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The aim of the *Bulletin* is to inform Society members, and the mathematical community at large, about the activities of the Society and about items of general mathematical interest. It appears twice each year. The *Bulletin* is supplied free of charge to members; it is sent abroad by surface mail. Libraries may subscribe to the *Bulletin* for 30 euro per annum.

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EDITORIAL

Patrick D. Barry, Professor Emeritus of Mathematics at University College Cork, has had significant influence on the development of Mathematics in Ireland. He did brilliant research work on complex analysis. In his autobiography, Walter Hayman wrote that Barry was "the only student I ever had who came to me with a PhD problem already prepared. It was on the minimum modulus of small integral and subharmonic functions, a subject on which Barry became the world expert." Barry was Hayman's second research student, and succeeded the first, Paddy Kennedy, in the chair at UCC. Thereafter he led the Department effectively for many years, as well as contributing in a major way to the university administration. Possessed of deep learning, and a strong sense of duty, he took seriously the responsibility of university mathematicians to monitor and assist with developments in the schools' programme. He was particularly concerned about changes to the geometry syllabus that took place in the nineteen-sixties. These changes were seriously misguided. The whole sorry story is almost unbelievable, and is documented in the 2007 Maynooth PhD thesis of Susan McDonald. Barry was tireless and relentless over a long period in his efforts to correct the problem. Of his writings about school geometry, the most significant is his book *Geometry with Trigonometry*. This text was eventually adopted by the NCCA as the bedrock underlying the geometry programme in the Project Maths syllabus. It was a fully rigorous text on Euclidean geometry, going substantially beyond the schools' programme, and suitable for study by university undergraduates. It has been out of print, and members will be glad to know that a second edition appeared last year, with extra material (ISBN: 978-0-12-805066-8). We hope to publish a review in the near future.

In the meantime, this dean of Irish mathematicians continues to flourish, and a digital copy of a draft of Barry's second book, *Generalization of Geometry*, has been uploaded under his name on a UCC website. The link is `http://euclid.ucc.ie/pbarry/SGiG2.pdf`. Barry writes that it is a 'second book' on geometry. He writes: "Down the centuries many adults have studied as a hobby further material on the geometry they learned at school, and derived great pleasure from this. With the decline in study of geometry over the last fifty years, this pool is probably greatly reduced but I have a strong hope that persons of this type will notice this book and be encouraged to study it. A major objective of the approach is to motivate in the familiar context of Euclidean geometry basic concepts of projective geometry." The book has almost 500 pages, and is a treasure-trove.

The 2017 Annual Scientific Meeting will be held at Sligo Institute of Technology, and the 2018 meeting will be held at UCD.

Links for Postgraduate Study

The following are the links provided by Irish Schools for prospective research students in Mathematics:

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http://www.dcu.ie/info/staff_member.php?id_no=2659
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NUIM: http://www.maths.nuim.ie/pghowtoapply
QUB: http://www.qub.ac.uk/puremaths/Funded_PG_2016.html
TCD: http://www.maths.tcd.ie/postgraduate/
UCC: http://www.ucc.ie/en/matsci/postgraduate/
UCD: mailto://nuria.garcia@ucd.ie
UU: http://www.compeng.ulster.ac.uk/rgs/

The remaining schools with Ph.D. programmes in Mathematics are invited to send their preferred link to the editor, a url that works. All links are live, and hence may be accessed by a click, in the electronic edition of this Bulletin[1].

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Deceased Members

It is with regret that we report the deaths of members:

Michael A. Hayes, MRIA, Emeritus Professor of Mathematical Physics at UCD. Died 1 January 2017. (Incorrectly reported as 2 Jan in Bulletin 78.)

Matthew McCarthy, MRIA, Emeritus Professor of Mathematical Physics and former Registrar and Deputy President of NUI Galway. Died 16 March 2017.

E-mail address: `subscriptions.ims@gmail.com`

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Homogeneous manifolds whose geodesics are orbits. Recent results and some open problems

ANDREAS ARVANITOYEORGOS

Abstract. A homogeneous Riemannian manifold $(M = G/K, g)$ is called a space with homogeneous geodesics or a G-g.o. space if every geodesic $\gamma(t)$ of M is an orbit of a one-parameter subgroup of G, that is $\gamma(t) = \exp(tX) \cdot o$, for some non zero vector X in the Lie algebra of G. We give an exposition on the subject, by presenting techniques that have been used so far and a wide selection of previous and recent results. We discuss generalization to two-step homogeneous geodesics. We also present some open problems.

1. Introduction

The aim of the present article is to give an exposition on developments about homogeneous geodesics in Riemannian homogeneous spaces, to present various recent results and discuss some open problems. One of the demanding problems in Riemannian geometry is the description of geodesics. By making some symmetry assumptions one could expect that certain simplifications may accur. Let (M, g) be a homogeneous Riemannian manifold, i.e. a connected Riemannian manifold on which the largest connected group G of isometries acts transitively. Then M can be expressed as a homogeneous space $(G/K, g)$, where K is the isotropy group at a fixed point o of M.

Motivated by well known facts such that, the geodesics in a Lie group G with a bi-invariant metric are the one-parameter subgroups

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Key words and phrases. Homogeneous geodesic, g.o. space, invariant metric, geodesic vector, geodesic lemma, naturally reductive space, generalized flag manifold, generalized Wallach space, M-space, δ-homogeneous space, pseudo-Riemannian manifold, two-step homogeneous geodesic,

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of G, or that the geodesics in a Riemannian symmetric space G/K are orbits of one-parameter subgroups in G/K, it is natural to search for geodesics in a homogeneous space, which are orbits. More precisely, a geodesic $\gamma(t)$ through the origin o of $M = G/K$ is called *homogeneous* if it is an orbit of a one-parameter subgroup of G, that is

$$\gamma(t) = \exp(tX) \cdot o, \quad t \in \mathbb{R}, \tag{1}$$

where X is a non zero vector in the Lie algebra $\mathfrak{g}$ of G. A non zero vector $X \in \mathfrak{g}$ is called a *geodesic vector* if the curve (1) is a geodesic. A homogeneous Riemannian manifold $M = G/K$ is called a *g.o. space* if all geodesics are homogeneous with respect to the largest connected group of isometries $I_o(M)$. Since their first systematic study by O. Kowalski and L. Vanhecke in [45], there has been a lot of research related to homogeneous geodesics and g.o spaces and in various directions.

Homogeneous geodesics appear quite often in physics as well. The equation of motion of many systems of classical mechanics reduces to the geodesic equation in an appropriate Riemannian manifold M. Homogeneous geodesics in M correspond to "relative equilibriums" of the corresponding system (cf. [6]). For further information about relative equilibria in physics we refer to [36] and references therein. In Lorentzian geometry in particular, homogeneous spaces with the property that all their *null* geodesics are homogeneous, are candidates for constructing solutions to the 11-dimensional supergravity, which preserve more than 24 of the available 32 supersymmetries. In fact, all Penrose limits, preserving the amount of supersymmetry of such a solution, must preserve homogeneity. This is the case for the Penrose limit of a reductive homogeneous spacetime along a null homogeneous geodesic ([35], [50], [55]). For a recent mathematical contribution in this topic see [28].

All naturally reductive spaces are g.o. spaces ([41]), but the converse is not true in general. In [39] A. Kaplan proved the existence of g.o. spaces that are in no way naturally reductive. These are generalized Heisenberg groups with two dimensional center. Another important class of g.o. spaces are the weakly symmetric spaces. These are homogeneous Riemannian manifolds $(M = G/K, g)$ introduced by A. Selberg in [57], with the property that every two points can be interchanged by an isometry of M. In [15] J. Berndt, O. Kowalski and L. Vanhecke proved that weakly symmetric spaces are g.o. spaces. In [42] O. Kowalski, F. Prüfer and L. Vanhecke gave an explicit classification of all naturally reductive spaces up to dimension five, and in [1] the authors classified naturally reductive homogeneous spaces up to dimension six. The classification in dimensions seven and eight was recently completed ([58]).

The term *g.o. space* was introduced by O. Kowalski and L. Vanhecke in [45], where they gave the classification of all g.o. spaces up to dimension six, which are in no way naturally reductive. Concerning the existence of homogeneous geodesics in a homogeneous Riemannian manifold, we recall the following. In ([38]) V.V. Kajzer proved that a Lie group endowed with a left-invariant metric admits at least one homogeneous geodesic. O. Kowalski and J. Szenthe extended this result to all homogeneous Riemannian manifolds ([44]). An extension of this result to reductive homogeneous pseudo-Riemannian manifolds was obtained ([31], [55]).

In [37] C. Gordon described g.o. spaces which are nilmanifolds and in [63] H. Tamaru classified homogeneous g.o. spaces which are fibered over irreducible symmetric spaces. In [26] and [30] O. Kowalski and Z. Dušek investigated homogeneous geodesics in Heisenberg groups and some H-type groups. Examples of g.o. spaces in dimension seven were obtained by Dušek, O. Kowalski and S. Nikčević in [32].

In [3] the author and D.V. Alekseevsky classified generalized flag manifolds which are g.o. spaces. Further, D.V. Alekseevsky and Yu. G. Nikonorov in [4] studied the structure of compact g.o. spaces and gave some sufficient conditions for existence and non existence of an invariant metric with homogeneous geodesics on a homogeneous space of a compact Lie group. They also gave a classification of compact simply connected g.o. spaces of positive Euler characteristic.

In [40] O. Kowalski, S. Nikčević and Z. Vlášek studied homogeneous geodesics in homogneous Riemannian manifolds, and in [49], [20] G. Calvaruso and R. Marinosci studied homogeneous geodesics in three-dimensional Lie groups. Homogeneous geodesics were also studied by J. Szenthe in [59], [60], [61], [62]. Also, D. Latifi studied homogeneous geodesics in homogeneous Finsler spaces ([46]), and the first author investigated homogeneous geodesics in the flag manifold $\mathrm{SO}(2l+1)/\,\mathrm{U}(l-m)\times\mathrm{SO}(2m+1)$ ([7]).

Homogeneous geodesics in the affine setting were studied in [26] and [33] (and in particular for any non reductive pseudo-Riemannian manifold).

Finally, a class of homogeneous spaces which satisfy the g.o. property are the δ-homogeneous spaces, which were introduced by V. Berestovskiĭ and C. Plaut in [14]. These spaces have interesting geometrical properties, but we will not persue here. We refer to the paper [13] by V. Berestovskiĭ and Yu.G. Nikonorov for more information in this direction. Further useful information about geodesic orbit spaces can be found in the recent work [53].

The paper is organized as follows. In Section 2 we present the basic techniques for finding homogeneous geodesics and detecting if a homogeneous space is a space with homogeneous geodesics (g.o. space). In Section 3 we present the classification up to dimension 6 and give examples in dimension 7. In Section 4 we discuss homogeneous g.o. spaces which are fibered over irreducible symmetric spaces and in Section 5 we present the classification of generalized flag manifolds which are g.o. spaces. In Section 6 we present results about another wide class of homogeneous spaces, the generalized Wallach spaces, and in Section 6 we discuss results related to M-spaces. These are homogeneous spaces G/K_1 so that $G/(S\times K_1)$ is a generalized flag manifold, where S a torus in a compact simple Lie group G. The pseudo-Riemannian setting is presented in Section 8. In Section 9 we discuss a generalization of homogeneous geodesics which we call two-step homogeneous geodesics. These are orbits of the product of two exponential factors. Finally, in Section 10 we present some open problems.

2. Homogeneous geodesics in homogeneous Riemannian manifolds - Techniques

A *homogeneous Riemannian manifold* is a Riemannian manifold M for which there exists a connected Lie group $G\subset I_0(M)$ which acts transitively on M as a group of isometries. Let $p\in M$ be a fixed point. If we denote by K the isotropy group at p, then M can be identified with the homogeneous space G/K. Note that there may exist more than one transitive isometry groups $G\subset I_0(M)$ so that M is represented as a coset space in more than one ways. For any fixed choice $M=G/K$, G acts effectively on G/K from the left. A G-invariant metric g on $M = G/K$ is a Riemannian metric so that the diffeomorphism $p \mapsto a \cdot p$ is an isometry.

It is known ([41]) that a Riemannian homogeneous space is always *reductive*. This means that if $\mathfrak{g}$, $\mathfrak{k}$ are the Lie algebras of G and K respectively, then there is a direct sum decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \tag{2}$$

with $\mathrm{Ad}(K)(\mathfrak{m}) \subset \mathfrak{m}$. The canonical projection $\pi : G \to G/K$ induces an isomorphism between the subspace $\mathfrak{m}$ of $\mathfrak{g}$ and the tangent space T_oM at the identity $o = eK$.

A G-invariant Riemannian metric g defines a scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$ which is $\mathrm{Ad}(K)$-invariant and vice-versa. If G is semisimple and compact and B denotes the negative of the Killing form of $\mathfrak{g}$, then any $\mathrm{Ad}(K)$-invariant scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$ can be expressed as $\langle x, y \rangle = B(\Lambda x, y)$ $(x, y \in \mathfrak{m})$, where Λ is an $\mathrm{Ad}(K)$-equivariant positive definite symmetric operator on $\mathfrak{m}$. Conversely, any such operator Λ determines an $\mathrm{Ad}(K)$-invariant scalar product $\langle x, y \rangle = B(\Lambda x, y)$ on $\mathfrak{m}$, which in turn determines a G-invariant Riemannian metric g on $\mathfrak{m}$. A Riemannian metric generated by the scalar product product B is called *standard metric*.

Definition 1. *A homogeneous Riemannian manifold* $(M = G/K, g)$ *is called a space with homogeneous geodesics, or* G*-g.o. space if every geodesic* γ *of* M *is an orbit of a one-parameter subgroup of* G*, that is* $\gamma(t) = \exp(tX) \cdot o$*, for some non zero vector* $X \in \mathfrak{g}$*. The invariant metric* g *is called* G*-g.o. metric. If* G *is the full isometry group, then the* G*-g.o. space is called a manifold with homogeneous geodesics, or a g.o. manifold.*

Notice that if all geodesics through the origin $o = eK$ are of the form $\gamma(t) = \exp(tX) \cdot o$, then the geodesics through any other point $a \cdot p$ $(a \in G, p \in M)$ is of the form $a\gamma(t) = \exp(t\,\mathrm{Ad}(a)X) \cdot (a \cdot p)$.

Definition 2. *A non zero vector* $X \in \mathfrak{g}$ *is called a geodesic vector if the curve (1) is a geodesic.*

All calculations for a g.o. space G/K can be reduced to algebraic computations using geodesic vectors. These can be computed by using the following fundamental result of the subject, still call it "lemma" by tradition:

Lemma 2.1 (Geodesic Lemma [45])**.** *A non zero vector* $X \in \mathfrak{g}$ *is a geodesic vector if and only if*

$$\langle [X,Y]_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle = 0, \tag{3}$$

for all $Y \in \mathfrak{m}$. Here the subscript $\mathfrak{m}$ denotes the projection into $\mathfrak{m}$.

A useful description of homogeneous geodesics (1) is provided by the following :

Proposition 2.2. ([3]) *Let $(M = G/K, g)$ be a homogeneous Riemannian manifold and Λ be the associated operator. Let $a \in \mathfrak{k}$ and $x \in \mathfrak{m}$. Then the following are equivalent:*

(1) The orbit $\gamma(t) = \exp t(a + x) \cdot o$ of the one-parameter subgroup $\exp t(a + x)$ through the point $o = eK$ is a geodesic of M.

(2) $[a + x, \Lambda x] \in \mathfrak{k}$.

(3) $\langle [a, x], y \rangle = \langle x, [x, y]_{\mathfrak{m}} \rangle$ for all $y \in \mathfrak{m}$.

(4) $\langle [a + x, y]_{\mathfrak{m}}, x \rangle = 0$ for all $y \in \mathfrak{m}$.

As a consequence, we obtain the following characterization of g.o. spaces:

Corollary 2.3 ([3]). *Let $(M = G/K, g)$ be a homogeneous Riemannian manifold. Then $(M = G/K, g)$ is a g.o. space if and only if for every $x \in \mathfrak{m}$ there exists an $a(x) \in \mathfrak{k}$ such that*

$$[a(x) + x, \Lambda x] \in \mathfrak{k}.$$

Therefore, the property of being a g.o. space G/K, depends only on the reductive decomposition and the G-invariant metric metric g on $\mathfrak{m}$. That is, if $(M = G/H, g)$ is a g.o. space, then any locally isomorphic homogeneous Riemannian space $(M = G/H, g')$ is a g.o. space. Also, a direct product of Riemannian manifolds is a manifold with homogeneous geodesics if and only if each factor is a manifold with homogeneous geodesics.

In order to find all homogeneous geodesics in a homogeneous Riemannian manifold $(M = G/K, g)$ it suffices to find a decomposition of the form (2) and look for geodesic vectors of the form

$$X = \sum_{i=1}^{s} x_i e_i + \sum_{j=1}^{l} a_j A_j. \tag{4}$$

Here $\{e_i : i = 1, 2, \ldots, s\}$ is a convenient basis of $\mathfrak{m}$ and $\{A_j : j = 1, 2, \ldots, l\}$ is a basis of $\mathfrak{k}$. By substituting $X = e_i$ $(i = 1, \ldots, s)$ into equation (3) we obtain a system of linear algebraic equations for the variables x_i and a_j. The geodesic vectors correspond to those solutions for which $x_1, \dots, x_s$ are not all equal to zero. For some applications of this method we refer to [40] and [49]. Also, $(M = G/K, g)$ is a g.o. space if and only if for every non zero s-tuple $(x_1, \dots, x_s)$ there is an l-tuple $(a_1, \dots, a_l)$ satisfying all quadratic equations. A useful technique used for the characterization of Riemannian g.o. spaces is based on the concept of the geodesic graph, originally introduced in [59]. We first need the following definition.

Definition 3. *A Riemannian homogeneous space* $(G/K, g)$ *is called naturally reductive if there exists a reductive decomposition (2) of* $\mathfrak{g}$ *such that*

$$\langle [X, Z]_{\mathfrak{m}}, Y\rangle + \langle X, [Z, Y]_{\mathfrak{m}}\rangle = 0, \quad \textit{for all} \ \ X, Y, Z \in \mathfrak{m}. \qquad (5)$$

It is well known that condition (5) implies that all geodesics in G/K are homogeneous (e.g. [54]).

Definition 4. *A homogeneous Riemannian manifold* (M, g) *is naturally reductive if there exists a transitive group* G *of isometries for which the correseponding Riemannian homogeneous space* $(G/K, g)$ *is naturally reductive in the sense of Definition 3.*

Therefore, it could be possible that a homogeneous space $M = G/K$ is not naturally reductive for some group $G \in I_0(M)$ (the connected component of the full isometry group of M), but it is naturally reductive if we write $M = G'/K'$ for some larger group of isometries $G' \subset I_0(M)$.

Let $(M = G/K, g)$ be a g.o. space and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be an $\mathrm{Ad}(K)$-invariant decomposition. Then

(1) There exists an $\mathrm{Ad}(H)$-equivariant map $\eta : \mathfrak{m} \to \mathfrak{k}$ (a *geodesic graph*) such that for any $X \in \mathfrak{m} \setminus \{0\}$, the curve $\exp t(X + \eta(X)) \cdot o$ is a geodesic.

(2) A geodesic graph is either linear (which is equivalent to natural reductivity with respect to some $\mathrm{Ad}(K)$-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}'$) or it is non differentiable at the origin o.

It can be shown ([43]) that a geodesic graph (for a g.o. space) is uniquely determined by fixing an $\mathrm{Ad}(H)$-invariant scalar product on $\mathfrak{k}$. Examples of g.o. spaces by using geodesic graphs are given in [29], [32], and [43]. Conversely, the property (1) implies that G/K is a g.o. space.

Another technique for producing g.o. metrics was given by C. Gordon as shown below:

Proposition 2.4. ([37], [63]) *Let G be a connected semisimple Lie group and $H \supset K$ be compact Lie subgroups in G. Let M_F and M_C be the tangent spaces of $F = H/K$ and $C = G/H$ respectively. Then the metric $g_{a,b} = aB\mid_{M_F} + bB\mid_{M_C}, (a, b \in \mathbb{R}^+)$ is a g.o. metric on G/K if and only if for any $v_F \in M_F$, $v_C \in M_C$ there exists $X \in \mathfrak{k}$ such that*

$$[X, v_F] = [X + v_F, v_C] = 0.$$

Actually, Gordon proved a more general result based on description of naturally reductive left-invariant metrics on compact Lie groups given by J.E. D'Atri and W. Ziller in [24].

3. Low dimensional examples

The problem of a complete classification of g.o. manifolds is open. Even the classification all g.o. metrics on a given Riemannian homogeneous space is not trivial (cf. for example [51]). A complete classification is known up to dimesion 6, given by O. Kowalski and L. Vanhecke:

Theorem 3.1. ([45]) 1) *All Riemannian g.o. spaces of dimension up to 4 are naturally reductive.*

2) *Every 5-dimensional Riemannian g.o. space is either naturally reductive, or of isotropy type* SU(2).

3) *Every 6-dimensional Riemannian g.o. space is either naturally reductive or one of the following:*

(a) *A two-step nilpotent Lie group with two-dimensional center, equipped with a left-invariant Riemannian metric such that the maximal connected isotropy subgroup is isomorphic to* SU(2) *or* U(2). *Corresponding g.o. metrics depend on three real parameters.*

(b) *The universal covering space of a homogeneous Riemannian manifold of the form $(M = \mathrm{SO}(5)/\mathrm{U}(2), g)$ or $(M = \mathrm{SO}(4,1)/\mathrm{U}(2), g)$, where* SO(5) *or* SO(4, 1) *is the identity component of the full isometry group, respectively. In each case, all corresponding invariant metrics g.o. metrics g depend on two real parameters.*

As pointed out by the authors in [45, p. 190], the g.o. spaces (a) and (b) are *in no way naturally reductive* in the following sence: whatever the representation of (M, g) as a quotient of the form

G'/K', where G' is a connected transitive group of isometries of (M, g), and whatever is the $\mathrm{Ad}(K)$-invariant decomposition $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{m}'$, the curve $\gamma(t) = \exp(tX) \cdot o$ is never a geodesic (for any $X \in \mathfrak{m} \setminus \{0\}$).

The first 7-dimensional example of a g.o. manifold was given by C. Gordon in [37]. This is a nilmanifold (i.e. a connected Riemannian manifold admitting a transitive nilpotent group of isometries), and it was obtained under a general construction of g.o. nilmanifolds. It took some time until some more 7-dimensional examples were given. In [32] Z. Dušek, O. Kowalski and S. Nikčević gave families of 7-dimensional g.o. metrics. Their main result is the following:

Theorem 3.2. ([32]) *On the* 7*-dimensional homogeneous space* $G/K = (\mathrm{SO}(5)\times\mathrm{SO}(2))/\,\mathrm{U}(2)$ *(or* $G/H = (\mathrm{SO}(4,1)\times\mathrm{SO}(2))/\,\mathrm{U}(2)$*) there is a family* $g_{p,q}$ *of invariant metrics depending on two parameters* p, q *(where the pairs* (p, q) *fill in an open subset of the plane) such that each homogeneous Riemannian manifold* $(G/H, g_{p,q})$ *is a locally irreducible and not naturally reductive Riemannian g.o. manifold.*

4. Fibrations over symmetric spaces

In the work [63] H. Tamaru classified homogeneous spaces $M = G/K$ satisfying the following properties: (i) M is fibered over irreducible symmetric spaces G/H and (ii) certain G-invariant metrics on M are G-g.o. metrics. More precisely, for G connected and semisimple, and H, K compact with $G \supset H \supset K$, he considered the fibration

$$F = H/K \to M = G/K \to B = G/H$$

and the G-invariant metrics $g_{a,b}$ on M determined by the scalar products

$$\langle\ ,\ \rangle = aB|_{\mathfrak{f}} + bB|_{\mathfrak{b}}\,, \quad a, b > 0.$$

Here $\mathfrak{f}$ and $\mathfrak{b}$ are the tangent spaces of F and B respectively, so that the tangent space of M at the origin is identified with $\mathfrak{f} \oplus \mathfrak{b}$. By using results from polar representations, he classifed all triplets (G, H, K) so that the metrics $g_{a,b}$ are G-g.o. metrics. The triplets of Lie algebras $(\mathfrak{g}, \mathfrak{h}, \mathfrak{k})$ so that $(\mathfrak{g}, \mathfrak{h})$ is a symmetric pair and $(\mathfrak{g}, \mathfrak{k})$ corresponds to a G-g.o. space G/K, are listed in Table 1.

