^{1/2} k(t)\,dt = 0.$$

So the desired inequality holds, with equality only in the case when $k(t)$ is identically 0, that is, when $h(t) = -h(1-t)$. But in that case $h(0) = -h(1) = -h(0)$, so $h(0) = h(1) = 0$, and since h is

convex with integral zero, we see that h is identically zero. Hence $f(t) = \alpha(2t - 1)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Finbarr points out that if f is twice continuously differentiable and $f''(t) \geq 0$ for all t, then there is a much shorter solution, which follows immediately from the identity below, which can be proved by integrating the left-hand integral by parts a couple of times:

$$\int_0^1 t^2(1-t)^2 f''(t)\, dt = 2\int_0^1 (1 - 6t + 6t^2) f(t)\, dt.$$

The second problem from *Bulletin* Number 77 was solved by Henry Ricardo of the Westchester Area Math Circle, New York, USA, and the North Kildare Mathematics Problem Club. The solution was also known to the proposer. Many have pointed out that the problem is well known. Henry notes that the problem is usually ascribed to Pierre Rémond de Montmort (1678–1719), and that *The Problem of Coincidences* by Lajos Takács (*Archive for History of Exact Sciences*, **21**, 1980) is an excellent survey on this problem and its generalisations. We give Henry's solution here, which coincides with that of the problem club, and which apparently is essentially due to Euler.

Problem 77.2. Each member of a group of n people writes his or her name on a slip of paper, and places the slip in a hat. One by one the members of the group then draw a slip from the hat, without looking. What is the probability that they all end up with a different person's name?

Solution 77.2. The problem is equivalent to counting the number D_n of permutations P of $\{1, \ldots, n\}$ that satisfy $P(k) \neq k$ for $1 \leqslant k \leqslant n$. Let us call such a permutation P a *derangement*. We use the notation $(j_1, j_2, \ldots, j_n)$ to represent a permutation, where j_k denotes the image of k.

For any derangement $(j_1, j_2, \ldots, j_n)$, we have $j_n \neq n$. Let $j_n = k$, where $k \in \{1, 2, \ldots, n-1\}$. Now we split the derangements on n elements into two cases.

Case 1: $j_k = n$ (so k and n map to each other). By removing elements k and n from the permutation, we have a derangement on $n - 2$ elements; and so, for fixed k, there are D_{n-2} derangements in this case.

Case 2: $j_k \neq n$. Swap the values of j_k and j_n, so that we have a new permutation with $j_k = k$ and $j_n \neq n$. By removing element k,

we have a derangement on $n-1$ elements; and so, for fixed k, there are D_{n-1} derangements.

Thus, with $n-1$ choices for k, we have, for $n \geqslant 2$,

$$D_n = (n-1)(D_{n-1} + D_{n-2}).$$

The probability P_n of a derangement is the number of derangements divided by the number of all possible permutations of n objects:

$$
\begin{aligned}
P_n &= \frac{D_n}{n!} = \frac{(n-1)}{n!}(D_{n-1} + D_{n-2}) \\
&= (n-1)\left(\frac{1}{n} \cdot \frac{D_{n-1}}{(n-1)!} + \frac{1}{n(n-1)} \cdot \frac{D_{n-2}}{(n-2)!}\right) \\
&= \left(1 - \frac{1}{n}\right)P_{n-1} + \frac{1}{n}P_{n-2} \\
&= P_{n-1} - \frac{1}{n}(P_{n-1} - P_{n-2}),
\end{aligned}
$$

or $P_n - P_{n-1} = -(1/n)(P_{n-1} - P_{n-2})$, with $P_1 = 0$ and $P_2 = 1/2$. It follows that

$$P_n - P_{n-1} = \frac{(-1)^n}{n!},$$

so

$$P_n = P_1 + \sum_{k=2}^{n}(P_k - P_{k-1}) = \sum_{k=2}^{n} \frac{(-1)^k}{k!}. \qquad \square$$

The third problem was incorrectly labelled 76.3, rather than 77.3, in issue 77. It was solved by Dixon Jones of the University of Alaska Fairbanks, USA, Niall Ryan of the University of Limerick, and the North Kildare Mathematics Problem Club. We give the solution of the problem club.

Problem 77.3. Evaluate

$$1 + \cfrac{1^2}{1 + \cfrac{2^2}{1 + \cfrac{3^2}{1 + \cdots}}}.$$

Solution 77.3. Consider the identity

$$\frac{1}{x} - \frac{1}{y} = \cfrac{1}{x + \cfrac{x^2}{y - x}}, \qquad \text{where } x, y \neq 0 \text{ and } x \neq y. \qquad (2)$$

Applying (2) with $x = n$ and $y = n+1$, where $n \geqslant 1$, we obtain

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n + \dfrac{n^2}{1}}.$$

Applying (2) with $x = n-1$ and $y = n+n^2$, where $n \geqslant 2$, we obtain

$$\frac{1}{n-1} - \frac{1}{n} + \frac{1}{n+1} = \frac{1}{(n-1) + \dfrac{(n-1)^2}{1 + \dfrac{n^2}{1}}}.$$

Continuing in this manner, we obtain

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^n \frac{1}{n+1} = \frac{1}{1 + \dfrac{1^2}{1 + \dfrac{2^2}{1 + \cdots + \dfrac{n^2}{1}}}}.$$

Then, taking limits, we see that

$$\log 2 = \frac{1}{1 + \dfrac{1^2}{1 + \dfrac{2^2}{1 + \cdots}}},$$

so the desired continued fraction is equal to $1/\log 2$. $\qquad \square$

The problem club point out that various generalisations of this continued fractions formula appear in the literature. One such generalisation is

$$\log(1 + x) = \frac{x}{1 + \dfrac{1^2 x}{2 - x + \dfrac{2^2 x}{3 - 2x + \dfrac{3^2 x}{4 - 3x + \cdots}}}}$$

(and there are more). The problem club's method comes from *Higher Algebra* by Hall and Knight, and seems to be due to Frobenius and Stickelberger (*J. Reine Angew. Math.*, **88**, 1880) originally.

A similar idea to that given in the proof can be used to establish that

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cdots}}}}.$$

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer Latex). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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