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Real Lie Algebras with Equal Characters

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Abstract. We recall Cartan's definition of characters of real forms of complex simple Lie algebras, based on Cartan decomposition. For a given complex simple Lie algebra, its real forms are uniquely determined by their characters in almost all cases. We work out the exceptions where non-isomorphic real forms have the same character.

1. Introduction

Let $\mathfrak{g}$ be a real form of a complex simple Lie algebra L. Let $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ be a Cartan decomposition, namely $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}$. In É. Cartan's classification of real simple Lie algebras, he defines the *character* of $\mathfrak{g}$ as

$$\text{character}(\mathfrak{g}) = \dim \mathfrak{p} - \dim \mathfrak{k}.$$

He observes that non-isomorphic real forms of exceptional Lie algebras have distinct characters [1, p.263-265], and uses them to denote these exceptional real forms. For example $\mathfrak{e}_{6(\delta)}$ denotes the real form of E_6 with character δ [2, p.518]. For the classical Lie algebras, Helgason notes that non-isomorphic real forms with equal character occur only in types A and D, and provides two examples [2, p.517]

$$\begin{array}{ll} \text{(a)} & \mathfrak{su}^*(14), \mathfrak{su}(9,5) \subset \mathfrak{sl}(14,\mathbb{C}), \\ \text{(b)} & \mathfrak{so}^*(18), \mathfrak{so}(12,6) \subset \mathfrak{so}(18,\mathbb{C}). \end{array} \tag{1}$$

The following theorem determines all non-isomorphic real forms with equal character.

Theorem 1.1. *All the cases of real forms* $\mathfrak{g}, \mathfrak{g}' \subset L$ *such that* $\mathfrak{g} \not\cong \mathfrak{g}'$ *and* $\mathfrak{g}, \mathfrak{g}'$ *have the same character are given as follows:*
(a) $\mathfrak{su}(2r^2+r-1, 2r^2-r-1), \mathfrak{su}^*(4r^2-2) \subset \mathfrak{sl}(4r^2-2,\mathbb{C})$, *where* $1 < r \in \mathbb{N}$*;*
(b) $\mathfrak{so}(r^2+r, r^2-r), \mathfrak{so}^*(2r^2) \subset \mathfrak{so}(2r^2,\mathbb{C})$, *where* $2 < r \in \mathbb{N}$.

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We see that $\mathfrak{su}^*(14)$ and $\mathfrak{su}(9,5)$ of (1)(a) are obtained by setting $r = 2$ in Theorem 1.1(a), while $\mathfrak{so}^*(18)$ and $\mathfrak{so}(12,6)$ of (1)(b) are obtained by setting $r = 3$ in Theorem 1.1(b).

If $\mathfrak{g}$ and $\mathfrak{g}'$ are real forms of L, then clearly $\dim \mathfrak{g} = \dim \mathfrak{g}'$. Hence the condition character$(\mathfrak{g})$ = character$(\mathfrak{g}')$ is equivalent to $\dim \mathfrak{k} = \dim \mathfrak{k}'$. It is known that $\mathfrak{g}$ is determined by $\mathfrak{k}$ and L [2, Ch.X-6,Thm.6.2]; and Theorem 1.1 says that $\mathfrak{g}$ is in fact determined by $\dim \mathfrak{k}$ and L except for the indicated cases.

2. Proof of Theorem 1.1

We now prove Theorem 1.1. We study $\mathfrak{sl}(n,\mathbb{C})$ in the proof of Theorem 1.1(a), and study $\mathfrak{so}(2n,\mathbb{C})$ in the proof of Theorem 1.1(b).

Proof of Theorem 1.1(a):

The Lie algebra $\mathfrak{sl}(n,\mathbb{C})$ has three classes of real forms $\mathfrak{g}$, whose maximal compact subalgebras $\mathfrak{k}$ are indicated in (2) (see for instance [2, p.518]). In (2)(a),

$$\begin{aligned}\dim \mathfrak{k} &= \dim \mathfrak{u}(p) + \dim \mathfrak{u}(n-p) - 1\\ &= p^2 + (n-p)^2 - 1 = 2p^2 - 2np + n^2 - 1.\end{aligned}$$

	$\mathfrak{g}$	$\mathfrak{k}$	$\dim \mathfrak{k}$
(a)	$\mathfrak{su}(p, n-p)$	$\mathfrak{s}(\mathfrak{u}(p) + \mathfrak{u}(n-p))$	$2p^2 - 2np + n^2 - 1$
(b)	$\mathfrak{su}^*(n)$, n even	$\mathfrak{sp}(\frac{n}{2}, \mathbb{R})$	$\frac{1}{2}(n^2+n)$
(c)	$\mathfrak{sl}(n,\mathbb{R})$	$\mathfrak{so}(n)$	$\frac{1}{2}(n^2-n)$