	$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{k}$
1	$\mathfrak{so}(2n+1), n \geq 2$	$\mathfrak{so}(2n)$	$\mathfrak{u}(n)$
2	$\mathfrak{so}(4n+1), n \geq 1$	$\mathfrak{so}(4n)$	$\mathfrak{su}(2n)$
3	$\mathfrak{so}(8)$	$\mathfrak{so}(7)$	$\mathfrak{g}_2$
4	$\mathfrak{so}(9)$	$\mathfrak{so}(8)$	$\mathfrak{so}(7)$
5	$\mathfrak{su}(n+1), n \geq 2$	$\mathfrak{u}(n)$	$\mathfrak{su}(n)$
6	$\mathfrak{su}(2n+1), n \geq 2$	$\mathfrak{u}(2n)$	$\mathfrak{u}(1) \oplus \mathfrak{sp}(n)$
7	$\mathfrak{su}(2n+1), n \geq 2$	$\mathfrak{u}(2n)$	$\mathfrak{sp}(n)$
8	$\mathfrak{sp}(n+1), n \geq 1$	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$	$\mathfrak{u}(1) \oplus \mathfrak{sp}(n)$
9	$\mathfrak{sp}(n+1), n \geq 1$	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$	$\mathfrak{sp}(n)$
10	$\mathfrak{su}(2r+n), r \geq 2, n \geq 1$	$\mathfrak{su}(r) \oplus \mathfrak{su}(r+n) \oplus \mathbb{R}$	$\mathfrak{su}(r) \oplus \mathfrak{su}(r+n)$
11	$\mathfrak{so}(4n+2), n \geq 2$	$\mathfrak{u}(2n+1)$	$\mathfrak{su}(2n+1)$
12	$\mathfrak{e}_6$	$\mathbb{R} \oplus \mathfrak{so}(10)$	$\mathfrak{so}(10)$
13	$\mathfrak{so}(9)$	$\mathfrak{so}(7) \oplus \mathfrak{so}(2)$	$\mathfrak{g}_2 \oplus \mathfrak{so}(2)$
14	$\mathfrak{so}(10)$	$\mathfrak{so}(8) \oplus \mathfrak{so}(2)$	$spin(7) \oplus \mathfrak{so}(2)$
15	$\mathfrak{so}(11)$	$\mathfrak{so}(8) \oplus \mathfrak{so}(3)$	$spin(7) \oplus \mathfrak{so}(3)$

TABLE 1. Riemannian g.o. spaces G/K fibered over irreducible symmetric spaces G/H ([63]).

5. Generalized flag manifolds

In the work [3] D.V. Alekseevsky and the author classified generalized flag manifolds with homogeneous geodesics. Recall that a generalized flag manifold is a homogeneous space G/K which is an adjoint orbit of a compact semisimple Lie group G. Equivalently, the isotropy subgroup K is the centralizer of a torus (i.e. a maximal abelian subgroup) in G. We assume that G acts effectively on M. A flag manifold $M = G/K$ is simply connected and has the canonically defined decomposition $M = G/K = G_1/K_1 \times G_2/K_2 \times \cdots \times G_n/K_n$, where $G_1, \ldots, G_n$ are simple factors of the (connected) Lie group G. This decomposition is the de Rham decomposition of M equipped with a G-invariant metric g. In particular, (M, g) is a g.o. space if and only if each factor $(M_i = G_i/K_i, g_i = g|_{M_i})$ is a g.o. space. This reduces the problem of the description of G-invariant metrics with homogeneous geodesics in a flag manifold $M = G/K$ to the case when the group G is simple.

Flag manifolds $M = G/K$ with G a simple Lie group can be classified in terms of their *painted Dynkin diagrams*. It turns out that for each classical Lie group there is an infinite series of flag manifolds, and for each of the exceptional Lie groups G_2, F_4, E_6, E_7, and E_8 there are 3, 11, 16, 31, and 40 non equivalent flag manifolds respectively (eg. [2], [16]). An important invariant of flag manifolds is their set of T-roots R_T. This is defined as the restriction of the root system R of $\mathfrak{g}$ to the center $\mathfrak{t}$ of the stability subalgebra $\mathfrak{k}$ of K. In [3] we defined the notion of *connected component* of R_T, namely two T-roots are in the same component if they can be connected by a chain of T-roots whose sum or difference is also a T-root. The set R_T is called *connected* if it has only one connected component.

Theorem 5.1. ([3]) *If the set of T-roots is connected then the standard metric on $M = G/K$ is the only G-invariant metric (up to scalar) which is a g.o. metric.*

Hence, for a flag manifold $M = G/K$ (G simple), a G-invariant g.o. metric may exist, only when R_T is not connected, so we only need to study those flag manifolds. It turns out that the system of T-roots is not connected only for three infinite series of a classical Lie group (namely the spaces $\mathrm{SO}(2\ell+1)/\mathrm{U}(\ell-m)\cdot\mathrm{SO}(2m+1)$, $\mathrm{Sp}(\ell)/\mathrm{U}(\ell-m)\cdot\mathrm{Sp}(m)$, and $\mathrm{SO}(2\ell)/\mathrm{U}(\ell-m)\cdot\mathrm{SO}(2m)$), and for 10 flag manifolds of an exceptional Lie group. An perpective of the above theorem is given by the following theorem:

Theorem 5.2. ([3]) *Let $M = G/K$ be a flag manifold of a simple Lie group. Then the set of T-roots is not connected if and only if the isotropy representation of M consists of two irreducible (non-equivalent) components.*

Therefore, the problem of the description of G-invariant metrics on flag manifolds with homogeneous geodesics reduces substantially to the study of this short list of prospective flag manifolds. To this end, we used the classification Table 1 of the work of H. Tamaru ([63]). Since any flag manifold can be fibered over a symmetric space ([17]), then by using Theorem 5.2 we obtain that the only flag manifolds which are in Table 1 are $\mathrm{SO}(2\ell+1)/\mathrm{U}(\ell)$ and $\mathrm{Sp}(\ell)/\mathrm{U}(1)\cdot\mathrm{Sp}(\ell-1)$.

On the other hand, in [5] D.N. Akhiezer and E.B. Vinberg classified all compact weakly symmetric spaces. Their classification shows that the only flag manifolds which are weakly symmetric spaces are $\mathrm{SO}(2\ell+1)/\mathrm{U}(\ell)$ and $C(1,\ell-1) = \mathrm{Sp}(\ell)/U(1)\cdot\mathrm{Sp}(\ell-1)$. This implies that any $\mathrm{SO}(2\ell+1)$-invariant metric g_λ on $\mathrm{SO}(2\ell+1)/\mathrm{U}(\ell)$

(depending, up to scale, on one real parameter λ) is weakly symmetric, hence it has homogeneous geodesics. Similarly for any $\mathrm{Sp}(\ell)$-invariant metric g_λ on $\mathrm{Sp}(\ell)/U(1)\cdot\mathrm{Sp}(\ell-1)$. In fact, the action of the group $\mathrm{SO}(2\ell+1)$ on $\mathrm{SO}(2\ell+1)/\mathrm{U}(\ell)$ can be extended to the action of the group $\mathrm{SO}(2\ell+2)$ with isotropy subgroup $U(\ell+1)$, which preserves the complex structure and the standard invariant metric g_0 (which corresponds to $\lambda = 1$). Hence, the Riemannian flag manifold $(\mathrm{SO}(2\ell+1)/\mathrm{U}(\ell), g_0)$ is isometric to the Hermitian symmetric space $\mathrm{Com}(\mathbb{R}^{2\ell+2}) = \mathrm{SO}(2\ell+2)/\mathrm{U}(\ell+1)$ of all complex structures in $\mathbb{R}^{2\ell+2}$. Similarly, the action of the group $\mathrm{Sp}(\ell)$ on $\mathrm{Sp}(\ell)/U(1)\cdot\mathrm{Sp}(\ell-1)$ can be extended to the action of the group $\mathrm{SU}(2\ell)$ with isotropy subgroup $\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(2\ell-1))$. As a consequence of the above we obtain the following:

Theorem 5.3. ([3]) *The only flag manifolds $M = G/K$ of a simple Lie group G admiting a non naturally reductive G-invariant metric g with homogeneous geodesics are the manifolds* $\mathrm{SO}(2\ell+1)/U(\ell)$ *and* $\mathrm{Sp}(\ell)/\mathrm{U}(1)\cdot\mathrm{Sp}(\ell-1)$ $(\ell \geq 2)$*, which admit (up to scale) a one-parameter family of* $\mathrm{SO}(2\ell+1)$ *(resp.* $\mathrm{Sp}(\ell)$*)-invariant metrics g_λ. Moreover, these manifolds are weakly symmetric spaces for $\lambda > 0$, and they are symmetric spaces with respect to Isom(g_λ) if and only if $\lambda = 1$, i.e. g_λ is a multiple of the standard metric.*

Note that for $\ell = 2$ we obtain $\mathrm{Sp}(2)/U(1)\cdot\mathrm{Sp}(1) \cong \mathrm{SO}(5)/\mathrm{U}(2)$, where the second quotient is an example of g.o. space in [45] which is not naturally reductive.

Finally, we mention a remarkable coincidence between Theorem 5.3 and a result by F. Podestà and G. Thorbergsson in [56], where they studied coisotropic actions on flag manifolds. One of their theorems states that if $M = G/K$ is a flag manifold of a simple Lie group then the action of K on M is coisotropic, if and only if M is up to local isomorphic either a Hermitian symmetric space, or one of the spaces obtained in Theorem 5.3.

6. Generalized Wallach spaces

Let G/K be a compact homogeneous space with connected compact semisimple Lie group G and a compact subgroup K with reductive decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m}$. Then G/K is called *generalized Wallach space* (known before as three-locally-symmetric spaces, cf.

[47]) if the module $\mathfrak{m}$ decomposes into a direct sum of three $\mathrm{Ad}(K)$-invariant irreducible modules pairwise orthogonal with respect to B, i.e. $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$, such that $[\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{k} \quad i = 1, 2, 3$. Every generalized Wallach space admits a three parameter family of invariant Riemannian metrics determined by $\mathrm{Ad}(K)$-invariant inner products $\langle \cdot, \cdot \rangle = \lambda_1 B(\cdot, \cdot) \mid_{\mathfrak{m}_1} + \lambda_2 B(\cdot, \cdot) \mid_{\mathfrak{m}_2} + \lambda_3 B(\cdot, \cdot) \mid_{\mathfrak{m}_3}$, where $\lambda_1, \lambda_2, \lambda_3$ are positive real numbers. The classification of generalized Wallach spaces was recently obtained by Yu.G. Nikoronov ([52]) (G semisimple) and Z. Chen, Y. Kang, K. Liang ([18]) (G simple) as follows:

Theorem 6.1 ([52])**.** *Let G/K be a connected and simply connected compact homogeneous space. Then G/K is a generalized Wallach space if and only if it is one of the following types:*

1) G/K is a direct product of three irreducible symmetric spaces of compact type.

2) The group is simple and the pair $(\mathfrak{g}, \mathfrak{k})$ is one of the pairs in Table 2.

3) $G = F \times F \times F \times F$ and $K = diag(F) \subset G$ for some connected, compact, simple Lie group F, with the following description on the Lie algebra level:

$$(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{f} \oplus \mathfrak{f} \oplus \mathfrak{f} \oplus \mathfrak{f}, \mathrm{diag}(\mathfrak{f})) = \{(X, X, X, X) \mid X \in f\},$$

where $\mathfrak{f}$ is the Lie algebra of F, and (up to permutation) $\mathfrak{m}_1 = \{(X, X, -X, -X) \mid X \in f\}$, $\mathfrak{m}_2 = \{(X, -X, X, -X) \mid X \in f\}$, $\mathfrak{m}_3 = \{(X, -X, -X, X) \mid X \in f\}$.

$\mathfrak{g}$	$\mathfrak{k}$	$\mathfrak{g}$	$\mathfrak{k}$
$\mathfrak{so}(k+l+m)$	$\mathfrak{so}(k) \oplus \mathfrak{so}(l) \oplus \mathfrak{so}(m)$	$\mathfrak{e}_7$	$\mathfrak{so}(8) \oplus 3\mathfrak{sp}(1)$
$\mathfrak{su}(k+l+m)$	$\mathfrak{su}(k) \oplus su(l) \oplus \mathfrak{su}(m)$	$\mathfrak{e}_7$	$\mathfrak{su}(6) \oplus \mathfrak{sp}(1) \oplus \mathbb{R}$
$\mathfrak{sp}(k+l+m)$	$\mathfrak{sp}(k) \oplus \mathfrak{sp}(l) \oplus \mathfrak{sp}(m)$	$\mathfrak{e}_7$	$\mathfrak{so}(8)$
$\mathfrak{su}(2l), l \geq 2$	$\mathfrak{u}(l)$	$\mathfrak{e}_8$	$\mathfrak{so}(12) \oplus 2\mathfrak{sp}(1)$
$\mathfrak{so}(2l), l \geq 4$	$\mathfrak{u}(l) \oplus \mathfrak{u}(l-1)$	$\mathfrak{e}_8$	$\mathfrak{so}(8) \oplus \mathfrak{so}(8)$
$\mathfrak{e}_6$	$\mathfrak{su}(4) \oplus 2\mathfrak{sp}(1) \oplus \mathbb{R}$	$\mathfrak{f}_4$	$\mathfrak{so}(5) \oplus 2\mathfrak{sp}(1)$
$\mathfrak{e}_6$	$\mathfrak{so}(8) \oplus \mathbb{R}^2$	$\mathfrak{f}_4$	$\mathfrak{so}(8)$
$\mathfrak{e}_6$	$\mathfrak{sp}(3) \oplus \mathfrak{sp}(1)$		

Table 2. The pairs $(\mathfrak{g}, \mathfrak{k})$ corresponding to generalized Wallach spaces G/K with G simple ([52]).

In [10] Yu Wang and the author investigated which of the families of spaces listed in Theorem 6.1 are g.o. spaces. By applying the method of searching for geodesics vectors shown at the end of Section 2 we obtained the following:

Theorem 6.2. ([10]) *Let $(G/K, g)$ be a generalized Wallach space as listed in Theorem 6.1. Then*

1) If $(G/K, g)$ is a space of type 1) then this is a g.o. space for any $\mathrm{Ad}(K)$*-invariant Riemannian metric.*

2) If $(G/K, g)$ is a space of type 2) or 3) then this is a g.o. space if and only if g is the standard metric.

However, in order to find all homogeneous geodesics in G/K it suffices to find all the real solutions of a system of $\dim\mathfrak{m}_1 + \dim\mathfrak{m}_2 + \dim\mathfrak{m}_3$ quadratic equations. By Theorem 6.2 we only need to consider homogeneous geodesics for spaces of types 2) and 3) given in Theorem 6.1, for the metric $(\lambda_1, \lambda_2, \lambda_3)$, where at least two of $\lambda_1, \lambda_2, \lambda_3$ are different. This is not easy in general. We obtained all homogeneous geodesics (for various values of the parameters $\lambda_1, \lambda_2, \lambda_3$ for the generalized Wallach space $SU(2)/\{e\}$, hence recovering a result on R.A. Marinosci ([49, p. 266]), and for the Stiefel manifolds $\mathrm{SO}(n)/\mathrm{SO}(n-2)$, $(n \geq 4)$.

7. M-spaces

Let G/K be a generalized flag manifold with $K = C(S) = S \times K_1$, where S is a torus in a compact simple Lie group G and K_1 is the semisimple part of K. Then the *associated M-space* is the homogeneous space G/K_1. These spaces were introduced and studied by H.C. Wang in [64].

In the works [11] and [12]Y. Wang, G. Zhao and the author investigated homogeneous geodesics in a class of homogeneous spaces called M-spaces. We proved that for various classes of M-spaces, the only g.o. metric is the standard metric. For other classes of M-spaces we give either necessary or necessary and sufficient conditions so that a G-invariant metric on G/K_1 is a g.o. metric. The analysis is based on properties of the isotropy representation $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s$ of the flag manifold G/K, in particular on the dimension of the submodules $\mathfrak{m}_i$. We summarize these results below.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of the Lie groups G and K respectively. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be an $\mathrm{Ad}(K)$-invariant reductive decomposition of the Lie algebra $\mathfrak{g}$, where $\mathfrak{m} \cong T_o(G/K)$. This is orthogonal with respect to $B = -$Killing from on $\mathfrak{g}$. Assume that

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s \tag{6}$$

is a B-orthogonal decomposition of $\mathfrak{m}$ into pairwise inequivalent irreducible $\mathrm{ad}(\mathfrak{k})$-modules.

Let G/K_1 be the corresponding M-space and $\mathfrak{s}$ and $\mathfrak{k}_1$ be the Lie algebras of S and K_1 respectively. We denote by $\mathfrak{n}$ the tangent space $T_o(G/K_1)$, where $o = eK_1$. A G-invariant metric g on G/K_1 induces a scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{n}$ which is $\mathrm{Ad}(K_1)$-invariant. Such an $\mathrm{Ad}(K_1)$-invariant scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{n}$ can be expressed as $\langle x, y\rangle = B(\Lambda x, y)$ $(x, y \in \mathfrak{n})$, where Λ is the $\mathrm{Ad}(K_1)$-equivariant positive definite symmetric operator on $\mathfrak{n}$.

The main results are the following:

Theorem 7.1. ([11]) *Let G/K be a generalized flag manifold with $s \geq 3$ in the decomposition (6). Let G/K_1 be the corresponding M-space.*

1) If $\dim \mathfrak{m}_i \neq 2$ $(i = 1, \ldots, s)$ and $(G/K_1, g)$ is a g.o. space, then

$$g = \langle\cdot,\cdot\rangle = \mu B(\cdot,\cdot)\mid_{\mathfrak{s}} + \lambda B(\cdot,\cdot)\mid_{\mathfrak{m}_1\oplus\mathfrak{m}_2\oplus\cdots\oplus\mathfrak{m}_s}, \ (\mu, \lambda > 0).$$

2) If there exists some $j \in \{1, \ldots, s\}$ such that $\dim \mathfrak{m}_j = 2$, then $(G/K_1, g)$ is a g.o. space if and only if g is the standard metric.

Theorem 7.2. ([12]) *Let G/K be a generalized flag manifold with two isotropy summands given by (6), and $(G/K_1, g)$ be the corresponding M-space. Then*

1) If $\dim \mathfrak{m}_2 = 2$, then the standard metric is the only g.o. metric on M-space $(G/K_1, g)$, unless $G/K_1 = SO(5)/SU(2)$ or $Sp(n)/Sp(n-1)$, $(n \geq 2)$.

2) If $\dim \mathfrak{m}_2 \neq 2$ and the M-space $(G/K_1, g)$ is a g.o. space, then $g = \langle\cdot,\cdot\rangle = \mu B(\cdot,\cdot)\mid_{\mathfrak{s}} + \lambda B(\cdot,\cdot)\mid_{\mathfrak{m}_1\oplus\mathfrak{m}_2}$, $(\mu, \lambda > 0)$, unless $G/K_1 = SO(2n+1)/SU(n), (n > 2)$.

However, the spaces $\mathrm{SO}(5)/\mathrm{SU}(2)$ and $\mathrm{Sp}(n)/\mathrm{Sp}(n-1)$ are included in Tamaru's Table 1, therefore they admit g.o. metrics. For the generalized flag manifolds with $s = 1$ or 2 in the decomposition (6) we use Theorem 6.2 and Tamaru's results in [63] to prove existence of non naturally reductive g.o. metrics in certain M-spaces, including the three isolated classes listed in parts 1) and 2) of Theorem 6.2.

We prove the following:

Theorem 7.3. ([12]) *The M-spaces $SU(n+1)/SU(n)$, $(n \geq 2)$, $SU(2r+n)/SU(r)\times SU(r+n)$, $(r \geq 2, n \geq 1)$, $SO(4n+1)/SU(2n)$, $(n \geq 1)$, $Sp(n)/Sp(n-1), (n \geq 2)$, $SO(4n+2)/SU(2n+1)$, $(n \geq 2)$ and $E_6/SO(8)$ admit non naturally reductive g.o. metrics.*

Finally, by using techniques from [11] we can prove the following:

Theorem 7.4. ([11]) *Let G/K be a generalized flag manifold with corresponding M-space $(G/K_1, g)$.*

1) If $G = G_2$, then $(G_2/K_1, g)$ is a g.o. space if and only if g is the standard metric.

2) If $G = F_4$, then the standard metric is the only g.o. metric on F_4/K_1, unless $K_1 = SU(2) \times SU(3)$, or $K_1 = SO(7)$.

3) If $G = E_6$, then the standard metric is the only g.o. metric on E_6/K_1, unless K_1 is one of $SU(3)\times SU(3)\times SU(2)$, $SU(5)\times SU(2)$, $SU(2) \times SU(2) \times SU(3)$, $SO(8)$, or $SO(10)$.

By a result of H. Tamaru [63] it follows that the M-space $E_6/SO(10)$ admits non-naturally reductive g.o. metrics.

8. Homogeneous geodesics in pseudo-Riemannian manifolds

It is well known that any homogeneous Riemannian manifold is reductive, but this is not the case for pseudo-Riemannian manifolds in general. In fact, there exist homogeneous pseudo-Riemannian manifolds which do not admit any reductive decomposition. Therefore, there is a dichotomy in the study of geometrical problems between reductive and non reductive pseudo-Riemannian manifolds. Due to the existence of null vectors in a pseudo-Riemannian manifold the definition of a homogeneous geodesic $\gamma(t) = \exp(tX)\cdot o$ needs to be modified by requiring that $\nabla_{\dot\gamma}\dot\gamma = k(\gamma)\dot\gamma$ (see also relevant discussion in [48, pp. 90-91]). It turns out that $k(\gamma)$ is a constant function (cf. [31]. Even though an algebraic characterization of geodesic vectors (that is an analogue of the geodesic Lemma 2.1) was known to physicists ([35], [55]), a formal proof was given by Z. Dušek and O. Kowalski in [31].

Lemma 8.1 ([31])**.** *Let $M = G/H$ be a reductive homogeneous pseudo-Riemannian space with reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$,* *and* $X \in \mathfrak{g}$. *Then the curve* $\gamma(t) = \exp(tX) \cdot o$ *is a geodesic curve with respect to some parameter* s *if and only if*

$$\langle [X, Z]_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle = k \langle X_{\mathfrak{m}}, Z_{\mathfrak{m}} \rangle, \quad \textit{for all} \ \ Z \in \mathfrak{m},$$

where k *is some real constant. Moreover, if* $k = 0$*, then* t *is an affine parameter for this geodesic. If* $k \neq 0$*, then* $s = e^{kt}$ *is an affine parameter for the geodesic. This occurs only if the curve* $\gamma(t)$ *is a null curve in a (properly) pseudo-Riemannian space.*

For applications of this lemma see [28]. The existence of homogeneous geodesics in homogeneous pseudo-Riemannian spaces (for both reductive and non reductive) was answered positively only recently by Z. Dušek in [27].

Two-dimensional and three-dimensional homogeneous pseudo Riemannian manifolds are reductive ([19], [34]). Four-dimensional non reductive homogeneous pseudo-Riemannian manifolds were classified by M.E. Fels and A.G. Renner in [34] in terms of their non reductive Lie algebras. Their invariant pseudo-Riemannian metrics, together with their connection and curvature, were explicitly described in by G. Calvaruso and A. Fino in [22].

The three-dimensional pseudo-Riemannian g.o. spaces were classified by G. Calvaruso and Marinosci in [21]. In the recent work [23], G. Calvaruso, A. Fino and A. Zaeim obtained explicit examples of four-dimensional non reductive pseudo-Riemannian g.o. spaces. They deduced an explicit description in coordinates for all invariant metrics of non reductive homogeneous pseudo-Riemannian four-manifolds. For those four-dimensional non reductive pseudo-Riemannia spaces which are not g.o., they determined the homogeneous geodesics though a point.

9. Two-step homogeneous geodesics

In the work [9] N.P. Souris and the author considered a generalisation of homogeneous geodesics, namely geodesics of the form

$$\gamma(t) = \exp(tX)\exp(tY) \cdot o, \quad X, Y \in \mathfrak{g}, \tag{7}$$

which we named *two-step homogeneous geodesics*. We obtained sufficient conditions on a Riemannian homogeneous space G/K, which imply the existence of two-step homogeneous geodesics in G/K. A Riemannian homogeneous spaces G/K such that any geodesic of

G/K passing through the origin is two-step homogeneous is called *two-step g.o. spaces.*

Geodesics of the form (7) had appeared in the work [65] of H.C. Wang as geodesics in a semisimple Lie group G, equipped with a metric induced by a Cartan involution of the Lie algebra $\mathfrak{g}$ of G. Also, in [25] R. Dohira proved that if the tangent space $T_o(G/K)$ of a homogeneous space splits into submodules $\mathfrak{m}_1, \mathfrak{m}_2$ satisfying certain algebraic relations, and if G/K is endowed with a special one parameter family of Riemannian metrics g_c, then all geodesics of the Riemannian space $(G/K, g_c)$ are of the form (7). The main result of [9] is the following:

Theorem 9.1. ([9]) *Let $M = G/K$ be a homogeneous space admitting a naturally reductive Riemannian metric. Let B be the corresponding inner product on $\mathfrak{m} = T_o(G/K)$. We assume that $\mathfrak{m}$ admits an* $\mathrm{Ad}(K)$*-invariant orthogonal decomposition*

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \cdots \oplus \mathfrak{m}_s, \tag{8}$$

with respect to B. We equip G/K with a G-invariant Riemannian metric g corresponding to the $\mathrm{Ad}(K)$*-invariant positive definite inner product $\langle \cdot, \cdot \rangle = \lambda_1 \, B|_{\mathfrak{m}_1} + \cdots + \lambda_s \, B|_{\mathfrak{m}_s}$, $\lambda_1, \ldots, \lambda_s > 0$. If $(\mathfrak{m}_a, \mathfrak{m}_b)$ is a pair of submodules in the decomposition (8) such that*

$$[\mathfrak{m}_a, \mathfrak{m}_b] \subset \mathfrak{m}_a, \tag{9}$$

then any geodesic γ of $(G/K, g)$ with $\gamma(0) = o$ and $\dot\gamma(0) \in \mathfrak{m}_a \oplus \mathfrak{m}_b$, is a two-step homogeneous geodesic. In particular, if $\dot\gamma(0) = X_a + X_b \in \mathfrak{m}_a \oplus \mathfrak{m}_b$, then for every $t \in \mathbb{R}$ this geodesic is given by

$$\gamma(t) = \exp t(X_a + \lambda X_b) \exp t(1-\lambda) X_b \cdot o, \quad \text{where } \lambda = \lambda_b/\lambda_a.$$

Moreover, if either $\lambda_a = \lambda_b$ or $[\mathfrak{m}_a, \mathfrak{m}_b] = \{0\}$ holds, then γ is a homogeneous geodesic, that is $\gamma(t) = \exp t(X_a + X_b) \cdot o$, for any $t \in \mathbb{R}$.

The following corollary provides a method to obtain many examples of two-step g.o. spaces.

Corollary 9.2. *Let $M = G/K$ be a homogeneous space admitting a naturally reductive Riemannian metric. Let B be the corresponding inner product of $\mathfrak{m} = T_o(G/K)$. We assume that $\mathfrak{m}$ admits an* $\mathrm{Ad}(K)$*-invariant, B-orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$, such* *that* $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_1$. *Then* M *admits an one-parameter family of* G*-invariant Riemannian metrics* g_λ, $\lambda \in \mathbb{R}^+$, *such that* (M, g_λ) *is a two-step g.o. space. Each metric* g_λ *corresponds to an* $\mathrm{Ad}(K)$*-invariant positive definite inner product on* $\mathfrak{m}$ *of the form* $\langle\ ,\ \rangle = B|_{\mathfrak{m}_1} + \lambda\, B|_{\mathfrak{m}_2}$, $\lambda > 0$.

The above Corollary 9.2 is a generalisation of Dohira's result [25].

The main tool for the proof of Theorem 9.1 is the following proposition.

Proposition 9.3. ([8])*Let* $M = G/K$ *be a homogeneous space and* $\gamma : \mathbb{R} \to M$ *be the curve* $\gamma(t) = \exp(tX)\exp(tY)\exp(tZ)\cdot o$, *where* $X, Y, Z \in \mathfrak{m}$. *Let* $T : \mathbb{R} \to \mathrm{Aut}(\mathfrak{g})$ *be the map given by*
$T(t) = \mathrm{Ad}(\exp(-tZ)\exp(-tY))$. *Then* γ *is a geodesic in* M *through* $o = eK$ *if and only if for any* $W \in \mathfrak{m}$, *the function* $G_W : \mathbb{R} \to \mathbb{R}$ *given by*

$$\begin{aligned} G_W(t) \quad &= \quad \langle\ (TX)_{\mathfrak{m}} + (TY)_{\mathfrak{m}} + Z_{\mathfrak{m}}, [W, TX + TY + Z]_{\mathfrak{m}}\ \rangle \\ &\quad + \ \langle\ W, [TX, TY + Z]_{\mathfrak{m}} + [TY, Z]_{\mathfrak{m}}\ \rangle, \end{aligned}$$

is identically zero, for every $t \in \mathbb{R}$.

The above proposition is a new tool towards the study of geodesics consisting of more than one exponential factors. In fact, for $X = Y = 0$ we obtain Lemma 2.1 of Kowalski and Vanhecke.

A natural application of Corollary 9.2 is for total spaces of homogeneous Riemannian submersions, as shown below.

Proposition 9.4. *Let* G *be a Lie group admitting a bi-invariant Riemannian metric and let* K, H *be closed and connected subgroups of* G, *such that* $K \subset H \subset G$. *Let* B *be the* Ad*-invariant positive definite inner product on the Lie algebra* $\mathfrak{g}$ *corresponding to the bi-invariant metric of* G. *We identify each of the spaces* $T_o(G/K)$, $T_o(G/H)$ *and* $T_o(H/K)$ *with corresponding subspaces* $\mathfrak{m}, \mathfrak{m}_1$ *and* $\mathfrak{m}_2$ *of* $\mathfrak{g}$, *such that* $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. *We endow* G/K *with the* G*-invariant Riemannian metric* g_λ *corresponding to the* $\mathrm{Ad}(K)$*-invariant positive definite inner product* $\langle\ ,\ \rangle = B|_{\mathfrak{m}_1} + \lambda\, B|_{\mathfrak{m}_2}$, $\lambda > 0$. *Then* $(G/K, g_\lambda)$ *is a two-step g.o. space.*

Example 9.5. ([8]) The odd dimensional sphere $\mathbb{S}^{2n+1}$ can be considered as the total space of the homogeneous Hopf bundle $\mathbb{S}^1 \to \mathbb{S}^{2n+1} \to \mathbb{C}P^n$. Let g_1 be the standard metric of $\mathbb{S}^{2n+1}$. We equip $\mathbb{S}^{2n+1}$ with an one parameter family of metrics g_λ, which "deform" the standard metric along the Hopf circles $\mathbb{S}^1$. By setting $G = \mathrm{U}(n+1)$, $K = \mathrm{U}(n)$ and $H = \mathrm{U}(n)\times\mathrm{U}(1)$, the Hopf bundle corresponds to the fibration $H/K \to G/K \to G/H$.

Since $\mathrm{U}(n+1)$ is compact, it admits a bi-invariant metric corresponding to an $\mathrm{Ad}(\mathrm{U}(n+1))$-invariant positive definite inner product B on $\mathfrak{u}(n+1)$. We identify each of the spaces $T_o\mathbb{S}^{2n+1} = T_o(G/K), T_o\mathbb{C}P^n = T_o(G/H)$, and $T_o\mathbb{S}^1 = T_o(H/K)$ with corresponding subspaces $\mathfrak{m}, \mathfrak{m}_1$, and $\mathfrak{m}_2$ of $\mathfrak{u}(n+1)$. The desired one parameter family of metrics g_λ corresponds to the one parameter family of positive definite inner products $\langle\ ,\ \rangle = B|_{\mathfrak{m}_1} + \lambda\, B|_{\mathfrak{m}_2}$, $\lambda > 0$ on $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$. Then Proposition 9.4 implies that $(\mathbb{S}^{2n+1}, g_\lambda)$ is a two-step g.o. space. In particular, let $X \in T_o\mathbb{S}^{2n+1}$. Then the unique geodesic γ of $(\mathbb{S}^{2n+1}, g_\lambda)$ with $\gamma(0) = o$ and $\dot\gamma(0) = X$, is given by $\gamma(t) = \exp t(X_1 + \lambda X_2)\exp t(1-\lambda)X_2\cdot o$, where X_1, X_2 are the projections of X on $\mathfrak{m}_1 = T_o\mathbb{C}P^n$ and $\mathfrak{m}_2 = T_o\mathbb{S}^1$ respectively. Note that if $\lambda = 1+\epsilon$, $\epsilon > 0$, then the metrics $g_{1+\epsilon}$ are Cheeger deformations of the natural metric g_1.

By using Proposition 9.2 it is possible to construct various classes of two-step g.o spaces. The recipe is the following:
(i) Let G/K be a homogeneous space with reductive decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m}$ admitting a naturally reductive metric corresponding to a positive definite inner product B on $\mathfrak{m}$.
(ii) We consider an $\mathrm{Ad}(K)$-invariant, orthogonal decomposition $\mathfrak{m} = \mathfrak{n}_1\oplus\cdots\oplus\mathfrak{n}_s$ with respect to B.
(iii) We separate the submodules $\mathfrak{n}_i$ into two groups as $\mathfrak{m}_1 = \mathfrak{n}_{i_1}\oplus\cdots\oplus\mathfrak{n}_{i_n}$ and $\mathfrak{m}_2 = \mathfrak{n}_{i_{n+1}}\oplus\cdots\oplus\mathfrak{n}_{i_s}$, so that $[\mathfrak{m}_1,\mathfrak{m}_2]\subset\mathfrak{m}_1$. The decomposition $\mathfrak{m} = \mathfrak{m}_1\oplus\mathfrak{m}_2$ is $\mathrm{Ad}(K)$-invariant and orthogonal with respect to B.
(iv) Then Corollary 9.2 implies that G/K admits an one parameter family of metrics g_λ so that $(G/K, g_\lambda)$ is a two-step g.o. space.

In [9] we applied the above recipe to the following classes of homogeneous spaces:
1) Lie groups with bi-invariant metrics, equipped with an one parameter family of left-invariant metrics.
2) Flag manifolds equipped with certain one parameter families of diagonal metrics.
3) Generalized Wallach spaces equipped with three different types of diagonal metrics (thus recovering some results from [8]).

4) k-symmetric spaces where k is even, endowed with an one parameter family of diagonal metrics.

10. Some open problems

It seems that the target for a complete classification of homogeneous g.o. spaces in any dimension greater than seven is far for being accomplished. In dimension seven there are several examples but a complete classification is still unknown. However, as shown in the present paper, for some large classes of homogeneous spaces it is possible to obtain some necessary conditions for the g.o. property. These conditions are normally imposed by the special Lie theoretic structure of corresponding homogeneous space. Also, the problem of an explicit description of homogeneous geodesics for spaces which are not g.o., is not trivial either. Eventhough it is mathematically simple, it requires high computational complexity. A more tractable problem could be to classify g.o. spaces with two or three irreducible isotropy summands.

Further, it is not usually an easy matter to show that the g.o. property of $(M = G/K, g)$ does not depend on the representation as a coset space and on the $\mathrm{Ad}(K)$-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Therefore, we often stress that we study G-g.o. spaces.

Also, it would be interesting to see how various results about Riemannian manifolds could be adjusted to pseudo-Riemannian manifolds, such as Propositions 2.2, 9.3.

Concerning generalizations of the g.o. property, we have introduced the concept of a two-step homogeneous geodesic and two-step g.o. space. We conjecture that a search for three-step (or more) homogeneous geodesics reduces to two-step homogeneous geodesics. Also, it would be interesting to study two-step homogeneous geodesics in pseudo-Riemannian manifolds (formulate corresponding geodesic lemma etc.).

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Sums of Polynomial Residues

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Abstract. In an article in the Monthly from 1904, Orlando Stetson studied the sums of distinct residues of triangular numbers modulo a prime. Rather curiously, this sum is always the same residue class independent of the prime chosen. We extend Stetson's theorem to all polygonal numbers and find similar phenomenon. Extensions to sums of residues of general polynomials are also discussed.

1. Introduction

Recall that the $n^{\underline{\text{th}}}$ s-gonal number is the number of points that are needed to create a regular polygon with s sides, each of length $n-1$ (see figure 1).

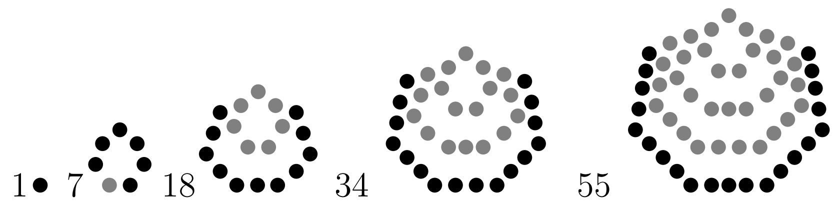

Figure 1. Heptagonal Numbers[1]

We denote these numbers by $P_s(n)$. Alternatively, we can use the algebraic description used by Stetson [5] and characterize these sequences with the recursions

$$\begin{aligned} P_s(1) &= 1 \\ P_s(n+1) - P_s(n) &= n(s-2) + 1. \end{aligned} \tag{1}$$

For example, in the *triangular numbers*, or 3-gonal numbers, the difference of consecutive terms follow the pattern $2, 3, 4, \ldots$, while the difference of consecutive squares (4-gonal) follows the sequence

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[1]Figure 1 created by Erica Maciejewski.

of odd numbers $3, 5, 7, \ldots$. Residues of squares, known as quadratic residues, have been well understood beginning as far back as the arithmeticae of Gauss [2].

Theorem 1.1 (Gauss, 1801)**.** *For an odd prime p, there are $(p-1)/2$ distinct quadratic residues modulo p. The sum of these residues is divisible by p.*

A similar property was discovered by Stetson [5] for the triangular numbers.

Theorem 1.2 (Stetson, 1904)**.** *For a prime $p \geq 5$, there are $(p-1)/2$ distinct triangular residues. The sum of these residues is congruent to $-1/16$ modulo p.*[2]

It is widely known that the sequence (1) of s-gonal numbers is generated by the function

$$P_s(n) = \frac{n^2(s-2) - n(s-4)}{2}. \tag{2}$$

We therefore observe that $2P_{2s+1}(x)$ and $P_{2s}(x)$ are quadratic polynomials in $\mathbb{Z}[x]$. It is natural to then ask about the residues of other quadratics, or about the residues of even more general polynomials. In Section 2 we revisit Stetson's work and in Section 3 we pick up where he left off in 1904 by investigating the more general sets of polygonal numbers and quadratics modulo a prime.

The question of sums of residues of more general polynomials is much more difficult. Although the results of the present work are mostly focused on sums of distinct residues of polygonal numbers, in Section 4 we provide conjectural result for a certain class of cubics, as well as a brief historical account of the complexity that arises in studying the residues of an arbitrary polynomial.

2. Stetson's Theorem

Being that Stetson's original work is over a century old, in this section we introduce our general notation, and reproduce the proof of Theorem 1.2 for completeness.

[2]We have adopted the convention of using fractions modulo p where it is understood that a number in the denominator represents the modular inverse of that number. For example, in Stetson's theorem above we mean the inverse of -16 modulo p.

Definition 2.1. Let p be a prime and $s \geq 3$ be an integer. The integer k is called an *s-gonal residue modulo p* if $k \not\equiv 0 \pmod{p}$ and $k \equiv P_s(n) \pmod{p}$ for some positive integer $n < p$. If no such n exists, we say that k is an *s-gonal non-residue* modulo p. Additionally, we define $\mathscr{S}_s(p)$ as the sum of the distinct s-gonal residues modulo p.

The following formulas can be found in most calculus texts, or obtained by induction.

Lemma 2.2. *Let n be a positive integer. Then*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2},$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} \frac{k(k+1)}{2} = \frac{n(n+1)(n+2)}{6}.$$

Proof of Theorem 1.2. Let $p \geq 5$ be a prime. Notice that $P_3(p-1) \equiv 0 \pmod{p}$. For the remaining integers n satisfying $0 < n < p-1$ we have

$$P_3(n) \equiv P_3(p-n-1) \pmod{p}.$$

Since $0 \leq p-n-1 \leq p-1$, we deduce that the triangular residues in the interval $[1, p-1)$ come in pairs, except for the case when $n = (p-1)/2$. It follows that the set

$$\{P_3(1), P_3(2), \ldots, P_3\left((p-1)/2\right)\}$$

is the complete set of $(p-1)/2$ distinct triangular residues modulo p. Using the formulas given in (2) and Lemma 2.2 we may calculate $\mathscr{S}_3(p)$ and obtain $\mathscr{S}_3(p) \equiv -\frac{1}{16} \pmod{p}$. $\square$

3. Generalizing Stetson's Theorem

Notice that the generating function for the s-gonal numbers given in (2) is a quadratic polynomial in n. With this observation, we prove an analogous result for all quadratic polynomials, and then apply this generalization to the polygonal numbers.

Definition 3.1. Let p be a prime and let $f(x)$ be a polynomial with integer coefficients. The integer k is called an *f-polynomial residue* *modulo* p if $k \not\equiv 0 \pmod{p}$ and $k \equiv f(n) \pmod{p}$ for some integer n. Additionally, we define $\mathscr{S}_f(p)$ to be the sum of the distinct f-polynomial residues modulo p.

Theorem 3.2. *Let $f(x) = ax^2 + bx + c$ be a quadratic polynomial with integer coefficients. For a prime $p \geq 5$ not dividing a we have*

$$\mathscr{S}_f(p) \equiv -\frac{b^2 - 4ac}{8a} \pmod{p}.$$

Proof. Let $f(x) = ax^2+bx+c$ be a quadratic polynomial with integer coefficients and let $p \geq 5$ be a prime not dividing a. For integers m and n we have that $f(n) \equiv f(m) \pmod{p}$ if and only if

$$\begin{aligned} 0 &\equiv an^2 + bn - am^2 - bm \\ &\equiv (n-m)\left(n + m + \frac{b}{a}\right), \end{aligned}$$

if and only if $n \equiv m \pmod{p}$ or $n \equiv -m - \frac{b}{a} \pmod{p}$, with both conditions occurring whenever $n \equiv -\frac{b}{2a} \pmod{p}$. Therefore outside of this last case the f-polynomial residues come in pairs. Using the equations from Lemma 2.2 we deduce that

$$\begin{aligned} \mathscr{S}_f(p) &\equiv \frac{\left(\sum_{i=0}^{p-1} f(i)\right) - f\left(-\frac{b}{2a}\right)}{2} + f\left(-\frac{b}{2a}\right) \pmod{p} \\ &\equiv \frac{1}{2}\left(-\frac{b^2}{4a} + c + \sum_{i=0}^{p-1} ai^2 + bi + c\right) \pmod{p} \\ &\equiv \frac{1}{2}\left(-\frac{b^2}{4a} + c + a \cdot \frac{p(p-1)(2p-1)}{6} + b \cdot \frac{p(p-1)}{2} + cp\right) \\ &\equiv -\frac{b^2 - 4ac}{8a} \pmod{p}. \end{aligned}$$

□

In the case of polygonal numbers, one may observe that the s-gonal residues still come in pairs. However, we no longer have the symmetry in the distribution of residues as in the 3-gonal case, where the residues occurred in pairs - one below $(p-1)/2$ and one above. For example, with the pentagonal numbers, $P_5(2) \equiv P_5(4) \pmod{17}$ with $(p-1)/2 = 8$. It is not even the case that the residues will come in pairs below and pairs above $(p-1)/2$, e.g. $P_5(7) \equiv P_5(15)$

(mod 17). Nonetheless, Theorem 3.2 provides an immediate corollary for polygonal numbers whenever s is even. In the case that s is odd and $P_s(n)$ has rational coefficients, it is enough to notice that the argument in Theorem 3.2 only requires that $2^{-1} \pmod p$ exists, which it does, and that $s \not\equiv 2 \pmod p$ in order to avoid division by 0.

Corollary 3.3. *Let $p \geq 5$ be a prime and $s \geq 3$ be an integer. If $s \not\equiv 2 \pmod p$, then there are $(p-1)/2$ distinct s-gonal residues modulo p, and*

$$\mathscr{S}_s(p) \equiv -\frac{1}{16}\frac{(s-4)^2}{(s-2)} \pmod p.$$

Remark 3.4. The special cases of $s = 2$ or p dividing a can be handled trivially. In the former, the 2-gonal numbers are simply $1, 2, 3, \ldots$. As such, the sum of distinct resudes modulo a prime p is 0. If p divides a, then $f(x) \equiv bx + c \pmod p$. If p also divides b, then $\mathscr{S}_f(p) \equiv c \pmod p$ as c is the only residue. On the other hand, if $\gcd(b, p) = 1$ then $0, b, 2b, \ldots, (p-1)b$ is a complete system of distinct residues, with sum 0 modulo p.

The case for higher degree polynomials is much more complicated, for reasons discussed in the next section. We have, however, attempted to investigate several classes of cubics, and we close this section with our most promising heuristic.

Conjecture 3.5. *Let a, b be integers and let $f(x) = ax^3 + bx^2$. For a prime $p \geq 5$ not dividing a,*

$$\mathscr{S}_f(p) = \begin{cases} \dfrac{2b^3}{81a^2} & \text{if } p \equiv 1 \pmod 6 \\ -\dfrac{2b^3}{81a^2} & \text{if } p \equiv 5 \pmod 6. \end{cases}$$

In the context of this Conjecture, it is easy to see that without loss of generality $gcd(a, b) = 1$ with $0 < a \leq p-1$ and $0 \leq b \leq p-1$. Moreover, if x, y are distinct integers in $[0, p-1]$, then $f(x) \equiv f(y) \pmod p$ if and only if (x, y) is a root modulo p of $a(x^2 + xy + y^2) + b(x + y)$. We have not yet found a closed form solution for these roots, however we have computationally verified [4] Conjecture 3.5 for all primes ≤ 1500.

4. General Polynomials

The difficulty in extending to more general polynomials lies in the complexity of listing, or even just counting the number of distinct f polynomial residues. This latter problem has a rich history in the literature in a variety of forms, and effectively remains unsolved to this day. We conclude with a summary of the work in this area to date.

Let $V_n(f)$ denote the number of distinct residues of $f(x)$ modulo n. In 1915 Kantor [3] computed $V_p(f)$ for all primes p and $\deg f = 3$. Precise values for $V_p(f)$ for degrees ≥ 4 are unknown at present, although partial solutions have been given for a specific class of quartics. In particular, Sun [6] determines the value of $V_p(x^4 + ax^2 + bx)$. The counting method of Kantor does not appear to lend itself to results on the sums of residues of cubics, and neither does the technique of Sun extend to sums of residues of $x^4 + ax^2 + bx$.

In the most general case, a complex generating function [7] for $V_n(f)$ is given by

$$V_n(f) = n \sum_{u=0}^{n-1} \left(\sum_{t=0}^{n-1} \sum_{v=0}^{n-1} \exp\left\{ 2\pi i \frac{t}{n} (f(u) - f(v)) \right\} \right)^{-1}, \tag{3}$$

which naturally lends itself to asymptotic estimates of $V_n(f)$. In 1954, Uchiyama [7] extended Weil's famous 1948 proof [11] of the Riemann Hypothesis for function fields and proved that if $q = p^k$ and $f^*(u, v) = (f(u) - f(v))/(u - v)$ is absolutely irreducible then $V_q(f) > q/2$. The example $f(x) = x^4 - x^2 + 1$ shows that the hypothesis on $f^*(u, v)$ cannot be dropped. However, a year later Carlitz proved [1] that on average $V_q(f)$ is indeed $> q/2$. A series of results followed [8, 9, 10] concerning the asymptotics for $V_q(f)$ over unitary polynomials and over polynomials of a fixed degree. We note that the main result of [10] depends on the Riemann Hypothesis.

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Real Lie Algebras with Equal Characters

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Abstract. We recall Cartan's definition of characters of real forms of complex simple Lie algebras, based on Cartan decomposition. For a given complex simple Lie algebra, its real forms are uniquely determined by their characters in almost all cases. We work out the exceptions where non-isomorphic real forms have the same character.

1. Introduction

Let $\mathfrak{g}$ be a real form of a complex simple Lie algebra L. Let $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition, namely $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}$. In É. Cartan's classification of real simple Lie algebras, he defines the *character* of $\mathfrak{g}$ as

$$\text{character}(\mathfrak{g}) = \dim \mathfrak{p} - \dim \mathfrak{k}.$$

He observes that non-isomorphic real forms of exceptional Lie algebras have distinct characters [1, p.263-265], and uses them to denote these exceptional real forms. For example $\mathfrak{e}_{6(\delta)}$ denotes the real form of E_6 with character δ [2, p.518]. For the classical Lie algebras, Helgason notes that non-isomorphic real forms with equal character occur only in types A and D, and provides two examples [2, p.517]

$$\begin{array}{ll}\text{(a)} & \mathfrak{su}^*(14), \mathfrak{su}(9,5) \subset \mathfrak{sl}(14,\mathbb{C}), \\ \text{(b)} & \mathfrak{so}^*(18), \mathfrak{so}(12,6) \subset \mathfrak{so}(18,\mathbb{C}).\end{array} \tag{1}$$

The following theorem determines all non-isomorphic real forms with equal character.

Theorem 1.1. *All the cases of real forms $\mathfrak{g}, \mathfrak{g}' \subset L$ such that $\mathfrak{g} \not\cong \mathfrak{g}'$ and $\mathfrak{g}, \mathfrak{g}'$ have the same character are given as follows:*
(a) $\mathfrak{su}(2r^2+r-1, 2r^2-r-1), \mathfrak{su}^*(4r^2-2) \subset \mathfrak{sl}(4r^2-2,\mathbb{C})$, *where* $1 < r \in \mathbb{N}$;
(b) $\mathfrak{so}(r^2+r, r^2-r), \mathfrak{so}^*(2r^2) \subset \mathfrak{so}(2r^2,\mathbb{C})$, *where* $2 < r \in \mathbb{N}$.

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We see that $\mathfrak{su}^*(14)$ and $\mathfrak{su}(9,5)$ of (1)(a) are obtained by setting $r = 2$ in Theorem 1.1(a), while $\mathfrak{so}^*(18)$ and $\mathfrak{so}(12,6)$ of (1)(b) are obtained by setting $r = 3$ in Theorem 1.1(b).

If $\mathfrak{g}$ and $\mathfrak{g}'$ are real forms of L, then clearly $\dim \mathfrak{g} = \dim \mathfrak{g}'$. Hence the condition character($\mathfrak{g}$) = character($\mathfrak{g}'$) is equivalent to $\dim \mathfrak{k} = \dim \mathfrak{k}'$. It is known that $\mathfrak{g}$ is determined by $\mathfrak{k}$ and L [2, Ch.X-6,Thm.6.2]; and Theorem 1.1 says that $\mathfrak{g}$ is in fact determined by $\dim \mathfrak{k}$ and L except for the indicated cases.

2. Proof of Theorem 1.1

We now prove Theorem 1.1. We study $\mathfrak{sl}(n,\mathbb{C})$ in the proof of Theorem 1.1(a), and study $\mathfrak{so}(2n,\mathbb{C})$ in the proof of Theorem 1.1(b).

Proof of Theorem 1.1(a):

The Lie algebra $\mathfrak{sl}(n,\mathbb{C})$ has three classes of real forms $\mathfrak{g}$, whose maximal compact subalgebras $\mathfrak{k}$ are indicated in (2) (see for instance [2, p.518]). In (2)(a),

$$\begin{aligned}\dim \mathfrak{k} &= \dim \mathfrak{u}(p) + \dim \mathfrak{u}(n-p) - 1\\ &= p^2 + (n-p)^2 - 1 = 2p^2 - 2np + n^2 - 1.\end{aligned}$$

	$\mathfrak{g}$	$\mathfrak{k}$	$\dim \mathfrak{k}$
(a)	$\mathfrak{su}(p, n-p)$	$\mathfrak{s}(\mathfrak{u}(p) + \mathfrak{u}(n-p))$	$2p^2 - 2np + n^2 - 1$
(b)	$\mathfrak{su}^*(n), n$ even	$\mathfrak{sp}(\frac{n}{2}, \mathbb{R})$	$\frac{1}{2}(n^2+n)$
(c)	$\mathfrak{sl}(n,\mathbb{R})$	$\mathfrak{so}(n)$	$\frac{1}{2}(n^2-n)$

(2)

If $\mathfrak{g}$ is a split form of L (i.e. $\mathfrak{g}$ has a Cartan subalgebra contained in $\mathfrak{p}$; also known as a normal form), then its character is strictly larger than that of other real forms of L [2, p.517]. Therefore, we can ignore the split form $\mathfrak{sl}(n,\mathbb{R})$, and consider only (2)(a,b). We recall the elementary fact

$$ap^2 + bp + c = 0 \implies p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{3}$$

It is easier to compare $\dim \mathfrak{k}$ instead of the characters. Suppose that (2)(a) and (2)(b) have equal $\dim \mathfrak{k}$. Then

$$0 = (2p^2 - 2np + n^2 - 1) - \frac{1}{2}(n^2+n) = 2p^2 - 2np + \frac{1}{2}(n^2 - n - 2). \tag{4}$$

By (3) and (4),

$$p = \frac{2n \pm \sqrt{(-2n)^2 - 4(n^2-n-2)}}{4} = \frac{1}{2}(n \pm \sqrt{n+2}). \tag{5}$$

This implies that $n+2$ is a perfect square. Furthermore since n is even in (2)(b), condition (5) also says that $\sqrt{n+2}$ is even, namely $n+2=(2r)^2$ for some $r \in \mathbb{N}$. Then (5) becomes $p = 2r^2 \pm r - 1$. For $r=1$, (2)(a,b) gives $\mathfrak{su}(2) \cong \mathfrak{su}^*(2)$. Hence we assume that $r>1$. This leads to the pairs of real forms in Theorem 1.1(a).

It remains to compare (2)(a) with itself for different values of p. If $\mathfrak{su}(p, n-p)$ and $\mathfrak{su}(q, n-q)$ have equal $\dim \mathfrak{k}$, then

$$0 = (2p^2 - 2np + n^2 - 1) - (2q^2 - 2nq + n^2 - 1) = 2(p^2 - np + (nq - q^2)).$$

By (3),

$$p = \frac{n \pm \sqrt{(-n)^2 - 4(nq - q^2)}}{2} = \frac{1}{2}(n \pm (n - 2q)) \in \{q, n-q\}.$$

This implies that $\mathfrak{su}(p, n-p) \cong \mathfrak{su}(q, n-q)$. We conclude that Theorem 1.1(a) gives all the cases of non-isomorphic real forms of $\mathfrak{sl}(n, \mathbb{C})$ with equal character. $\square$

Proof of Theorem 1.1(b):

The Lie algebra $L = \mathfrak{so}(2n, \mathbb{C})$ has two classes of real forms $\mathfrak{g}$, with $\mathfrak{k}$ and $\dim \mathfrak{k}$ indicated in (6).

	$\mathfrak{g}$	$\mathfrak{k}$	$\dim \mathfrak{k}$
(a)	$\mathfrak{so}(p, 2n-p)$	$\mathfrak{so}(p) + \mathfrak{so}(2n-p)$	$p^2 - 2np + 2n^2 - n$
(b)	$\mathfrak{so}^*(2n)$	$\mathfrak{u}(n)$	n^2

(6)

Suppose that (6)(a) and (6)(b) have equal $\dim \mathfrak{k}$. Then

$$p^2 - 2np + n^2 - n = 0.$$

By (3),

$$p = \frac{2n \pm \sqrt{(-2n)^2 - 4(n^2 - n)}}{2} = n \pm \sqrt{n}.$$

It implies that n is a perfect square, say $n = r^2$ for some $r \in \mathbb{N}$. Then $p = r^2 \pm r$. For $r=1$, (6)(a,b) gives $\mathfrak{so}(2) \cong \mathfrak{so}^*(2)$. Similarly for $r=2$, it gives $\mathfrak{so}(6,2) \cong \mathfrak{so}^*(8)$. Hence we assume that $r>2$. This leads to the pairs of real forms in Theorem 1.1(b).

We also compare (6)(a) with itself for different values of p. If $\mathfrak{so}(p, 2n-p)$ and $\mathfrak{so}(q, 2n-q)$ have equal $\dim \mathfrak{k}$, then

$$0 = (p^2 - 2np + 2n^2 - n) - (q^2 - 2nq + 2n^2 - n) = p^2 - 2np + (2nq - q^2).$$

By (3),

$$p = \frac{2n \pm \sqrt{(-2n)^2 - 4(2nq - q^2)}}{2} = n \pm (n - q) \in \{2n - q, q\}.$$

This implies that $\mathfrak{so}(p, 2n-p) \cong \mathfrak{so}(q, 2n-q)$. We conclude that Theorem 1.1(b) gives all the cases of non-isomorphic real forms of $\mathfrak{so}(2n, \mathbb{C})$ with equal character. $\square$

Since non-isomorphic real forms with equal character may occur only in types A and D [2, p.517], this completes the proof of Theorem 1.1.

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An interview with Vincent Hart

COLM MULCAHY

Abstract. An interview with Vincent Hart, a very early Irish mathematics doctorate, who has spent half a century in Brisbane after starting his career at Cork and at the DIAS.

1. Introduction

Vincent Gerald Hart was born in Hull in 1930, and later brought up in Cork. He attended UCC, and taught there from 1951 to 1966, with forays to DIAS, MIT and the University of Queensland along the way. His January 1958 PhD, earned under the guidance of John L. Synge, seems to have made him the third Irish person to complete a doctorate by research in the mathematical sciences in the Republic of Ireland. (Maynooth's James McMahon and UCD's Cormac Smith had earned theirs in 1952 and 1954, respectively, under J. L. Synge and J. R. Timoney.)

Now, half a century after he resettled in Australia—where his career included serving as department head, supervising research, and collaborating in Diarmuid Ó Mathúna's book *Integrable Systems in Celestial Mechanics* (Birkäuser, 2008)—Vincent Hart looks back on seven decades of scholarship and life in academia.

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2. Interview

1. Where did you grow up, what background did your parents have, and what schools did you go to?

I was born in Hull, Yorkshire, England in 1930, and would probably still be there were it not for the second World War. My mother was a primary school teacher, and my father was an accountant in a shipping office. After primary school, I attended Marist College in Hull, and then, for one term, Wyggeston Grammar School in Leicester—whence we had moved due to the bombing. This lasted until December, 1940, Leicester having been bombed even closer to us in November. My father, who was in the Army by then, decided that my mother, myself, and two younger brothers, should move to family members in Cork, in neutral Ireland. And there I grew up very happily. In Cork I attended the Christian Brothers' College until 1947.

2. What first drew you to maths, and how old were you when you realized it was something you wanted to pursue above other options?

At Christians' I received a sound education, my best subject being Latin. I could do mathematics also, but was not enthused by it—until at age seventeen, I read a book explaining how the Bohr atom was described by mathematics applicable also to the solar system: George Gamow's *Mr Tompkins Explores the Atom* (Cambridge, 1945). Then the scales fell from my eyes, and I became, and remained, very interested in the application of mathematics to the problems of the real world.

3. Tell us about your days as a student at UCC, including noteworthy teachers and fellow students?

At University College Cork, to which I was admitted on a scholarship in October, 1947, I enrolled for an Honours BSc in Experimental Physics and Mathematical Physics. This meant that I had to attend also the lectures in Mathematics—which gave me about 20 hours per week of contact. I graduated in 1950, with a medal, and took an MSc in Mathematics in 1951. I had good teachers: In Experimental Physics: J J McHenry, C Ó Ceallaigh, D J Stevens. In Mathematics: T M Carey, H St J Atkins. In Mathematical Physics: M D McCarthy for the first two years, and P M Quinlan for the third year.

Fellow students were T M Cronin, P J Donohoe, and P B Kennedy. All were very able, particularly Kennedy who won the 1951 Travelling Studentship in Maths; he was a year ahead of our group of three only. He was in a class by himself in every way (he was the only BSc student in maths who graduated in 1949). We caught up with him for the MSc in 1951. He had two years to prepare for the Studentship; we had but one year to prepare. All four of us (Kennedy, Cronin, Donohoe, me) obtained our MSc (or MA for Donohoe) on our answering on the Studentship examination. P B Kennedy later became Professor of Mathematics, first in Cork, and then in York. I learned a lot from my contemporaries and enjoyed their company very much.

UCC graduation, 1951 [1]

Tim Cronin was a very good mathematician, who had been widely educated. I shared accommodation with him in Dublin, and was impressed by the large number of books on English poetry on his bookshelf. His health was not very good I believe. A very congenial colleague.

[1](Photo courtesy V G Hart) Front row: Professor H St J Atkins, P B Kennedy, Dr Tadhg (T M) Carey. Back row : T M Cronin, P J Donohoe, V G Hart, one unknown person. PBK is in the honour position since he has just been awarded the 1951 Travelling Studentship.

P J Donohoe had a chequered career. On winning the studentship in 1954 he went to M J Lighthill in Manchester, and was given a project in fluid dynamics. After some months he just gave up without notification and retired to home in Rathmore, County Kerry, to everyone's consternation. After temporary jobs in UCC he obtained a lectureship in QUB—after which he seemed to have returned to the straight and narrow. He was ebullient and brilliant, but erratic. He was outgoing, with a quick intelligence coupled with erudition. But having brought this formidable apparatus to bear, and having achieved something—studentship, DIAS scholarship—he seemed to lose interest, with the obvious consequences. I don't think he published anything, but I'm sure his PhD [QUB, 1966] was good work. He was probably under some pressure to complete it. He probably had too many interests, and the period when he was at QUB was certainly anything but restful for academics.

After the 1950 group, the next maths and maths physics graduates were Kevin O'Donnell and Siobhán O'Shea in 1952. Kevin became an actuary, worked in London for a stockbroker, and moved to Dublin to head a big Irish Insurance Company. I know Kevin well. He and his brother, Des, swept all before them in the 1947 Entrance Scholarship examinations at UCC. At that time, in a College of about 1000 students, there were only about 8 or 10 scholarships offered yearly. Both O'Donnell brothers declined their awards, with Des (who died a few years ago) going to a bank, and Kevin to the Jesuits. After the two year novitiate, Kevin left and came back to maths and maths physics at UCC. I believe he is still happily retired at Ballybride, County Dublin. Both are Cork boys.

Siobhán was a worthy colleague. The stimulus needed to get her moving in research was provided by P B Kennedy after he became Professor in Cork.

4. After your masters, in 1951, you taught for a while at UCC and began your association with DIAS. How did that come about?

After BSc graduation in October, 1950, I was offered a teaching post (as Assistant I believe) at UCC. This I held for nearly two years until I applied for and was granted a position as Scholar at DIAS. This was a research position which I held for two years, being supervised by J L Synge, until I was offered a Temporary Lectureship back at UCC in mid 1954. This became a Statutory Lectureship in 1958, which I held until resignation in 1966.

5. You were a DIAS Scholar from 1952 to 1954, studying with John Lighton Synge, and then you returned to UCC. Your PhD was conferred in early 1958. What was it like working with Synge, and what was the nature of that research? Did you visit DIAS a lot to do more work with Synge in the period 1954-1957?

Professor Synge was impressive in various ways. He was an excellent lecturer with a very clear style, as a supervisor he was excellent, very experienced and understanding. I was fortunate to have him as my PhD supervisor: by contrast to the experience of a friend, who was given a problem much too difficult for him by a different supervisor.

From October 1954 I worked at UCC at Lecturer level. And I mean worked: I had 14 lectures per week for a long period of years-with one memorable term when I was asked to give McHenry's lectures while he was in hospital. That gave me 17 lectures per week. All the while I was trying to complete my PhD thesis—with frequent letters to J L Synge. There were only a few visits to Dublin, and those for seminars or lectures.

My research was entirely personal in Cork–except for letters to and fro with JLS. The topic concerned the Hypercircle method; I had previously assisted in the production of Synge's book: *The Hypercircle Method in Mathematical Physics* (Cambridge, 1957).

Mercifully, the lecture load dropped to 11 per week after about half my years as Lecturer at UCC had expired. And I should add that I got away to MIT for the calendar year 1959, and to the University of Queensland for another year, 1964/65. At MIT, I worked with Professor Eric Reissner on problems involving solid mechanics. Two papers on the bending of an annular plate resulted–with D J Evans who contributed numerical skills. I benefited from contact with Professor L N Howard (who we subsequently invited to the University of Queensland). I enjoyed meeting some very able graduate students, including Charles Conley. And my former student Diarmuid Ó Mathúna was pleasant company, since he was there doing a PhD under Reissner. Norbert Wiener's office was two doors away from mine, and we had several chats. There were some excellent lecture courses, particularly one by Jürgen Moser, which broadened my knowledge base considerably. A very fruitful year for me.

6. Who else did you know at DIAS?

At DIAS I met Schrödinger, Synge and Cornelius Lanczos, three Senior Professors, of course. And there was a constant stream of eminent people passing through the Institute: Dirac, Polanyi, Kilmister, Heitler, for example—all very good for us young students. Nearer to my level were Jim Pounder, John Roche, Paddy Donohoe, Fr McHugh, and as postdoc visitors, Martin J Klein from USA, Ernesto Corinaldesi from Italy and Daykin from Canada.

I also knew Henry Sandham *[(1917-1963), another PhD student of Synge's]*; he was not a well man. He was recovering from a lung problem, probably TB which was not unusual for the time, and one could hear his laboured breathing across the room. Nevertheless he was a very pleasant man, always very helpful in discussing mathematics—of which he had a wide knowledge particularly in analysis. He was a mine of information on integrals, series, etc.

7. Who were your UCC colleagues in the period 1950-1966?

P M Quinlan (Prof of Math Phys), George Kelly, P B Kennedy, P D Barry, Siobhán O'Shea, Finbarr Holland, Tadhg Carey; all colleagues of varying qualities–mostly good to excellent.

8. What notable students were at UCC in the period 1950-1966?

Many. Matt McCarthy, Michael Mortell, Tony Hollingsworth, Brendan McWilliams, Finbarr Holland, Jim Flavin, P D Barry, Diarmuid Ó Mathúna, Richard Scott. M Mortell became President of UCC. P Barry became Head of Mathematics at UCC after Kennedy, and F Holland later a professor there. J Flavin became professor and HOD at UCG, and M McCarthy professor and Registrar there. R Scott became professor at Caltech after PhD there, and D Ó Mathúna , after PhD at MIT, served with NASA in its heroic moon landing stage.

Frank Hodnett became Head of Department at University of Limerick, and Michael J O'Callaghan became Head of Department at UCC—after P M Quinlan's retirement. Tony Hollingsworth and Brendan McWilliams became notable meteorologists, Hollingsworth being chief of the Reading research institute in England. McWilliams remained in the Irish Met service, contributed lively and topical regular columns to the *Irish Times*. Sadly both have died, but McWilliams's wife has compiled her late husband's columns into an excellent book: *The Book of Weathereye* (Gill and Macmillan, 2008).

Both men were in my Honours BSc class in 1964, which I particularly remember since the students gave me a parting present of six Waterford sherry glasses—still in constant use!

9. Was there any "institutional memory" of George Boole's legacy?

Not really, except for a talk by Sir Geoffrey Taylor in mid 1964. He was a descendant of Boole.

10. You spent the academic year 1964-5 in Brisbane, were you testing the waters for your permanent move there in 1966? What attracted you to the University of Queensland?

No. I met in UCC by chance the Head of the Maths Department at the University of Queensland, Clive Davis. He came to Britain and Ireland in February 1964 on a recruiting expedition. My wife and I went to Queensland in mid 1964 for a year, and came back to Cork in mid 1965. Then in late 1965, I got the Readership offer to go back to Queensland—which we did in mid 1966.

11. How was the adjustment from academia in UCC? It must have been a big culture shock in general?

Yes, it was a culture shock—but a very agreeable one. The University of Queensland department was much bigger and more diverse than the two departments at UCC (maths and maths physics). The Queensland department comprised Pure Maths, Applied Maths and Statistics, and later Numerical Analysis sections, and had about 15 academics in the mid 1960s. This compared with just Maths and Maths Physics Depts at UCC with a total of just four staff between them.

And conditions at Queensland were better, allowing more time for research and construction of new courses. Another welcome feature was the provision of paid study leave overseas at the rate of three months every three years. *Bozhe moi!*[2]

12. Did you know you were going to spend the rest of your life there?

No. I enjoyed the break as an excursion from my normal job at UCC. However it was a most favourable time for academics, with

[2] A Tom Lehrer reference.

Universities expanding everywhere, and by late 1965 I had three offers of jobs, one in Canada, and two in Australia. And so our minds tended to favour perhaps a three year stint overseas—with the acceptance of a Readership offer at the Uiniversity of Queensland. But this three year period expanded irrevocably. . . .

13. How did your research interests develop down under? You've had several PhD students?

Very well. Our department expanded to about 50 academics by the 1970s, and there were excellent opportunities for designing new courses and for engaging with able students. Also the overseas study leave periods, six of which I enjoyed during my 28 years, were very fruitful in generating new contacts and ideas for research. They were taken in Universities of East Anglia (twice), UCD (twice), Nottingham, Auckland and Leeds, the last two on the same leave in 1992.

I had several PhD students, one of whom, J M Hill, became very prolific in research. He has produced a great deal of work, and has held professorial positions at Wollongong and Adelaide Universities.

In administration, I spent eleven years as Head of Department, during which I had a very able Chinese research assistant, Jingyu Shi. He and I turned out ten papers on the stresses in grafted arteries under pulsatile flow. I also spent 16 years as Treasurer of the Australian Mathematical Society. So, all in all, I was very happy with our serendipitous decision to move to Queensland in 1966.

14. Tell us about your brothers.

Like me, Julian (1938-2012) and Ian (1939-1980) were born in Hull and moved to Ireland in 1940. Julian preferred the old form of surname: MacAirt. Both both won keenly contested entrance scholarships to UCC: Julian won a Honan entrance scholarship, and Ian won a Keliher entrance. In comparison I won only a College entrance scholarship, after the named awards were distributed. They were bright boys.

Julian also received the gold medal of the UCC Graduates' Club as the most distinguished graduate of 1959—as I did in for 1950—he drowned tragically at The Meeting of the Waters, Wicklow. Julian got BSc (Maths and Stats) and PhD (Economics) at UCC), and a Dunlop Fellowship in Economics at Oxford (1960-63), then worked at Aer Lingus before becoming Lecturer and Senior Lecturer in Statistics at TCD (1967-95). He wrote two books and 25 papers in statistics.

Ian got his BA, MA, and PhD (in Psychology) at UCC, and a HDip. He worked at the Economic and Social Research Institute in Dublin (1967-80), and wrote 16 papers in clinical psychology. He was regarded as one of pioneers in social work in Ireland on prisoners, deprived children, and drugs, and was one of the early workers in the Simon community for homeless people.

15. Have you been back to Cork (or Yorkshire) much over the years? What do you miss most about those places?

Yes. We had six visits to Yorkshire and Ireland to see relatives before retirement in 1995, and three trips after retirement. Having grown up in both England and Ireland I have always been interested in the history, literature and development of both countries. There is a lively conversational style in each that is hard to equal.

16. What was your role in the book Integrable Systems in Celestial Mechanics *(Birkäuser, 2008) by Diarmuid Ó Mathúna of the DIAS?*

This book achieves the complete analytic solution to the problem of a body moving in the gravitational field of two fixed centres, thus completing Euler's solution. Also, by a simple change of sign in the governing equations, it provides the solution of a quite different dynamical problem—that of Vinti, which concerns the motion of a body in a realistic model of the earth's gravitational field. I showed the need to expand the range of parameters that was at first considered in Chapter 3, and I wrote the Appendix, in which illustrations are given of the various orbits occurring. *Maple* codes were supplied by Sean Murray of the DIAS; he helped to get the formulae involving Jacobian Elliptic functions into suitable form. Not many items can improve on the great Euler; I think that is its great strength, and I'm very pleased to have contributed to it.

17. What have been your favourite courses to teach?

I liked the small honours classes best I think; special functions, asymptotic methods, fluid mechanics, elasticity.

18. What course did you create that you are most pleased with?

I think fourth year honours non-linear elasticity. This started Jim Hill up on his research career, to considerable effect.

19. What advice do you have for today's students who are interested in applying maths to the real world?

I think they have to read widely, and, if possible, study the approach of some able applier of mathematics. I think that I benefited by following Synge's geometrical method rather than an alternative abstract one. But people differ of course. The Study Groups for Industry I attended in Oxford and Australia were fruitful-and these are now widespread of course.

20. You've seen applications of maths change a lot in your lifetime. What has surprised or excited you the most?

Both surprise and excitement come from the great facility the computer gives us to research the literature from home. Together with the fine packages such as *Maple* and *Matlab*, which enable much more powerful computation than in the past. This is heartfelt from one who struggled with the Facit, Marchant, and Brunsvigas of the 1960s.

Excitement is not quite the word when I contemplate the change in delivery of instruction. The mode of delivery of courses has changed greatly. Nowadays perhaps only thirty percent of students attend the contact period, the details being available on the screen. This means that the students miss interaction with each other and with the instructor. This is surely a serious detriment. And whatever the educationalists say about self instruction, study of the careful exposition of the great mathematical works is still the only way to learn one's trade.

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Beyond Dominated Convergence: Newer Methods of Integration

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Abstract. Lebesgue's dominated convergence theorem is a crucial pillar of modern analysis, but there are certain areas of the subject where this theorem is deficient. Deeper criteria for convergence of integrals are described in this article.

1. What Is Integration?

We learn calculus in secondary school: first, differentiation of functions, and later integration as the inverse or opposite of differentiation—the integral is the anti-derivative or primitive function, from which *definite integrals* can be easily deduced.

In more advanced mathematics courses we learn Riemann's definition of definite integrals which enables us to integrate more functions. The Riemann method does not make use of differentiation; it is similar to the ancient method of finding areas and volumes "by exhaustion"—that is, estimating the area or volume by dividing it up into pieces which are more easily estimated, and then taking the aggregate of the pieces.

Specialists in mathematical analysis go on from this to study the Lebesgue method of integration. Why? Again, one of the stock answers to this question is that the Lebesgue method enables us to integrate functions which cannot be integrated by more familiar methods such as the calculus integral and the Riemann integral.

The Dirichlet function is sometimes mentioned. In the unit interval $[0,1]$ this function takes value one at the rational points, and zero at the other points. The Dirichlet function is not the derivative of some other function, so it cannot be integrated by the method we learn at school in calculus lessons. Also, it cannot be integrated

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by Riemann's method. But the Lebesgue integral of the Dirichlet function exists: the definite integral of the Dirichlet function on the unit interval is 0.

But—so what? Apart from some specialists and experts, is there anybody else who has any real use for the Dirichlet function, and who really cares whether or not it is amenable to calculus? It cannot be pictured as lines in a graph, it does not have a straightforward formulation in polynomial, trigonometric or exponential terms. Unlike the area and volume calculations of antiquity, and unlike the calculus of Newton and Leibnitz which explained the world in mechanical terms, what difference does the Dirichlet function make to anyone outside the narrow and rarified world of a tiny number of people in pure mathematics?

The same can be said of many of the other arcane and exotic functions, such as the Devil's Staircase, invented during the long nineteenth century gestation of Lebesgue integration, measure theory and set theory. Such functions have counter-intuitive and challenging qualities that we can admire and wonder at. But they were described by Hermite and Poincaré as unwelcome monsters causing mayhem in the rich and fertile garden of mathematics.

"Does anyone believe that the difference between the Lebesgue and the Riemann integrals can have physical significance, and that whether, say, an airplane would or would not fly could depend on this difference? If such were claimed, I should not care to fly in that plane." (Richard W. Hamming [5]).

This critique is understandable, but unhistorical. By the early nineteenth century, the rich and fertile garden was on the verge of becoming a barren and dangerous wilderness—and not because of trespassing monster-functions.

2. Monstrous Functions

In the tradition of Newton and Leibnitz, Fourier series representation of functions had opened up the analysis of wave motion, crucial to an understanding of sound, light, electricity and so on. But strange and paradoxical things could happen when the integral of a function was obtained by integrating its Fourier series term by term. Certain questions could no longer be avoided. To what extent, and under what conditions, is a function identical to its corresponding Fourier series representation? When is the integral of a function equal to the series obtained by integrating the terms of the corresponding Fourier series?

This boils down to whether a convergent series of integrable functions has integrable limit, and whether the integral of the limit is equal to the limit of the integrals whenever the latter limit exists.

Such issues motivated decades of investigation of the notion of integration, until a satisfactory resolution was found in the convergence theorems—uniform, monotone, and dominated convergence—of Lesbesgue integration theory. In particular the dominated convergence theorem tells us that if a sequence of integrable functions f_j converges to f, and if the sequence satisfies $|f_j| < g$ for all j, where g is integrable, then f is integrable and $\int f_j$ converges to $\int f$ as $j \to \infty$.

The integral here is the definite Lebesgue integral on some domain. But the theorem holds for functions which are integrable in the older and more familiar senses of Riemann, Cauchy, and Newton/Leibnitz, since, broadly speaking, functions which are integrable in the latter senses are, *a fortiori*, integrable in the Lebesgue sense.

From this point onwards somebody—not necessarily expert in Lebesgue's integration—who is contentedly doing some familiar integral operations, and who encounters some issue of convergence such as term-wise integration of a Fourier series, can proceed in safety if a dominant integrable function g can be found for the convergence.

This is the practical significance of Lebesgue's theory. It is a reason why "it is safe to fly in airplanes", so to speak. It is why the fertile garden did not turn into a barren wilderness. And the "monster-functions" were in reality guard dogs that played their part in protecting the garden.

3. A Non-monstrous Function

But is this the end of the story? Did Lebesgue's 1901 and 1902 papers [12, 13] give the last word on the subject?

Here is a sequence of non-monstrous functions formed by combining some familiar polynomial with trigonometric functions. For $j = 2, 3, 4, \ldots$, let $f_j(x) = 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}$ if $\frac{1}{j} \leq x \leq 1$ and $= 0$ if $0 \leq x < \frac{1}{j}$.

An impression of function f_j can be obtained from Figure 3 below, which contains the graph of $f(x) = 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}$ for $0 < x \leq 1$.

Figure 2 resembles Figure 3 in the neighbourhood of $x = 0$. The difference between the two is $2x \sin x^{-2}$, whose graph is in Figure 1.

But the values of the latter function are very small in the neighbourhood of $x = 0$. Figure 1 demonstrates its "visual insignificance", so to speak. Note that the vertical scale of Figure 1 is much more magnified than the vertical scales of Figures 2 and 3.

Figure 4 is the graph of the primitive function or anti-derivative of f, which will play a big part in our discussion.

Each f_j has a single discontinuity (at $x = \frac{1}{j}$), but is differentiable at every other point. Each f_j is integrable (in the sense of Riemann and Lebesgue), and the sequence f_j is convergent at each x to $f(x) = 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}$, $f(0) = 0$, whose graph is Figure 3.

The limit function f has a single discontinuity (from the right) at $x = 0$; and it has a primitive function $F(x) = x^2 \sin x^{-2}$ $(x > 0)$, $F(0) = 0$ (Figure 4); so in fact f has a definite integral $F(1) - F(0) = \sin 1$ on the domain $[0, 1]$ **provided** the Newton-Leibnitz definition of the integral is used.

But f is unbounded on $[0, 1]$ and therefore is not Riemann integrable on $[0, 1]$. And, though clearly non-monstrous, and understandable to a beginning calculus student, the function f is **not** Lebesgue integrable. See below for discussion of this point.

On the face of it, this example indicates a step backwards, as it were, where the old school method of Newton/Leibnitz is actually more effective than more modern methods. Lebesgue's theory of the integral threw up anomalies of this kind, and accordingly investigation of the theory continued through the twentieth century.

To recapitulate, Lebesgue's dominated convergence theorem can be said to be the cutting edge of modern integration theory. But it fails to capture the convergence of sequences such as f_j and $\int_0^1 f_j(x)\, dx$. The graph in Figure 3 suggests $2x^{-2}$ as a conceivable candidate for dominating function g for the terms $|f_j|$, $j = 1, 2, 3, \ldots$, at least in a neighbourhood of the critical point $x = 0$. But $2x^{-2}$ is not integrable in a neighbourhood of 0, and it seems that the dominated convergence theorem is not applicable here.

This failure must sound some alarm bells, because while many working mathematicians can get by without the Lebesgue integral, we cannot really do without convergence theorems which allow us, for instance, to perform routine operations on Fourier series; or, more generally, to safely find the integral of the limit of a sequence of integrable functions by taking the limit of the corresponding sequence of integrals.

The purpose of this essay is to dip into some aspects of modern integration theory in order to introduce Theorems 5.1, 5.2, and 5.3 below, which are delicate enough to deal with, for instance, the convergence of the functions f_j above and their integrals; while also covering the ground already captured by the convergence theorems of Lebesgue's theory.

For ease of reference, here again are the sequence f_j and related functions, $j = 1, 2, 3, \ldots$:

$$f(x) = \begin{cases} 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0, \end{cases} \tag{1}$$

$$f_j(x) = \begin{cases} 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2} & \text{if } \frac{1}{j} \leq x \leq 1, \\ 0 & \text{if } 0 \leq x < \frac{1}{j}, \end{cases} \tag{2}$$

$$F(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0, \end{cases} \tag{3}$$

$$F_j(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } \frac{1}{j} \leq x \leq 1, \\ 0 & \text{if } 0 \leq x < \frac{1}{j}. \end{cases} \tag{4}$$

Figures 3 and 4 are, respectively, the graphs of the functions f and F. The graphs of f_j and F_j are easily substituted—just insert a horizontal line segment at height 0 from $x = 0$ to $x = \frac{1}{j}$.

The function f has a single discontinuity at $x = 0$, while F is continuous. The reader should verify that F is differentiable, including at the point $x = 0$ (from the right), and that $F'(0) = 0 = f(0)$. The fact that $F'(x) = f(x)$ for $x > 0$ is easily verified.

This establishes that $f, = \lim f_j$, has a primitive or anti-derivative, and F is the calculus- or Newton-indefinite integral of f. Also the calculus- or Newton-definite integral on $[0, 1]$ is

$$\int_0^1 f(x)\, dx = F(1) - F(0) = \sin 1 - 0 = \sin 1.$$

For each j both f_j and F_j have a discontinuity at $x = 1/j$, but provided $x \neq 1/j$, we have $F'(x) = f_j(x)$. Thus, for each j, f_j is Riemann and Lebesgue integrable on $[0, 1]$, but not calculus- or Newton-integrable on $[0, 1]$. However f_j **is** calculus-integrable on

$[j^{-1}, 1]$ for each j, and

$$\int_0^1 f_j(x)\,dx = \int_{\frac{1}{j}}^1 f_j(x)\,dx = F(1) - F\left(\frac{1}{j}\right) = \sin 1 - \frac{1}{j^2}\sin j^2,$$

provided we interpret $\int_0^1$ as a Riemann (or Lebesgue) integral. Thus, as $j \to \infty$,

$$\int_0^1 f_j(x)\,dx \to \int_0^1 f(x)\,dx \tag{5}$$

provide we interpret the left hand integrals in the sense of Riemann or Lebesgue, and the right hand one as a calculus or Newton/Leibnitz integral.

Unless we can interpret it in some other way, (5) is deficient as it stands, since we cannot ascribe the same meaning to the symbol $\int_0^1$ in the left- and right-hand terms. In fact we will establish later that (5) is valid for an adapted[1] version of the Riemann integral; and that the convergence—including integrability of the limit function f—though unrelated to any kind of dominated convergence, satisfies a new kind of Riemann sum convergence criterion which goes beyond the Lebesgue dominated convergence theorem.

To recapitulate, for $0 \leq x \leq 1$ the function f_j is bounded and continuous—except for a discontinuity at $x = j^{-1}$. Also it is differentiable except at $x = j^{-1}$. It has anti-derivative $F_j(x)$—except at $x = j^{-1}$.

By familiar standard results, f_j is Riemann integrable and Lebesgue integrable on domain $[0, 1]$. But, as discussed above, its limit function f is **not** Lebesgue (or Riemann) integrable.

There are theorems which tell us when to expect Lebesgue integrability of the limit function of Lebesgue integrable functions. If the convergence of the functions f_j to the function f is uniform, monotone, or dominated by a Lebesgue integrable function, then Lebesgue integrability of the functions f_j implies Lebesgue integrability of their limit function f, with

$$\lim_{j\to\infty}\left(\int_0^1 f_j(x)dx\right) = \int_0^1 \left(\lim_{j\to\infty} f_j(x)\right) dx = \int_0^1 F(x)dx.$$

Inspection of the graphs indicates that convergence of functions f_j is not uniform, monotone or dominated. And, even though

[1]That is, the Riemann-complete integral, also known as the generalized Riemann or Henstock-Kurzweil integral.

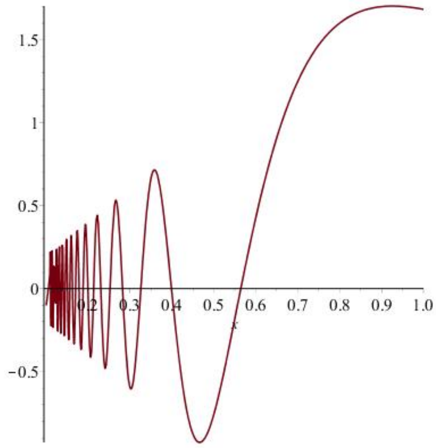

FIGURE 1. $2x \sin x^{-2}$

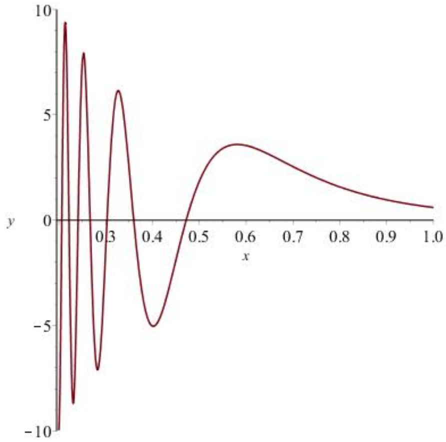

FIGURE 3. $2x \sin x^{-2} - 2x^{-1} \cos x^{-2}$

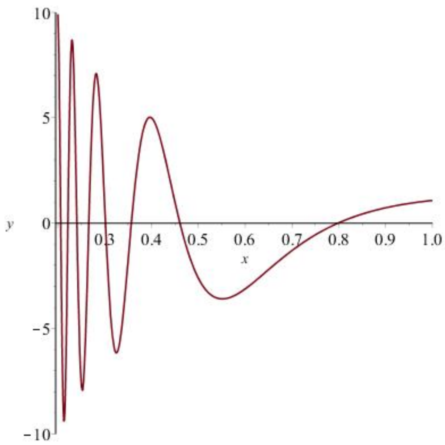

FIGURE 2. $2x^{-1} \cos x^{-2}$

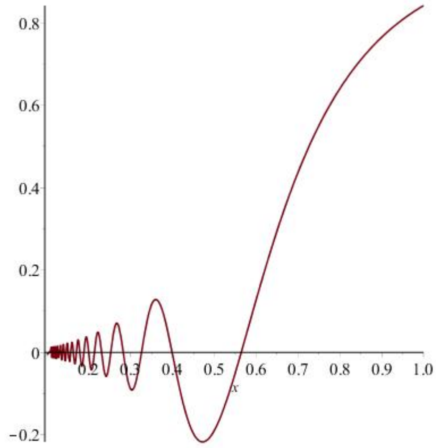

FIGURE 4. $x^2 \sin x^{-2}$

each f_j is Lebesgue integrable, the limit function f is not Lebesgue integrable—as demonstrated below.

Is this a big problem for the garden of mathematics, or is it just a minor incursion by, not a monster, but an atypical creature which is easily contained? This article attempts to provide some perspective.

The names of Denjoy, Perron, Kolmogorov and others are associated with twentieth century efforts [4] to pursue the implications of problems such as the convergence of functions f_j and their integrals. This article will examine the Riemann sum approach of R. Henstock and J. Kurzweil.

4. Riemann-complete Integration

Kurzweil came to this subject through his investigations of differential equations. Henstock was interested in convergence issues in integration. Independently, each of them focussed on careful construction of Riemann sums for integrands f.

Here is a broad outline of Riemann sum construction. A partition $\mathcal{P}$ of a domain such as $[0,1]$ is a set of points $u_0 < u_1 < u_2 < \cdots < u_n = 1$. Identify $\mathcal{P}$ with the corresponding set of disjoint intervals I_i:

$$\mathcal{P} = \{[u_0, u_1],\ ]u_1, u_2],\ ]u_2, u_3], \ldots,]u_{n-1}, u_n]\} = \{I_i \ : \ i = 1, 2, \ldots, n\}.$$

For each $I_i \in \mathcal{P}$ let $|I_i|$ denote the length $u_i - u_{i-1}$ of I_i. Given a function $f(x)$ defined for $x \in [0,1]$, evaluation points x_i are selected for the intervals $I_i \in \mathcal{P}$ in accordance with certain rules (such as $u_{i-1} \leq x_i < u_i$), and then a Riemann sum for f is

$$\sum_{i=1}^{n} f(x_i) \times |I_i|, \quad \text{or, more briefly,} \quad (\mathcal{P}) \sum f(x)|I|.$$

The integral $\int_0^1 f(x)\, dx$ of f on the domain $[0,1]$, denoted by α, exists if α satisfies a condition which is broadly of the following form. Given $\varepsilon > 0$, partitions $\mathcal{P}$ can be chosen such that

$$\left| \alpha - \sum_{i=1}^{n} f(x_i)|I_i| \right| < \varepsilon \ \text{ for specified choices of } \ x_i \text{ and } I_i, \ 1 \leq i \leq n. \tag{6}$$

This inequality is reminiscent of the Riemann integral of f, but it is **not** the full definition that is required here. There must be some rule (sometimes called a *gauge*) for selecting the elements $\{(x_i, I_i)\}$ (corresponding to the partition $\mathcal{P} = \{I_i\}$ or $\{(x_i, I_i)\}$) that can be admitted in the inequality.

For Riemann integration, the rule is that, given $\varepsilon > 0$ there exists a constant $\delta > 0$ such that, for every partition $\mathcal{P} = \{(x_i, I_i)\}$ for which $|I_i| = u_i - u_{i-1} < \delta$ and $u_{i-1} \leq x_i \leq u_i$ for each $I_i \in \mathcal{P}$, the above[2] inequality (6) holds. Denote such a rule by γ, and denote a partition $\mathcal{P}$ which satisfies an appropriate instance of the rule γ by $\mathcal{P}_\gamma$.

[2]If f is continuous on $[0,1]$ then its Riemann integral exists there.

An integral constructed from such a rule can be identified by notation ${}^{\gamma}\!\int$. Then the definition of the integral $\alpha = {}^{\gamma}\!\int_0^1 f(x)\,dx$ is as follows. There is a number α for which, given any $\varepsilon > 0$, there exists a corresponding[3] instance $\gamma(\varepsilon)$ of γ such that every partition $\mathcal{P}_{\gamma(\varepsilon)}$ satisfies

$$\left|\alpha - (\mathcal{P}_{\gamma(\varepsilon)}) \sum f(x)|I|\right| < \varepsilon. \tag{7}$$

We will omit the ${}^{\gamma}$ in ${}^{\gamma}\!\int$, and allow the context to demonstrate which version of the integral is being discussed.

While the Riemann sum rule for ordinary Riemann integration is "$|I| < \delta$", the primary innovations of Kurzweil and Henstock were:

(1) to replace selection of intervals $\{I_i\}$ by selection of linked pairs $\{(x_i, I_i)\}$ in constructing Riemann sums,[4] and
(2) to replace the constant δ by variable $\delta(x) > 0$, where $x = x_i$ is the evaluation point in the term $f(x_i)|I_i|$ of the Riemann sum in (7).

To distinguish this from the Riemann integral, call it[5] the *Riemann-complete* integral. Clearly, every Riemann integrable function is also integrable in the Riemann-complete sense.

A Stieltjes-type definition of the integral of a function f can be expressed as follows. Suppose $f(x)$ and $g(x)$ are point functions defined on the domain $[0, 1]$. The Riemann-Stieltjes integral of f with respect to g is got by replacing the length function $|I|$ by the increment function $g(I) = g(u_i) - g(u_{i-1})$ in the above definitions. A standard result is that if f is continuous and g is monotone (or has bounded variation), then the Riemann-Stieltjes integral $\int_0^1 f(x)\,dg$ exists. If the constant $\delta > 0$ in the definition is replaced by the function $\delta(x) > 0$ of the Riemann-complete construction, call the resulting integral the *Stieltjes-complete* integral of f with respect to g.

[3]For the ordinary Riemann integral, read "there exists a corresponding number $\delta > 0$".

[4]The familiar condition $u_{i-1} \leq x_i \leq x_i$ is sometimes altered. Also the new approach often gives priority to the evaluation points x_i; and then it can be a more subtle and difficult task to determine the linked partitioning elements I_i. See [9] and [14].

[5]It is also called the Henstock-Kurzweil integral, generalized Riemann integral, and gauge integral.

To see how the Riemann-complete integral matches the calculus or Newton/Leibnitz integral, suppose a point function $f(x)$ has an anti-derivative or primitive function $F(x)$ for $0 \leq x \leq 1$, so its definite integral in the Newton/Leibnitz sense is $F(1)-F(0)$. Proceeding as follows, it is easy to deduce that f is Riemann-complete integrable.

If $\mathcal{P}$ is a partition of $[0,1]$ with partition points u_i, $0 = u_0 < u_1 < \cdots < u_n = 1$, and if $u_{i-1} \leq x_i \leq u_i$, then

$$\begin{array}{rcl}
(\mathcal{P})\sum f(x)|I| & = & \sum_{i=1}^{n} f(x_i)(u_i - u_{i-1}) \\
 & = & \sum_{i=1}^{n} \left(f(x_i)(x_i - u_{i-1}) + f(x_i)(u_i - x_i)\right), \\
F(1) - F(0) & = & \sum_{i=1}^{n} F(u_i) - F(u_{i-1}) \\
 & = & \sum_{i=1}^{n} \left((F(x_i) - F(u_{i-1}) + (F(u_i) - F(x_i))\right)
\end{array}$$

Let $\varepsilon > 0$ be given. Then, for each x, $0 < x < 1$, there exists a number $\delta(x) > 0$ such that, for $|x - a| < \delta(x)$,

$$\left|\frac{F(x) - F(a)}{x - a} - f(x)\right| < \varepsilon. \tag{8}$$

Now choose a partition $\mathcal{P}$ so that each term $f(x_i)(u_i - u_{i-1})$ satisfies

$$x_i - u_{i-1} < \delta(x_i), \qquad u_i - x_i < \delta(x_i).$$

The existence of such partitions is a consequence of the Heine-Borel theorem. For such a partition (8) implies

$$|(F(x_i) - F(u_{i-1})) - f(x_i)(x - u_{i-1})| \;<\; \varepsilon(x_i - u_{i-1}), \tag{9}$$
$$|(F(u_i) - F(x_i)) - f(x_i)(u_i - x_i)| \;<\; \varepsilon(u_i - x_i); \tag{10}$$

with corresponding inequalities for $x = 0$ and $x = 1$. Writing $\alpha = F(1) - F(0)$ and $R = (\mathcal{P})\sum f(x)|I|$,

$$
\begin{aligned}
|\alpha - R| &= \left| \sum_{i=1}^{n} (F(u_i) - F(x_i)) + (F(x_i) - F(u_{i-1})) \right. - \\
&\qquad \left. - \sum_{i=1}^{n} (f(x_i)(x_i - u_{i-1}) + f(x_i)(u_i - x_i)) \right| \\
&\leq \sum_{i=1}^{n} \{ |(F(x_i) - F(u_{i-1}) - f(x_i)(x_i - u_{i-1})| + \\
&\qquad + |(F(u_i) - F(x_i)) - f(x_i)(u_i - x_i)| \} \\
&< \sum_{i=1}^{n} \{ \varepsilon(x_i - u_{i-1}) + \varepsilon(u_i - x_i) \} \\
&= \varepsilon \sum_{i=1}^{n} (x_i - u_{i-1} + u_i - x_i) \quad = \quad \varepsilon.
\end{aligned}
$$

Thus the Riemann-complete integral of f exists and equals the Newton/Leibnitz definite integral $F(1) - F(0)$.

In the case that f is given by (1), while the Newton/Leibnitz and Riemann-complete integrals exist, it has been asserted above that the Lesbesgue integral does not exist.[6] This assertion remains to be demonstrated.

The definition of the Lebesgue integral of a function can be addressed in various equivalent ways, e.g. [18]. For instance, given a real-valued, measurable function f defined on an arbitrary measurable space S, with measure μ defined on the family of measurable subsets of S, [15] shows how to define the Lebesgue integral of f on S as a Riemann-Stieltjes integral in $\mathbf{R}$, the set of real numbers. In fact, writing

$$
g(x) = \mu\left(f^{-1}(]-\infty, x])\right),
$$

the Lebesgue integral $\int_S f\, d\mu$ is the Riemann-Stieltjes integral $\int_{-\infty}^{\infty} f\, dg$. If $S \subseteq \mathbf{R}$ and μ is Lebesgue measure in $\mathbf{R}$, and if the Lebesgue integral of f exists, then the Riemann-complete integral of f exists and the two integrals are equal. Every Lebesgue integrable function is integrable in the Riemann-complete sense (see [14]).

A key point is that the Lebesgue integral is an absolute integral, while the Riemann-complete is non-absolute. Writing $f_+(x) = f(x)$

[6] In that case the Riemann integral of f does not exist either.

if $f(x) \geq 0$, with $f_+(x) = 0$ otherwise, and $f_-(x) = f(x) - f_+(x)$, absolute integrability implies that f is Lebesgue integrable if and only if both f_+ and f_- are Lebesgue integrable.[7] We use this point to demonstrate that the function f defined by (1) is not Lebesgue integrable.

With that in mind, Figure 3 provides an indication of how the function f defined by (1)

- fails to be Lebesgue integrable, while
- its Riemann-complete integral exists.

In fact Figure 3 shows that, in neighbourhoods of $x = 0$, the graph of f oscillates increasingly rapidly, in positive (f_+, above the x-axis) loops and negative (f_-, below the x-axis) loops whose amplitude (or height/depth) increases without limit as $x \to 0$. This creates the suspicion, or expectation, that the sum of areas of the positive loops diverges to $+\infty$, while the sum of areas of the negative loops diverges to $-\infty$.

But if, instead of treating positive and negative loops separately, we add up their areas in their natural sequence, then positive and negative areas will tend to cancel each other out, and the resulting sequence of net values may converge.[8] The latter is what happens in the Riemann sum construction of the Riemann-complete integral of f.

The following discussion seeks to add substance to these speculations. In any interval, not including zero, but with small values of x, Figure 1 shows that the contribution from the term $2x \sin x^{-2}$ to the area under the graph of f is vanishingly small in neighbourhoods of $x = 0$, while the corresponding contribution from the term $2x^{-1} \cos x^{-2}$ in f is relatively large. Therefore, disregarding the term $2x \sin x^{-2}$, the zeros of (1) can, for present purposes, be estimated approximately as

$$x = \sqrt{\frac{2}{(2n+1)\pi}} \text{ as } x \to 0 \ \text{ (or integer } n \to \infty).$$

[7]This restriction does not apply to the Riemann-complete integral of f, which does **not** require the Riemann-complete integrability of f_+ and f_-.

[8]For example, with $b_i = (-1)^i i^{-1}$, the series $\sum_{i=1}^{\infty} b_i$ converges, but the series consisting of only the positive terms (or only negative terms) diverges; so $\sum_{i=1}^{\infty} |b_i|$ diverges.

Accordingly we may estimate that, for large, even values of n,

$$\int_{\sqrt{\frac{2}{(2n+3)\pi}}}^{\sqrt{\frac{2}{(2n+1)\pi}}} f_+(x)\,dx \quad \text{is approximately} \quad \frac{2}{\pi}\left(\frac{1}{2n+1}+\frac{1}{2n+3}\right)$$

while for large and odd values of n

$$\int_{\sqrt{\frac{2}{(2n+3)\pi}}}^{\sqrt{\frac{2}{(2n+1)\pi}}} f_-(x)\,dx \quad \text{is approximately} \quad \frac{2}{\pi}\left(\frac{1}{2n+1}+\frac{1}{2n+3}\right).$$

Writing

$$a_n = \frac{2}{\pi}\left(\frac{1}{2n+1}+\frac{1}{2n+3}\right),$$

each of the two series

$$a_2 + a_4 + a_6 + \cdots, \qquad a_1 + a_3 + a_5 + \cdots$$

diverges, so it is clear that f is not Lebesgue integrable in $[0,1]$. But it is easy to see that the series

$$-a_1 + a_2 - a_3 + a_4 - \cdots$$

is non-absolutely convergent, even if we did not already know, from existence of the primitive function $F(x)$ for $0 \leq x \leq 1$, that f is Riemann-complete integrable.

This is because the Riemann-complete convergence is obtained from the cancellation effects produced by successively summing the positive and negative parts in their natural sequence.

We can ensure this by choosing $\delta(x)$ as follows. When x lies between adjacent roots $\sqrt{\frac{2}{(2n+1)\pi}}$ and $\sqrt{\frac{2}{(2n+3)\pi}}$ let

$$\delta(x) < \min\left\{x - \sqrt{\frac{2}{(2n+3)\pi}},\quad \sqrt{\frac{2}{(2n+1)\pi}} - x\right\};$$

and if x is one of the roots $\sqrt{\frac{2}{(2n+1)\pi}}$, take

$$\delta(x) < \min\left\{\sqrt{\frac{2}{(2n+1)\pi}},\quad \sqrt{\frac{2}{(2n+1)\pi}} - \sqrt{\frac{2}{(2n+3)\pi}}\right\};$$

and when $x = 0$ let $\delta(0) > 0$ be arbitrary. Any partition corresponding to this definition of $\delta(x)$ $(0 \leq x \leq 1)$ will contain a term with $f(0) = 0$, and the terms for non-zero x will each contain an arbitrarily close estimate of the area of the corresponding positive or negative loop in Figure 3. This provides the required cancellation and convergence of Riemann sums, since the alternating loops are monotone decreasing in area as x approaches 0.

In the case of the Lebesgue integral this cancellation effect is absent, and convergence fails.

This establishes that, just as there are Lebesgue integrable functions that are not Riemann integrable, there are Riemann-complete integrable functions that are not Lebesgue integrable.

5. Convergence Criteria

Anybody experienced in the theory of integration will be aware that most of the preceding discussion covers fairly well-worn ground which has already been worked through in many excellent publications, such as [2].

But at the outset of this article it was stated that, while Lebesgue's dominated convergence theorem is a crucial pillar of modern analysis, there are certain areas of the subject where this theorem is deficient. The sequence $\{f_j\}$ of (2) demonstrates that the dominated convergence theorem provides no illumination in this particular instance of converging non-absolute integrals. This section addresses the deficit.

There are convergence conditions and criteria which encompass and surpass the dominated convergence, monotone and uniform convergence theorems of standard integration theory. These are the convergence criteria of Theorems 5.1, 5.2, and 5.3 below. They are valid for Riemann-complete integrals (which include integrals of the Newton/Leibnitz, Riemann, and Lebesgue kinds). Measurability of the integrand functions is not assumed.

Theorem 5.1. *Suppose f_j is integrable on $[0,1]$ and $f_j(x)$ converges to $g(x)$ for $x \in [0,1]$. Suppose, given arbitrary $\varepsilon > 0$, there exist a number α_1 and, for $x \in [0,1]$, a gauge $\delta(x)$, and integers $p = p(x)$ depending on ε, so that, for every partition $\{I_i\}$ of $[0,1]$ with linked elements $\{(x_i, I_i)\}$ satisfying $|I_i| < \delta(x_i)$ $(i = 1, \ldots, n)$, the condition*

$$\left|\alpha_1 - \sum_{i=1}^{n} f_{j(x_i)}(x_i) \times |I_i|\right| \quad < \quad \varepsilon$$

holds for all choices of $j = j(x_i) > p(x_i)$ $(i = 1, \ldots, n)$ *in the Riemann sum. Then the limit function* $g(x)$ *is integrable on* $[0, 1]$, *with* $\int_0^1 g(x)\,dx = \alpha_1$.

Theorem 5.2. *Suppose* f_j *is integrable on* $[0, 1]$ *and* $f_j(x)$ *converges to* $g(x)$ *for* $x \in [0, 1]$. *Suppose, given arbitrary* $\varepsilon > 0$, *there exist a number* α_2 *and a positive integer* $q = q(\varepsilon)$ *depending only on* ε, *so that, for every partition* $\{I_i\}$ *of* $[0, 1]$ *with linked elements* $\{(x_i, I_i)\}$ *satisfying* $|I_i| < \delta(x_i)$ $(i = 1, \ldots, n)$, *the condition*

$$\left| \alpha_2 - \sum_{i=1}^{n} f_j(x_i) \times |I_i| \right| \quad < \quad \varepsilon$$

holds for every choice of $j > q(\varepsilon)$ *with* j *constant for each term of the Riemann sum. Then* $\int_0^1 f_j(x)dx$ *converges as* $j \to \infty$.

Theorems 5.1 and 5.2 can be expressed in converse form (see [14]), so they are criteria for their respective conclusions. Apart from the opening sentence of each they apply independently of each other; in the sense that either one of them may hold for particular integrands while the other one does not hold.

Theorem 5.3. *If both of Theorems 5.1 and 5.2 hold (so both of* $\int_0^1 g$ *and* $\lim_{j\to\infty} \int_0^1 f_j$ *exist), then*

$$\int_0^1 g(x)\,dx = \lim_{j\to\infty} \int_0^1 f_j(x)\,dx$$

if and only if $\alpha_1 = \alpha_2$.

For anybody more familiar with the classical integration theorems on passage to a limit, these theorems or criteria may appear somewhat indigestible at first sight.

Their starting point is that a convergent sequence of functions f_j is given. These function are assumed to be Riemann-complete integrable, which is a weaker assumption than Lebesgue integrability. There is no assumption of properties like continuity or measurability.

To answer questions about convergence of the corresponding sequence of Riemann-complete integrals, and about the Riemann-complete integrability of the limit function, from previous experience of integration we might be led to expect some condition, not about Riemann sums, but only about the functions f_j—such as monotonicity, domination by a fixed integrable function g; or the like. But nothing like this appears in the above convergence criteria. Instead we have various statements about Riemann sums.

However, setting aside for a moment the conception of integral as primitive function, or anti-derivative, the original and more durable meaning of integral involves slicing up (partitioning), followed by summation, followed by taking a limit of the sums.

From this perspective, it may be less of a surprise that Riemann sums appear in the formulation of conditions for limits of integrals, since Riemann sums are central to the concept of integral.

Consider Theorem 5.1. Given integrability of the terms f_j in the sequence, this theorem addresses the integrability of the limit function f, which, essentially, involves the question of convergence of Riemann sums $\sum f(x)|I|$.

To make an initial stab at this question, we might consider sequences of Riemann sums $\sum f_j(x)|I|$, $j = 1, 2, 3, \ldots$. We know that, for each x, the sequence of values $f_j(x)$ converges to $f(x)$ as $j \to \infty$. We also know that, for each j, Riemann sums of the form $\sum f_j(x)|I|$ converge to the integral of f_j. Can we somehow put these two facts together to deduce, as an immediate consequence, convergence of Riemann sums $\sum f(x)|I|$ to the integral of f?

Of course, we know that the answer to this is **no**. The answer is **yes** if the terms f_j satisfy some conditions such as $|f_j| < g$ where g is integrable. But if we want a condition expressed in the form of a condition on Riemann sums, clearly something more delicate than convergence of $\sum f_j(x)|I|$ is required.

For instance, with $\varepsilon > 0$ given, the condition we need is **not** that, for all j greater than some $j_0 = j_0(\varepsilon)$, every Riemann sum $\sum f_j(x)|I|$ will be contained within some ball $B(\varepsilon)$ of the form $]\beta - \varepsilon, \beta + \varepsilon[$.

This is too crude for our purpose. All it says is that f_j is integrable—which we already know. The convergence of $f_j(x)$ to $f(x)$ may be very fast at some points x, and very slow at other points x. This behaviour is provided for in Theorem 5.1, by choosing, **not** $j_0 = j_0(\varepsilon)$, but $j_0 = j_0(\varepsilon, x)$ different for each x.

This formulation is sufficient, and necessary, for integrability of the limit function f. Once this point is established, the criteria of Theorems 5.2 and 5.3 are fairly obvious, and less subtle. But are these conditions of Theorems 5.1, 5.2 and 5.3 "workable" in the way that the dominated convergence condition $|f_j| < g$ is?

After all, Riemann sums are fine for defining the meaning of the integral of a function. But when we actually want to find the value of an integral we do not typically work with Riemann sums. Instead we revert to the integral as primitive or anti-derivative, using the substitution method or integration by parts. Or we use some less direct method, such as solving a related differential equation; or a myriad of other[9] *ad hoc* methods.

To respond to such questions, and to demonstrate that Riemann sums **can** actually be of use here, we can as an example take the sequence f_j defined in (2). Remember, for each j the function f_j is Riemann integrable and Lebesgue integrable, but not Newton/Leibnitz integrable, and their limit function f is Newton/Leibnitz integrable but not Riemann or Lebesgue integrable. For each j, f_j is Riemann-complete integrable.[10] This discussion of the convergence criteria of Theorems 5.1, 5.2 and 5.3 is set in the context of Riemann-complete integrability.

The subject of the first criterion is the (Riemann-complete) integrability of the limit function f, and it is established by examining Riemann sums of the form

$$\sum_{i=1}^{n} f_{j(x_i)}(x) \times |I_i|, \quad \text{or} \quad \sum_{i=1}^{n} f_{j(x_i)}(x)(u_i - u_{i-1}).$$

We already know, by various means, including a direct investigation of the Riemann sums $\sum f(x)|I|$, that $f(x)$ is (Riemann-complete) integrable on $[0, 1]$.

The function $f(x)$ is the limit of functions $f_j(x)$. Is it possible to confirm further the integrability of f by direct examination, not just of $\sum f(x)|I|$, but of Riemann sums $\sum f_{j(x)}(x)|I|$ involving functions f_j instead of f, where the factor f_j in the sum has variable index $j = j(x)$, depending on the element x of the division $\mathcal{D} = \{(x, I)\}$ used to construct the Riemann sum?

This is the essence of Theorem 5.1. And according to Theorem 5.1 the answer to this question should be yes. Given $\varepsilon > 0$, and

[9]Which is **not** to say that Riemann sums are "merely" a device of fundamental theory, and nothing else. Versions of them have had other uses; such as the ancient techniques of quadrature; or computer programs for estimating numerical values of an integral. Simpson's rule is another example.

[10]As is f, from earlier discussion. But for present purposes we wish to **deduce** Riemann-complete integrability of f from Theorem 5.1.

with a suitable gauge $\delta(x)$, provided factors $f_{j(x)}(x)$ are chosen appropriately we should be able to demonstrate that the value of each corresponding Riemann sum $\sum f_{j(x)}(x)|I|$ will lie within some ball B of radius ε where ε is arbitrarily small.

Since we are already convinced of the integrability of f in this case, what we are really trying to do here is to get a sense of the behaviour of sums $\sum f_{j(x)}(x)|I|$. So, given the integrability of f, write $\alpha_1 = \int_{]0,1]} f(x)dx$ and choose a gauge δ so that, for every δ-fine partition $\{(x_i, I_i)\}$ of $]0,1]$,

$$\left|\alpha_1 - \sum_{i=1}^{n} f(x_i)|I_i|\right| < \varepsilon, \quad \text{or} \quad \sum_{i=1}^{n} f(x_i)|I_i| \in B(\alpha_1, \varepsilon),$$

the ball with centre α_1 and radius ε.

Consider any one of these Riemann sums $\sum_{i=1}^{n} f(x_i)|I_i|$, corresponding to a particular δ-fine partition with $\{(x_i, I_i) \ : \ i = 1, \dots, n\}$. For each x choose

$$r(x) \geq \frac{1}{x}, \quad \text{so} \quad r(x_i) \geq \frac{1}{x_i} \ \text{ for each } \ i.$$

Then, by definition of f_j, if $j = j(x) \geq r(x)$,

$$f_j(x) = f_{j(x)}(x) = f(x),$$

so, for all choices $j(x_i) \geq r(x_i) = r(x_i, \varepsilon)$,

$$\sum_{i=1}^{n} f_{j(x_i)}(x_i)|I_i| = \sum_{i=1}^{n} f(x_i)|I_i| \in B(\alpha_1, \varepsilon),$$

as required by Theorem 5.1.

In general, the convenient equation $f_j(x) = f_{j(x)}(x) = f(x)$ cannot be appealed to. But if, with suitable choices of $j = j(x)$, the differences

$$f_j(x) - f(x) = f_{j(x)}(x) - f(x),$$

can make sufficiently small contributions to the Riemann sum, then it may be plausible that

$$\sum_{i=1}^{n} f_{j(x)}(x_i)|I_i| \in B'(\varepsilon) \quad \text{implies} \quad \sum_{i=1}^{n} f(x_i)|I_i| \in B''(\varepsilon)$$

so f is integrable. This is the intuitive content of Theorem 5.1.

Now to Theorem 5.2. The preceding remarks are concerned with the integrability of $\lim_{j\to\infty} f_j$. The fundamental assumption is that each function f_j in the sequence $\{f_j\}$ is integrable. In the case of our example (2) the anti-derivatives (4) are the sequence $\{F_j\}$, giving a sequence of integrals

$$\int_{]0,1]} f_j(x)dx = F_j(1) - F_j(j^{-1}) = \sin 1 - F_j(j^{-1}),$$

which can be denoted by β_j. Note that continuity of F implies $\beta_j \to 0$ as $j \to \infty$. In this case it is already clear that the sequence of integrals $\int_{]0,1]} f_j(x)dx$ converges as $j \to \infty$, the limit being (in this case) $\sin 1$; what we want is confirmation, including intuitive confirmation, that Theorem 5.2 actually works.

The convergence of a sequence of integrals is the subject of Theorem 5.2, and it is again expressed in terms of Riemann sums. The criterion implies that, with arbitrarily small $\varepsilon > 0$ given, there is a ball $B(\alpha_2, \varepsilon)$ with centre α_2 and radius ε, and a corresponding integer q depending only on ε, so that for each $j \geq q = q(\varepsilon)$, a gauge $\delta_j(x) > 0$ can be found such that for every δ_j-fine partition of $[0,1]$ the corresponding Riemann sum $\sum_{i=1}^n f_j(x_i)|I_i|$ is contained in $B(\alpha_2, \varepsilon)$; so for every δ_j-fine $\{(x_i, I_i)\}$,

$$\left|\alpha_2 - \sum_{i=1}^n f_j(x_i)|I_i|\right| < \varepsilon$$

whenever $j \geq q$. Unlike Theorem 5.1, here j is the same for each term of any particular Riemann sum.

Again, this is easy to demonstrate since we already know in this case that the integrals $\int_{]0,1]} f_j(x)dx$ converge to the value $\sin 1$ as $j \to \infty$. Just take

$$\alpha_2 = \sin 1 = \lim_{j\to\infty} g(1) - g(j^{-1}) = \int_{]0,1]} f_j(x)dx,$$

and choose q so that $j \geq q$ implies

$$\left|\alpha_2 - \int_{]0,1]} f_j(x)dx\right| < \varepsilon.$$

For each $j \geq q$ choose a gauge $\delta_j(x)$ $(0 \leq x \leq 1)$ so that, for any δ_j-fine partition $\{(x_i, I_i)\}$ of $]0,1]$,

$$\left|\int_{]0,1]} f_j(x)dx - \sum_{i=1}^n f_j(x_i)|I_i|\right| < \varepsilon.$$

Then, by the triangle inequality,

$$\left|\alpha_2 - \sum_{i=1}^{n} f_j(x_i)|I_i|\right| < 2\varepsilon, \quad \text{or} \quad \sum_{i=1}^{n} f_j(x_i)|I_i| \in B(\alpha_2, 2\varepsilon)$$

for all $j \geq q = q(\varepsilon)$ and all δ_j-fine partitions of $]0,1]$. In other words, the criterion of Theorem 5.2 confirms the convergence of the sequence of integrals

$$\left\{\int_{]0,1]} f_j(x)dx\right\};$$

and this demonstration illustrates the intuitive content of Theorem 5.2.

Finally, the question arises whether the integral of the limit

$$\int_{]0,1]} \lim_{j\to\infty} f_j(x)dx$$

equals the limit of the integrals

$$\lim_{j\to\infty}\int_{]0,1]} f_j(x)dx.$$

For the sequence f_j of (2), we already know by direct evaluation that these two quantities have the same value, namely $\sin 1$. This agrees with the criterion of Theorem 5.3, which requires that α_1 and α_2 have the same value. In this case

$$\alpha_1 = \sin 1 = \alpha_2;$$

so the intuitive content of Theorem 5.3 is clear in the context of this example.

6. Conclusion

So, does it really matter whether aviation designers work out their aerodynamic equations using old-fashioned Riemann integrals or the latest fancy Lebesgue integrals?

Probably not much. But it matters a lot if the value 22/7 for π were hard-wired into every computer in the world. Or if the wrong value for elasticity of O-rings at freezing temperature was used in space shuttle design. And it certainly matters whether our aviation designers make tricky, unjustifiable calculations involving, for instance, term-by-term integration of Fourier series.

It is thanks to the intellectual diligence of the nineteenth century, not to mention its monster-functions, that we have the dominated convergence theorem to keep the garden of mathematics safe and fertile—and, indeed, to keep airplanes flying safely.

But do we really need anything more than the dominated convergence theorem for absolutely convergent integrals? Why bring up the convergence criteria of Theorems 5.1, 5.2, and 5.3? Is the sequence $\{f_j\}$ described in (2) above just an exceptional one-off, or is it representative of something more significant? If the latter, where are all these non-absolute integrals?

In fact they are very widespread. Modern stochastic calculus [16, 17] is based on integrals for which absolute convergence fails, but which may converge weakly or, in some cases, non-absolutely. These are described in [14, 15].

A very significant formulation of quantum mechanics is in terms of path integrals [3] which also fail to converge absolutely. Famously, the dominated convergence theorem does not work for these integrals, and, as described in [14], the non-absolute convergence criteria must be invoked.

"Does anyone believe ... I would not care to fly in that plane." A healthy scepticism must be welcomed. But what is certain is that, while integration is central to mathematical analysis, there are no certain and definite ways of tackling any problem of integration, and even a beginning student has to exercise imagination and ingenuity. From the ancient methods of quadrature, to the methods of Newton/Leibnitz, Cauchy, Riemann, Lebesgue, Denjoy, Perron, Kolmogorov, Kurzweil, or Henstock, it is unwise to disregard any resource or insight that can be called upon.

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Composition operators between weighted Bergman spaces and weighted Banach spaces of holomorphic functions

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Abstract. An analytic self-map ϕ of the open unit disk $\mathbb{D}$ in the complex plane induces the so-called composition operator $C_\phi : H(\mathbb{D}) \to H(\mathbb{D})$, $f \mapsto f \circ \phi$, where $H(\mathbb{D})$ denotes the set of all analytic functions on $\mathbb{D}$. Motivated by [5] we analyze under which conditions on the weight v all composition operators C_ϕ acting between the weighted Bergman space and the weighted Banach space of holomorphic functions both generated by v are bounded.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$ endowed with the compact-open topology *co*. Moreover, let ϕ be an analytic self-map of $\mathbb{D}$. Such a map induces through composition the classical composition operator

$$C_\phi : H(\mathbb{D}) \to H(\mathbb{D}), \; f \mapsto f \circ \phi.$$

Composition operators acting on various spaces of analytic functions have been studied by many authors, since this kind of operator appears naturally in a variety of problems, such as e.g. in the study of commutants of multiplication operators or the study of dynamical systems, see the excellent monographs [8] and [17]. For a deep insight in the recent research on (weighted) composition operators we refer the reader to the following papers as well as the references therein: [4], [5], [6], [7], [11], [13], [14], [15], [16].
Let us now explain the setting in which we are interested. We say that a function $v : \mathbb{D} \to (0, \infty)$ is a *weight* if it is bounded and continuous. For a weight v we consider the following spaces:

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(1) The *weighted Banach spaces of holomorphic functions* defined by

$$H_v^\infty := \{f \in H(\mathbb{D});\ \|f\|_v := \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\}.$$

Endowed with the weighted sup-norm $\|.\|_v$ this is a Banach space. These spaces arise naturally in several problems related to e.g. complex analysis, spectral theory, Fourier analysis, partial differential and convolution equations. Concrete examples may be found in [3]. Weighted Banach spaces of holomorphic functions have been studied deeply in [2] and also in [1].

(2) The *weighted Bergman spaces* given by

$$A_v^2 := \left\{ f \in H(\mathbb{D});\ \|f\|_{v,2} := \left(\int_{\mathbb{D}} |f(z)|^2 v(z)\, dA(z) \right)^{\frac{1}{2}} < \infty \right\},$$

where $dA(z)$ is the normalized area measure such that area of $\mathbb{D}$ is 1. Endowed with norm $\|.\|_{v,2}$ this is a Hilbert space. An introduction to Bergman spaces is given in [10] and [9].

In [19] we characterized the boundedness of composition operators acting beween weighted Bergman spaces and weighted Banach spaces of holomorphic functions in terms of the involved weights as well as the symbols. In this article we are interested in the question, for which weights v all composition operators $C_\phi : A_v^2 \to H_v^\infty$ are bounded.

2. Background and basics

2.1. **Theory of weights.** In this part of the article we want to give some background information on the involved weights. A very important role play the so-called *radial* weights, i.e. weights which satisfy $v(z) = v(|z|)$ for every $z \in \mathbb{D}$. If additionally $\lim_{|z|\to 1} v(z) = 0$ holds, we refer to them as *typical* weights. Examples include all the famous and popular weights, such as

(a) the *standard weights* $v(z) = (1-|z|)^\alpha$, $\alpha \geq 1$,
(b) the *logarithmic weights* $v(z) = (1-\log(1-|z|))^\beta$, $\beta > 0$,
(c) the *exponential weights* $v(z) = e^{-\frac{1}{(1-|z|)^\alpha}}$, $\alpha \geq 1$.

In [12] Lusky studied typical weights satisfying the following two conditions

$$(L1) \qquad \inf_{n\in\mathbb{N}} \frac{v(1-2^{-n-1})}{v(1-2^{-n})} > 0$$

and

$$(L2) \qquad \limsup_{n\to\infty} \frac{v(1-2^{-n-j})}{v(1-2^{-n})} < 1 \text{ for some } j \in \mathbb{N}.$$

In fact, weights having $(L1)$ and $(L2)$ are *normal* weights in the sense of Shields and Williams, see [18]. The standard weights are normal weights, the logarithmic weights satisfy $(L1)$, but not $(L2)$ and the exponential weights satisfy neither $(L1)$ nor $(L2)$. In our context $(L2)$ is not of interest, while $(L1)$ will play a secondary role. The formulation of results on weighted spaces often requires the so-called associated weights. For a weight v its associated weight is given by

$$\tilde{v}(z) := \frac{1}{\sup\{|f(z)|;\ f \in H(\mathbb{D}),\ \|f\|_v \le 1\}}, \quad z \in \mathbb{D}.$$

See e.g. [2] and the references therein. Associated weights are continuous, $\tilde{v} \ge v > 0$ and for every $z \in \mathbb{D}$ there is $f_z \in H(\mathbb{D})$ with $\|f_z\|_v \le 1$ such that $f_z(z) = \frac{1}{\tilde{v}(z)}$. Since it is quite difficult to really calculate the associated weight we are interested in simple conditions on the weight that ensure that v and $\tilde{v}$ are equivalent weights, i.e. there is a constant $C > 0$ such that

$$v(z) \le \tilde{v}(z) \le Cv(z) \text{ for every } z \in \mathbb{D}.$$

If v and $\tilde{v}$ are equivalent, we say that v is an *essential* weight. By [5] condition $(L1)$ implies the essentiality of v.

2.2. **Setting.** This section is devoted to the description of the setting we are working in. In the sequel we will consider weighted Bergman spaces generated by the following class of weights. Let ν be a holomorphic function on $\mathbb{D}$ that does not vanish and is decreasing as well as strictly positive on $[0,1)$. Moreover, we assume that $\lim_{r\to 1} \nu(r) = 0$. Now, we define the weight as follows:

$$v(z) := \nu(|z|) \text{ for every } z \in \mathbb{D}. \qquad (1)$$

Obviously such weights are bounded, i.e. for every weight v of this type we can find a constant $C > 0$ such that $\sup_{z\in\mathbb{D}} v(z) \le C$. Moreover, we assume additionally that $|\nu(z)| \ge \nu(|z|)$ for every

$z \in \mathbb{D}$.
Now, we can write the weight v in the following way

$$v(z) = \min\{|g(\lambda z)|,\ |\lambda| = 1\},$$

where g is a holomorphic function on $\mathbb{D}$. Since ν is a holomorphic function, we obviously can choose $g = \nu$. Then we arrive at

$$\begin{aligned}\min\{|\nu(\lambda z)|,\ |\lambda| = 1\} &= \min\{|\nu(\lambda r e^{i\Theta})|,\ |\lambda| = 1\}\\ &\leq |\nu(e^{-i\Theta} r e^{i\Theta})| = |\nu(r)| = |\nu(|z|)| = v(z)\end{aligned}$$

for every $z \in \mathbb{D}$. Conversely, by hypothesis, for every $\lambda \in \partial\mathbb{D}$ we obtain for every $z \in \mathbb{D}$

$$|\nu(\lambda z)| \geq \nu(|\lambda z|) \geq \nu(|z|) = v(z).$$

Thus, the claim follows. The standard, logarithmic and exponential weights can all be defined like that.

2.3. **Composition operators between weighted Bergman spaces and weighted Banach spaces of holomorphic functions.** In the setting of weighted Banach spaces of holomorphic functions the classical composition operator has been studied by Bonet, Domański, Lindström and Taskinen in [4] and [5]. Among other things they proved that in case that v and w are arbitrary weights the boundedness of the operator $C_\phi : H_v^\infty \to H_w^\infty$ is equivalent to

$$\sup_{z\in\mathbb{D}} \frac{w(z)}{\tilde{v}(\phi(z))} < \infty.$$

Moreover, they showed that v satisfies condition $(L1)$ if and only if every composition operator $C_\phi : H_v^\infty \to H_v^\infty$ is bounded.

This was the motivation to study the boundedness composition operators acting between weighted Bergman spaces and weighted Banach spaces of holomorphic functions. Doing this we obtain the following results which we need in the sequel. For the sake of understanding and completeness we give the proof here.

For $p \in \mathbb{D}$ let

$$\alpha_p(z) := \frac{p - z}{1 - \bar{p}z},\ z \in \mathbb{D},$$

be the Möbius transformation that interchanges p and 0.

Lemma 2.1 ([20], Lemma 1)**.** *Let* $v(z) = \nu(|z|)$ *for every* $z \in \mathbb{D}$ *with* $\nu \in H(\mathbb{D})$ *be a weight as defined in Section 2.2. Then there is a constant* $M > 0$ *such that*

$$|f(z)| \leq M \frac{\|f\|_{v,2}}{(1-|z|^2)v(z)^{\frac{1}{2}}}$$

for every $f \in A^2_v$.

Proof. As we have seen in Section 2.2 a weight as defined above may be written as

$$v(z) := \min\{|g(\lambda z)|;\ |\lambda| = 1\} \text{ for every } z \in \mathbb{D},$$

where g is a holomorphic function on $\mathbb{D}$. In the sequel we will write $g_\lambda(z) := g(\lambda z)$ for every $z \in \mathbb{D}$. Now, fix $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Moreover, let $p \in \mathbb{D}$ be arbitrary. Then, we consider the map

$$T_{p,\lambda} : A^2_v \to A^2_v,\ T_{p,\lambda} f(z) = f(\alpha_p(z))\alpha_p'(z) g_\lambda(\alpha_p(z))^{\frac{1}{2}}.$$

Let $f \in A^2_v$. Then a change of variables yields

$$\begin{aligned}
\|T_{p,\lambda} f\|^2_{v,2} &= \int_{\mathbb{D}} v(z)|f(\alpha_p(z))|^2|\alpha_p'(z)|^2|g_\lambda(\alpha_p(z))|\, dA(z)\\
&\leq \int_{\mathbb{D}} v(z)|f(\alpha_p(z))|^2|\alpha_p'(z)|^2|v(\alpha_p(z))|\, dA(z)\\
&\leq \sup_{z\in\mathbb{D}} v(z)\int_{\mathbb{D}} |f(\alpha_p(z))|^2|\alpha_p'(z)|^2|v(\alpha_p(z))|\, dA(z)\\
&\leq C\int_{\mathbb{D}} v(t)|f(t)|^2\, dA(t) = C\|f\|^2_{v,2}.
\end{aligned}$$

Next, put $h_{p,\lambda}(z) := T_{p,\lambda} f(z)$ for every $z \in \mathbb{D}$. By the Mean Value Property we obtain

$$v(0)|h_{p,\lambda}(0)|^2 \leq \int_{\mathbb{D}} v(z)|h_{p,\lambda}(z)|^2\, dA(z) \leq \|h_{p,\lambda}\|^2_{v,2} \leq C\|f\|^2_{v,2}.$$

Since λ was arbitrary, we obtain

$$v(0)|f(p)|^2(1-|p|^2)v(p)^{\frac{1}{2}} \leq C\|f\|^2_{v,2}.$$

Finally,

$$|f(p)| \leq M\frac{\|f\|_{v,2}}{(1-|p|^2)v(p)^{\frac{1}{2}}} < \infty.$$

□

The following result is obtained by using the previous lemma and following exactly the proof of [19] Theorem 2.2. Again, for a better understanding we give the proof.

Theorem 2.2. *Let* $v(z) = \nu(|z|)$ *for every* $z \in \mathbb{D}$ *with* $\nu \in H(\mathbb{D})$ *be a weight as defined in Section 2.2. Then the operator* $C_\phi : A_v^2 \to H_v^\infty$ *is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{v(z)}{(1-|\phi(z)|^2)v(\phi(z))^{\frac{1}{2}}} < \infty. \tag{2}$$

Proof. First, we assume that (2) holds. Applying Lemma 2.1 for every $f \in A_v^2$ we have

$$|f(z)| \leq C \frac{\|f\|_{v,2}}{(1-|z|^2)v(z)^{\frac{1}{2}}}$$

for every $z \in \mathbb{D}$. Thus, for every $f \in A_v^2$:

$$\|C_\phi f\|_v = \sup_{z\in\mathbb{D}} v(z)|f(\phi(z))| \leq C \sup_{z\in\mathbb{D}} \frac{v(z)\|f\|_{v,2}}{(1-|\phi(z)|^2)v(\phi(z))^{\frac{1}{2}}} < \infty.$$

Hence the operator must be bounded.
Conversely, let $p \in \mathbb{D}$ be fixed. Then there is $f_p^2 \in H_v^\infty$, $\|f_p^2\|_v \leq 1$ with $|f_p^2(p)| = \frac{1}{\tilde{v}(p)}$. Now, put

$$g_p(z) := f_p(z)\alpha_p'(z) \text{ for every } z \in \mathbb{D}.$$

Changing variables we obtain

$$\begin{aligned}\|g_p\|_{v,2}^2 = \int_{\mathbb{D}} |g_p(z)|^2 v(z)\, dA(z) &= \int_{\mathbb{D}} |f_p(z)|^2|\alpha_p'(z)|^2 v(z)\, dA(z) \\ &\leq \sup_{z\in\mathbb{D}} v(z)|f_p(z)|^2 \int_{\mathbb{D}} |\alpha_p'(z)|^2\, dA(z) = 1\end{aligned}$$

Next, we assume to the contrary that there is a sequence $(z_n)_n \subset \mathbb{D}$ such that $|\phi(z_n)| \to 1$ and

$$\frac{v(z_n)}{(1-|\phi(z_n)|^2)v(\phi(z_n))^{\frac{1}{2}}} \geq n \text{ for every } n \in \mathbb{N}.$$

Now, we consider

$$g_n(z) := g_{\phi(z_n)}(z) \text{ for every } z \in \mathbb{D} \text{ and every } n \in \mathbb{N}$$

as defined above. Then $(g_n)_n$ is contained in the closed unit ball of A_v^2 and we can find a constant $c > 0$ such that

$$c \geq v(z_n)|g_n(\phi(z_n))| = \frac{v(z_n)}{(1-|\phi(z_n)|^2)v(\phi(z_n))^{\frac{1}{2}}} \geq n$$

for every $n \in \mathbb{N}$. Since we know that under the given assumptions we have $v = \tilde{v}$ this is a contradiction. $\square$

Having now characterized the boundedness of the composition operator acting between A_v^2 and H_v^∞ we take the second result of Bonet, Domański, Lindström and Taskinen as a motivation to ask the question: For which weights v are all operators $C_\phi : A_v^2 \to H_v^\infty$ bounded?

3. Results

Lemma 3.1. *Let $v(z) = \nu(|z|)$ for every $z \in \mathbb{D}$ with $\nu \in H(\mathbb{D})$ be a weight as defined in Section 2.2. Moreover, let $\sup_{z\in\mathbb{D}} \frac{v(z)^{\frac{1}{2}}}{1-|z|^2} < \infty$ and $C_{\alpha_p} : A_v^2 \to H_v^\infty$ be bounded for every $p \in \mathbb{D}$. Then all composition operators $C_\phi : A_v^2 \to H_v^\infty$ are bounded.*

Proof. Let $\phi : \mathbb{D} \to \mathbb{D}$ be an arbitrary analytic function. We have to show that $C_\phi : A_v^2 \to H_v^\infty$ is bounded. Now, $\phi = \alpha_p \circ \psi$ where $p = \phi(0)$, $\psi = \alpha_p \circ \phi$ and $\psi(0) = 0$. Since $\psi(0) = 0$, by the Schwarz Lemma we obtain that $|\psi(z)| \leq |z|$. Hence we get

$$\sup_{z\in\mathbb{D}} \frac{v(z)}{(1-|\psi(z)|^2)v(\psi(z))^{\frac{1}{2}}} \leq \sup_{z\in\mathbb{D}} \frac{v(z)}{(1-|z|^2)v(z)^{\frac{1}{2}}} = \sup_{z\in\mathbb{D}} \frac{v(z)^{\frac{1}{2}}}{(1-|z|^2)}$$

$< \infty$. Thus, C_ψ is bounded. Finally, we can conclude that C_ϕ is bounded since it is a composition of bounded operators. $\square$

The proof of the following theorem is inspired by [5].

Theorem 3.2. *Let $v(z) = \nu(|z|)$ for every $z \in \mathbb{D}$ with $\nu \in H(\mathbb{D})$ be a weight as defined in Section 2.2. Moreover, let $\sup_{z\in\mathbb{D}} \frac{v(z)^{\frac{1}{2}}}{1-|z|^2} < \infty$. Then the composition operator $C_\phi : A_v^2 \to H_v^\infty$ is bounded for every analytic self-map ϕ of $\mathbb{D}$ if and only if*

$$\inf_{n\in\mathbb{N}} \frac{(2^{-n} - 2^{-2n-2})v(1-2^{-n-1})^{\frac{1}{2}}}{v(1-2^{-n})} > 0. \tag{3}$$

Proof. By Lemma 3.1 we have to show that condition (3) holds if and only if $C_{\alpha_p} : A_v^2 \to H_v^\infty$ is bounded for every $p \in \mathbb{D}$.

First, let each $C_{\alpha_p} : A^2_v \to H^\infty_v$ be bounded. Then we have that for every $p \in \mathbb{D}$ there is $M_p > 0$ such that

$$v(z) \leq M_p v(\alpha_p(z))^{\frac{1}{2}}(1 - |\alpha_p(z)|^2) \text{ for every } z \in \mathbb{D}.$$

Since $\sup_{|z|=r} |\alpha_p(z)| = \frac{|p|+r}{1+|p|r}$ it follows that $v(z) \leq M_p v\left(\frac{|p|+r}{1+|p|r}\right)^{\frac{1}{2}}\left(1 - \left(\frac{|p|+r}{1+|p|r}\right)^2\right)$ for all $|z| = r$. Let $l(r) = v(1-r)^{\frac{1}{2}}(1-(1-r)^2)$ and $s = 1 - r$. Now, since $1 - \frac{|p|+1-s}{1+|p|(1-s)} = \frac{s(1-|p|)}{1+|p|-|p|s}$, for $s < \frac{1}{2}$ we obtain

$$l\left(s\frac{1-|p|}{1+|p|}\right) \leq l\left(1 - \frac{|p|+1-s}{1+|p|(1-s)}\right) \leq l\left(s\frac{1-|p|}{1-\frac{|p|}{2}}\right) \tag{4}$$

Next, choose $p = \frac{2}{5}$ and find $M > 0$ and $s_0 > 0$ such that

$$v(1-s) \leq Ml\left(\frac{s}{2}\right) = Mv\left(1-\frac{s}{2}\right)^{\frac{1}{2}}\left(1-\left(1-\frac{s}{2}\right)^2\right) \text{ for all } s \in]0, s_0[.$$

Hence the claim follows.

Conversely we assume that (3) holds. Then l as defined above has the property that there are $M > 0$ and $t_0 \in]0,1[$ with

$$v(1-t) \leq Ml\left(\frac{t}{2}\right) \text{ for all } t \geq t_0.$$

Hence, for any $c < \infty$ we find $n \in \mathbb{N}$ such that $c < 2^n$ and thus $l(t) \leq M^n l\left(\frac{t}{c}\right)$. We take $c = \frac{1+|p|}{1-|p|}$. By the first inequality in (4) for all $p \in \mathbb{D}$ there is $M_p > 0$ such that

$$v(1-t) \leq M_p l\left(1 - \frac{|p|+1-t}{1+|p|(1-t)}\right)$$

for all $t > t_0$. Clearly this implies that for all $p \in \mathbb{D}$ there exists $M_p > 0$ such that for every $|z| = r$ we have that $v(z) < M_p v(\alpha_p(z))^{\frac{1}{2}}(1 - |\alpha_p(z)|^2)$. $\square$

Example 3.3. (a) Let $v(z) = (1-|z|)^n$, $n \geq 2$. Then all composition operators $C_\phi : A^2_v \to H^\infty_v$ are bounded. To prove this we have to show that the weight v satisfies the following conditions

(1) $\sup_{z\in\mathbb{D}} \frac{v(z)^{\frac{1}{2}}}{1-|z|^2} < \infty$,

(2) $\inf_{k\in\mathbb{N}} \frac{(2^{-k}-2^{-2k-2})v(1-2^{-k-1})^{\frac{1}{2}}}{v(1-2^{-k})} > 0$.

Indeed,

$$\sup_{z\in\mathbb{D}} \frac{v(z)^{\frac{1}{2}}}{1-|z|^2} = \sup_{z\in\mathbb{D}} \frac{(1-|z|)^{\frac{n}{2}}}{(1-|z|)(1+|z|)} = \sup_{z\in\mathbb{D}} \frac{(1-|z|)^{\frac{n-2}{2}}}{(1+|z|)}$$

$\leq \sup_{z\in\mathbb{D}}(1-|z|)^{\frac{n-2}{2}} < \infty$ since $n \geq 2$. Moreover,

$$\begin{aligned} \inf_{k\in\mathbb{N}} \frac{(2^{-k}-2^{-2k-2})v(1-2^{-k-1})^{\frac{1}{2}}}{v(1-2^{-k})} &= \inf_{k\in\mathbb{N}}(2^{-k}-2^{-2k-2})2^{\frac{1}{2}(kn-n)} \\ &= \inf_{k\in\mathbb{N}} 2^{-\frac{n}{2}}(2^{k(\frac{n}{2}-1)}-2^{k(-2+\frac{n}{2})-2}) \\ &> 0 \text{ for every } n. \end{aligned}$$

(b) The weight $v(z) = 1-|z|$ satisfies neither (1) nor (2). We obtain

$$\sup_{z\in\mathbb{D}} \frac{v(z)^{\frac{1}{2}}}{1-|z|^2} = \sup_{z\in\mathbb{D}} \frac{1}{(1-|z|)^{\frac{1}{2}}(1+|z|)} \geq \frac{1}{2}\sup_{z\in\mathbb{D}} \frac{1}{(1-|z|)^{\frac{1}{2}}} = \infty.$$

Furthermore easy calculations show

$$\begin{aligned} \inf_{k\in\mathbb{N}} \frac{(2^{-k}-2^{-2k-2})v(1-2^{-k-1})^{\frac{1}{2}}}{v(1-2^{-k})} &= \inf_{k\in\mathbb{N}}(2^{-k}-2^{-2k-2})2^{\frac{1}{2}(k-1)} \\ &= \frac{1}{\sqrt{2}}\inf_{k\in\mathbb{N}}(2^{-\frac{k}{2}}-2^{\frac{-3}{2}k-2}) = 0. \end{aligned}$$

Hence, in this case there exists a composition operator $C_\phi : A_v^2 \to H_v^\infty$ that is not bounded. For example, the operator C_ϕ generated by the map $\phi(z) = z$ for every $z \in \mathbb{D}$ is not bounded, since

$$\sup_{z\in\mathbb{D}} \frac{v(z)}{(1-|\phi(z)|^2)v(\phi(z))^{\frac{1}{2}}} = \sup_{z\in\mathbb{D}} \frac{1}{(1+|z|)(1-|z|)^{\frac{1}{2}}} = \infty.$$

(c) The exponential weights $v(z) = e^{-\frac{1}{(1-|z|)^n}}$, $n > 0$, satisfy condition (1), but not condition (2). First, we get

$$\sup_{z\in\mathbb{D}} \frac{v(z)^{\frac{1}{2}}}{1-|z|^2} = \sup_{z\in\mathbb{D}} \frac{e^{-\frac{1}{2(1-|z|)^n}}}{1-|z|} < \infty.$$

It follows, that (1) is fulfilled. Now,

$$\inf_{k\in\mathbb{N}} \frac{(2^{-k}-2^{-2k-2})v(1-2^{-k-1})^{\frac{1}{2}}}{v(1-2^{-k})} = \inf_{k\in\mathbb{N}}(2^{-k}-2^{-2k-2})\frac{e^{-2^k n}}{e^{-2^k n}} = 0.$$

Thus, (2) is not satisfied. There must be a composition operator $C_\phi : A^2_v \to H^\infty_v$ that is not bounded.

(d) The logarithmic weights $v(z) = \frac{1}{(1-\log(1-|z|))^n}$, $n > 0$, neither satisfy (1) nor (2). Indeed,

$$\sup_{z\in\mathbb{D}} \frac{v(z)^{\frac{1}{2}}}{1-|z|^2} = \sup_{z\in\mathbb{D}} \frac{1}{(1-|z|^2)(1-\log(1-|z|))^{\frac{n}{2}}} = \infty$$

and

$$\inf_{k\in\mathbb{N}} \frac{(2^{-k}-2^{-2k-2})v(1-2^{-k-1})^{\frac{1}{2}}}{v(1-2^{-k})} = \inf_{k\in\mathbb{N}} (2^{-k}-2^{-2k-2}) \frac{(1-\log 2^{-k})^n}{(1-\log 2^{-k-1})^{\frac{n}{2}}}$$

$= 0$. With the criteria above we cannot decide, whether all composition operators $C_\phi : A^2_v \to H^\infty_v$ are bounded or not. But, again selecting $\phi(z) = z$ we see that

$$\sup_{z\in\mathbb{D}} \frac{v(z)}{(1-|\phi(z)|^2)v(\phi(z))^{\frac{1}{2}}} = \sup_{z\in\mathbb{D}} \frac{1}{(1-|z|^2)(1-\log(1-|z|))^{\frac{n}{2}}} = \infty.$$

Hence the corresponding composition operator is not bounded.

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Klaus Truemper: The Construction of Mathematics, Leibniz Company, 2017. ISBN:9-780-9663-5543-7, USD 14.95, 293+viii pp.

REVIEWED BY PETER LYNCH

Is mathematics discovered or created? The Platonic view is that mathematical ideas such as numbers and geometric forms have an a priori existence independent of humanity and gradually come to light as they are discovered through research and investigation. The contrary view is that mathematics is a creation of the human intellect. The question has been debated for centuries. The author of this book, Klaus Truemper, addresses this question and comes to a definite conclusion, strongly in favour of mathematics as a human creation, justifying the subtitle of his book, *The Human Mind's Greatest Achievement.*

Mathematics has emerged over thousands of years, in several civilizations. The first part of the book (Chapters 2 to 7) traces the development of the struggle for insight. Where do mathematical ideas come from? Are they somehow already present in the physical world, hidden and awaiting discovery by inquisitive minds? Or are they the products of human ingenuity? The second half of the book investigates this question from several perspectives, reaching a definite, although hardly definitive, conclusion.

Following the Introduction, Chapter 2 traces the development of numbers from the natural or counting numbers through rational to real and complex numbers. What is the origin of all these numbers? The general thrust is that each successive layer is a result of creation. The question then occurs to this reviewer: if we start with the natural numbers and the additional numbers already exist in some realm awaiting discovery, there seems to be only one way forward. But if we are free to create at will, is there not a multitude of possible extensions, not trivially equivalent and all internally consistent? Are there such alternative number systems, and is it perhaps

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that the standard number system is the one most suited to physical applications? This is not considered in any depth in the book.

Chapter 3 discusses mathematical notation. It is beyond doubt that well-chosen notation can facilitate advances while badly-chosen symbols can severely inhibit it. Truemper discusses the contrast between Newton's awkward fluents and fluxions and Leibniz's elegant notation. The former certainly held up progress in analysis in Britain for more than a century. In Chapter 4 (Infinity) Truemper shows how the application of mathematical arguments in physical contexts can produce nonsensical results. One example is Torricelli's Trumpet, which has finite volume but infinite surface area. Indeed, infinity frequently leads to paradoxical results when we try to apply it to physical systems. The Banach-Tarski Theorem is a particularly sharp example.

In Chapter 5, some classical problems (squaring the circle, etc.) are considered. The key argument here is that all these problems, outstanding for 2000 years, were resolved in the nineteenth century only when mathematics broke free from the natural world. Truemper writes (pg. 77): "mathematics is different from nature, does not need nature and should not be confused with nature."

Chapter 6 examines the role of proof in mathematics from Babylon and Ancient Greece to modern times, when Hilbert's dream of a rock-solid foundation for mathematics was shattered by Gödel's Incompleteness Theorems. The Zermelo-Fraenkel Axioms (ZF), with or without the Axiom of Choice and Continuum Hypothesis, form the basis of most mathematics today. Modern researchers have no real choice but to accept the potential inconsistency of these foundations, hoping — indeed expecting — that if an inconsistency is ever found it will be remedied by suitable modification of the underlying axioms.

The proof by Paul Cohen that the Axiom of Choice may be added to ZF without affecting (in)consistency shows how more than one mathematical system is possible. So, if we consider a single physical universe, at most one of these systems can describe it, implying that the other systems somehow have an existence independent of the physical world.

A chapter on computing machines is interesting but seems inessential to the dominant theme of the book. Still, I cannot resist the temptation to quote Leibniz, inventor of the binary system and of some mechanical calculators: It is beneath the dignity of excellent men to waste their time on calculations when any peasant could do the work just as accurately with the aid of a machine (perhaps mathematicians should avoid quoting this to their colleagues in computer science).

Chapter 8 opens the second part of the book asking in its title "Is Mathematics Created or Discovered?" Mathematical platonism posits that all of mathematics resides in a realm of abstract objects that is separate from the sensible world. This implies that mathematical truths are discovered, not invented. From 1800 onwards many mathematicians departed from this view, starting with Gauss who wrote that "number is purely a product of our mind". Kronecker's famous dictum is that "God made the integers; all else is the work of man". Yet, many twentieth century mathematicians supported the view that mathematical results are discovered.

The concept of "language games", devised by Ludwig Wittgenstein, is introduced in Chapter 9. It is argued that the technique can resolve many thorny philosophical problems. A language game is "a controlled setting of language use that brings a particular facet of a given philosophical problem into focus" and provides insight into the problem. To apply this technique many examples are required and these are drawn from the earlier chapters. In each instance, it is assumed that mathematics is discovered. Then contradictions arising during the course of the game indicate that this assumption must be abandoned.

Chapter 10 looks at several stages in the historical emergence of mathematics, using the language games framework. Before the concept of numbers, came one-one correspondences or bijections, for example between pebbles and sheep or fingers of the hand and children. Soon names were made up for groups of pebbles or fingers, leading to the counting numbers. All this could be regarded as creative. More species of numbers negatives, fractions, square roots followed as the need for them arose. Again, all could be described as invention rather than discovery. Other areas considered include logarithms, calculus, function theory, Lebesgue integration and the hierarchy of infinities. In each case, the assumption of discovery leads the author to bizarre and untenable consequences.

Analogies between mathematics and art are considered. Truemper gives a strange argument constructing a mathematical function that precisely specifies Michaelangelo's David: the function is defined in 3-dimensional Cartesian space and takes the value 1 for points within the statue and 0 for points outside. He then argues that, if the function existed before the statue was made, Michaelangelo must have discovered rather than created David. But the same argument holds if we replace *David* (created) by *Mount Everest* (discovered). I did not find this example illuminating. A comparison of the development of music and mathematics is more enlightening. Musical compositions are universally held to be creations, not discoveries. Why then should mathematical results be regarded as discoveries?

Truemper next addresses the proposition "Mathematical concepts are created, whereas the consequences provable from these concepts are discovered". I might paraphrase this: "Definitions are created, theorems/proofs are discovered". By analogies with sculpture, music and literature, the author shows that such a proposition leads to unreasonable conclusions. But I cannot easily accept such analogies as valid, or as vitiating the proposition. Indeed, this idea (definitions created, theorems discovered) has arguments and evidence in its favour (See "Invention or Discovery?" at `https://thatsmaths.com`).

The "unreasonable effectiveness" of mathematics in the physical sciences is examined in Chapter 11. This concept is often advanced in support of the idea that mathematics is discovered. Evidence is amassed in the book that, contrary to a widespread view, mathematics is actually quite ineffective in providing solutions to many problems in the modern world. There is selection bias: failed mathematical models tend to be ignored in favour of successful ones.

There are many natural processes for which we have not been able to construct useful mathematical models. Truemper considers these as evidence of the "reasonable ineffectiveness" of mathematics. However, our inability to solve the non-linear Navier-Stokes equations in closed form does not diminish the remarkable power of these equations to describe accurately a huge range of fluid phenomena. Truemper then compares the limitations of weather forecasting and economic prediction. This is to miss a crucial distinction: there are no Navier-Stokes equations for the economy!

Mathematics appears to be essential to civilization and is often considered to be an inherent part of nature. However, this view is disputed in Chapter 12, which gives the absence of any mathematics in an Amazonian tribe as an agument against the discovery of mathematics. I found this unconvincing and feel that the entire chapter is irrelevant and superfluous. In the last substantive chapter, recent advances in brain science are used to account for divergent opinions amongst experts concerning creation/discovery. In the past, Gauss and Cantor argued for creation, while Frege and Gödel supported discovery. It is claimed that differences arise from "embodiment of different learning experiences". Modern neuroscience is hardly needed to see that scholars with different backgrounds, knowledge and experience may reach different conclusions, and appeal to recent research does not really provide any additional insight into the creation/discovery dilemma. As with the previous chapter, I feel that this one could have been omitted without loss.

Braoadly speaking, mathematics involves the study of quantity (number), structure (algebra), space (geometry) and change (analysis). This book concentrates mostly on the first and last categories. The concept of symmetry is not mentioned. It would be interesting to examine the concept of symmetry in the context of creation/discovery.

The main text of the book covers 207 pages and is supplemented by 67 pages of endnotes containing much fascinating background material. A good bibliography follows this.

In summary, I found the book well-written with generally clear and convincing arguments (despite the counterexamples mentioned above). If there is a general criticism it is that the author has tried too hard to support his main conclusion, giving more weight to arguments supporting it and less to those that might refute it. Notwithstanding this, the book is an interesting, enjoyable and thought-provoking read. Of course, it cannot provide the final word on the central question, which I feel has the characteristics of a Gödelian problem, irresolvable with our current tools of thought.

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Elena Shchepakina, Vladimir Sobolev and Michael P. Mortell: Singular Perturbations. Introduction to System Order Reduction Methods with Applications, Springer, 2014.
ISBN:978-3-319-09569-1, USD 59.99, 212 pp.

REVIEWED BY DMITRII RACHINSKII

This book will appeal to undergraduate and graduate students, and researchers in the area of applications of singular perturbations in various fields such as chemical kinetics, combustion, control theory and nonlinear dynamics. It combines analytic singular perturbation methods with the geometric approach based on analysis on integral manifolds. The authors are known experts in this field. The reduction to a low dimensional slow integral manifold underpins the order reduction technique presented in the book.

The book is specially constructed to allow a non-expert to read it from beginning to end. In particular, it could serve as the basis for an excellent graduate course on singular perturbations. It begins very simply and is self-contained to such an extent that it is accessible to upper class undergraduate student in mathematics or physics. New topics are introduced by a smooth transition from previous topics. The presentation proceeds by dealing with progressively more difficult problems, where the theory and the solution techniques are laid out. These techniques are illustrated with a large number of examples drawn from widely diverse areas including reaction kinetics of organometallic compounds, combustion problems, population models, control of gyroscopic and robotic systems, and laser dynamics. The examples are completely worked out and become more sophisticated and challenging as the text moves on, until finally they are at the research level where many of the details of calculations in published papers are given. In the earlier chapters there are many simpler examples enabling the reader to get a good grasp of underlying ideas. This exposition style is comparable to that of another major book in the subject area by P. Kokotovic, H. K. Khalil and J. O'Reilly [1].

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In each chapter, the basics are introduced, some theory is presented in a relatively simple way, and then there are examples showing how to apply the method.

Chapters 1 and 2 present an introduction to perturbation methods and integral manifolds.

Chapter 3 gives an overview of examples of increasing complexity with up to two slow variables and up to two fast variables.

Chapter 4 presents a method of finding integral manifolds in parametric and implicit form.

Chapter 5 lays out a technique of scaling transformations and gauge functions, which regularizes singular singularly perturbed problems in order to approximate integral manifolds.

In Chapter 6, the techniques developed so far are applied to order reduction problems.

The full strength of the methodology presented in the book is seen in Chapters 7 and 8 in which the usual hypotheses of integral manifold theory such as the conditions of Tikhonov's theorem are violated. These chapters should be of particular interest to those researchers who need perturbation methods for solving systems of strongly nonlinear equations. The exposition includes mathematical treatment of fascinating slow-fast phenomena represented by *cascades of canard* solutions and integral surfaces with variable stability called *black swans*.

References

[1] P. Kokotovic, H. K. Khalil and J. O'Reilly. *Singular Perturbation Methods in Control: Analysis and Design*. Academic Press, 1986.

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PROBLEMS

IAN SHORT

Problems

The first problem this issue was contributed by Peter Danchev of Plovdiv University, Bulgaria.

Problem 79.1. Suppose that k and n are positive integers with $1 \leqslant k \leqslant n$. Find the largest integer m such that the binomial coefficient $\binom{2^n}{k}$ is divisible by 2^m.

The next problem was suggested by Prithwijit De of the Homi Bhabha Centre for Science Education, Mumbai, India.

Problem 79.2. Let f be a function that is continuous on the interval $[0, \pi/2]$ and satisfies $f(x) + f(\pi/2 - x) = 1$ for each x in $[0, \pi/2]$. Evaluate the integral

$$\int_0^{\pi/2} \frac{f(x)}{(\sin^3 x + \cos^3 x)^2}\, dx.$$

We finish with an elegant identity involving sums of powers of integers. It would be pleasing to see a simple geometric proof of this classic identity, but perhaps that is asking too much.

Problem 79.3. Prove that, for any positive integer n,

$$(1^5 + \cdots + n^5) + (1^7 + \cdots + n^7) = 2(1 + \cdots + n)^4.$$

Solutions

Here are solutions to the problems from *Bulletin* Number 77. The first problem was solved by the North Kildare Mathematics Problem Club as well as the proposer, Finbarr Holland of University College Cork. The two solutions were similar in spirit; we give the solution of the problem club.

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Problem 77.1. Suppose that $f : [0,1] \to \mathbb{R}$ is a convex function and $\int_0^1 f(t)\,dt = 0$. Prove that

$$\int_0^1 t(1-t)f(t)\,dt \leqslant 0,$$

with equality if and only if $f(t) = a(2t-1)$ for some real number a.

Solution 77.1. Let

$$\alpha = \tfrac{1}{2}(f(1)-f(0)), \quad g(t) = \alpha(1-2t), \quad \text{and} \quad h(t) = f(t)+g(t).$$

Then h is convex, $h(0) = h(1)$, and

$$\int_0^1 h(t)\,dt = 0.$$

Let $k(t) = h(t) + h(1-t)$. Then k is convex, $k(t) = k(1-t)$, and

$$\int_0^{1/2} k(t)\,dt = \int_0^1 k(t)\,dt = 0. \tag{1}$$

Also,

$$\int_0^{1/2} t(1-t)k(t)\,dt = \int_0^1 t(1-t)f(t)\,dt.$$

It cannot be that $k(0) < 0$, because if that were so then $k(1) < 0$, and hence (by convexity) $k(t) < 0$ on $[0,1]$, which contradicts (1). Reasoning similarly, we see that if $k(0) = 0$, then $k(t) = 0$ for all t, and

$$\int_0^{1/2} t(1-t)k(t)\,dt = 0.$$

The remaining possibility is that $k(0) > 0$. In this case, since k is convex with integral zero, there must be exactly two zeros of k between 0 and 1, and by symmetry they are at points β and $1-\beta$ for some $\beta \in (0,\frac{1}{2})$. Moreover, k is strictly decreasing on the interval $(0,\frac{1}{2})$ and $t(1-t)$ is positive and increasing on $(0,\frac{1}{2})$. Thus one can check that the inequality $t(1-t)k(t) < \beta(1-\beta)k(t)$ is satisfied on both intervals $(0,\beta)$ and $(\beta,1/2)$. Therefore

$$\int_0^{1/2} t(1-t)k(t)dt < \beta(1-\beta)\int_0^{1/2} k(t)dt = 0.$$

So the desired inequality holds, with equality only in the case when $k(t)$ is identically 0, that is, when $h(t) = -h(1-t)$. But in that case $h(0) = -h(1) = -h(0)$, so $h(0) = h(1) = 0$, and since h is convex with integral zero, we see that h is identically zero. Hence $f(t) = \alpha(2t-1)$. $\square$

Finbarr points out that if f is twice continuously differentiable and $f''(t) \geq 0$ for all t, then there is a much shorter solution, which follows immediately from the identity below, which can be proved by integrating the left-hand integral by parts a couple of times:

$$\int_0^1 t^2(1-t)^2 f''(t)\,dt = 2\int_0^1 (1-6t+6t^2)f(t)\,dt.$$

The second problem from *Bulletin* Number 77 was solved by Henry Ricardo of the Westchester Area Math Circle, New York, USA, and the North Kildare Mathematics Problem Club. The solution was also known to the proposer. Many have pointed out that the problem is well known. Henry notes that the problem is usually ascribed to Pierre Rémond de Montmort (1678–1719), and that *The Problem of Coincidences* by Lajos Takács (*Archive for History of Exact Sciences*, **21**, 1980) is an excellent survey on this problem and its generalisations. We give Henry's solution here, which coincides with that of the problem club, and which apparently is essentially due to Euler.

*Problem 77.2***.** Each member of a group of n people writes his or her name on a slip of paper, and places the slip in a hat. One by one the members of the group then draw a slip from the hat, without looking. What is the probability that they all end up with a different person's name?

Solution 77.2. The problem is equivalent to counting the number D_n of permutations P of $\{1,\dots,n\}$ that satisfy $P(k) \neq k$ for $1 \leqslant k \leqslant n$. Let us call such a permutation P a *derangement*. We use the notation $(j_1, j_2, \dots, j_n)$ to represent a permutation, where j_k denotes the image of k.

For any derangement $(j_1, j_2, \dots, j_n)$, we have $j_n \neq n$. Let $j_n = k$, where $k \in \{1, 2, \dots, n-1\}$. Now we split the derangements on n elements into two cases.

Case 1: $j_k = n$ (so k and n map to each other). By removing elements k and n from the permutation, we have a derangement on $n-2$ elements; and so, for fixed k, there are D_{n-2} derangements in this case.

Case 2: $j_k \neq n$. Swap the values of j_k and j_n, so that we have a new permutation with $j_k = k$ and $j_n \neq n$. By removing element k, we have a derangement on $n-1$ elements; and so, for fixed k, there are D_{n-1} derangements.

Thus, with $n-1$ choices for k, we have, for $n \geqslant 2$,

$$D_n = (n-1)\left(D_{n-1} + D_{n-2}\right).$$

The probability P_n of a derangement is the number of derangements divided by the number of all possible permutations of n objects:

$$\begin{aligned} P_n &= \frac{D_n}{n!} = \frac{(n-1)}{n!}\left(D_{n-1} + D_{n-2}\right) \\ &= (n-1)\left(\frac{1}{n}\cdot\frac{D_{n-1}}{(n-1)!} + \frac{1}{n(n-1)}\cdot\frac{D_{n-2}}{(n-2)!}\right) \\ &= \left(1-\frac{1}{n}\right)P_{n-1} + \frac{1}{n}P_{n-2} \\ &= P_{n-1} - \frac{1}{n}\left(P_{n-1} - P_{n-2}\right), \end{aligned}$$

or $P_n - P_{n-1} = -(1/n)(P_{n-1} - P_{n-2})$, with $P_1 = 0$ and $P_2 = 1/2$. It follows that

$$P_n - P_{n-1} = \frac{(-1)^n}{n!},$$

so

$$P_n = P_1 + \sum_{k=2}^{n}(P_k - P_{k-1}) = \sum_{k=2}^{n}\frac{(-1)^k}{k!}. \qquad \square$$

The third problem was incorrectly labelled 76.3, rather than 77.3, in issue 77. It was solved by Dixon Jones of the University of Alaska Fairbanks, USA, Niall Ryan of the University of Limerick, and the North Kildare Mathematics Problem Club. We give the solution of the problem club.

Problem 77.3. Evaluate

$$1 + \cfrac{1^2}{1 + \cfrac{2^2}{1 + \cfrac{3^2}{1 + \cdots}}}.$$

Solution 77.3. Consider the identity

$$\frac{1}{x} - \frac{1}{y} = \cfrac{1}{x + \cfrac{x^2}{y-x}}, \quad \text{where } x, y \neq 0 \text{ and } x \neq y. \qquad (2)$$

Applying (2) with $x = n$ and $y = n+1$, where $n \geqslant 1$, we obtain

$$\frac{1}{n} - \frac{1}{n+1} = \cfrac{1}{n + \cfrac{n^2}{1}}.$$

Applying (2) with $x = n-1$ and $y = n+n^2$, where $n \geqslant 2$, we obtain

$$\frac{1}{n-1} - \frac{1}{n} + \frac{1}{n+1} = \cfrac{1}{(n-1) + \cfrac{(n-1)^2}{1 + \cfrac{n^2}{1}}}.$$

Continuing in this manner, we obtain

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^n \frac{1}{n+1} = \cfrac{1}{1 + \cfrac{1^2}{1 + \cfrac{2^2}{1 + \cdots + \cfrac{n^2}{1}}}}.$$

Then, taking limits, we see that

$$\log 2 = \cfrac{1}{1 + \cfrac{1^2}{1 + \cfrac{2^2}{1 + \cdots}}},$$

so the desired continued fraction is equal to $1/\log 2$. □

The problem club point out that various generalisations of this continued fractions formula appear in the literature. One such generalisation is

$$\log(1+x) = \cfrac{x}{1 + \cfrac{1^2 x}{2 - x + \cfrac{2^2 x}{3 - 2x + \cfrac{3^2 x}{4 - 3x + \cdots}}}}$$

(and there are more). The problem club's method comes from *Higher Algebra* by Hall and Knight, and seems to be due to Frobenius and Stickelberger (*J. Reine Angew. Math.*, **88**, 1880) originally.

A similar idea to that given in the proof can be used to establish that

$$\frac{\pi}{4}=\cfrac{1}{1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cdots}}}}.$$

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer Latex). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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