(2)

If $\mathfrak{g}$ is a split form of L (i.e. $\mathfrak{g}$ has a Cartan subalgebra contained in $\mathfrak{p}$; also known as a normal form), then its character is strictly larger than that of other real forms of L [2, p.517]. Therefore, we can ignore the split form $\mathfrak{sl}(n,\mathbb{R})$, and consider only (2)(a,b). We recall the elementary fact

$$ap^2 + bp + c = 0 \implies p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \qquad (3)$$

It is easier to compare $\dim \mathfrak{k}$ instead of the characters. Suppose that (2)(a) and (2)(b) have equal $\dim \mathfrak{k}$. Then

$$0 = (2p^2 - 2np + n^2 - 1) - \frac{1}{2}(n^2 + n) = 2p^2 - 2np + \frac{1}{2}(n^2 - n - 2). \quad (4)$$

By (3) and (4),

$$p = \frac{2n \pm \sqrt{(-2n)^2 - 4(n^2 - n - 2)}}{4} = \frac{1}{2}(n \pm \sqrt{n+2}). \qquad (5)$$

This implies that $n+2$ is a perfect square. Furthermore since n is even in (2)(b), condition (5) also says that $\sqrt{n+2}$ is even, namely $n+2=(2r)^2$ for some $r \in \mathbb{N}$. Then (5) becomes $p = 2r^2 \pm r - 1$. For $r = 1$, (2)(a,b) gives $\mathfrak{su}(2) \cong \mathfrak{su}^*(2)$. Hence we assume that $r > 1$. This leads to the pairs of real forms in Theorem 1.1(a).

It remains to compare (2)(a) with itself for different values of p. If $\mathfrak{su}(p, n-p)$ and $\mathfrak{su}(q, n-q)$ have equal $\dim \mathfrak{k}$, then

$$0 = (2p^2-2np+n^2-1)-(2q^2-2nq+n^2-1) = 2(p^2-np+(nq-q^2)).$$

By (3),

$$p = \frac{n \pm \sqrt{(-n)^2 - 4(nq-q^2)}}{2} = \frac{1}{2}(n \pm (n-2q)) \in \{q, n-q\}.$$

This implies that $\mathfrak{su}(p, n-p) \cong \mathfrak{su}(q, n-q)$. We conclude that Theorem 1.1(a) gives all the cases of non-isomorphic real forms of $\mathfrak{sl}(n, \mathbb{C})$ with equal character. □

Proof of Theorem 1.1(b):

The Lie algebra $L = \mathfrak{so}(2n, \mathbb{C})$ has two classes of real forms $\mathfrak{g}$, with $\mathfrak{k}$ and $\dim \mathfrak{k}$ indicated in (6).

	$\mathfrak{g}$	$\mathfrak{k}$	$\dim \mathfrak{k}$
(a)	$\mathfrak{so}(p, 2n-p)$	$\mathfrak{so}(p) + \mathfrak{so}(2n-p)$	$p^2 - 2np + 2n^2 - n$
(b)	$\mathfrak{so}^*(2n)$	$\mathfrak{u}(n)$	n^2

(6)

Suppose that (6)(a) and (6)(b) have equal $\dim \mathfrak{k}$. Then

$$p^2 - 2np + n^2 - n = 0.$$

By (3),

$$p = \frac{2n \pm \sqrt{(-2n)^2 - 4(n^2-n)}}{2} = n \pm \sqrt{n}.$$

It implies that n is a perfect square, say $n = r^2$ for some $r \in \mathbb{N}$. Then $p = r^2 \pm r$. For $r = 1$, (6)(a,b) gives $\mathfrak{so}(2) \cong \mathfrak{so}^*(2)$. Similarly for $r = 2$, it gives $\mathfrak{so}(6, 2) \cong \mathfrak{so}^*(8)$. Hence we assume that $r > 2$. This leads to the pairs of real forms in Theorem 1.1(b).

We also compare (6)(a) with itself for different values of p. If $\mathfrak{so}(p, 2n-p)$ and $\mathfrak{so}(q, 2n-q)$ have equal $\dim \mathfrak{k}$, then

$$0 = (p^2-2np+2n^2-n)-(q^2-2nq+2n^2-n) = p^2-2np+(2nq-q^2).$$

By (3),

$$p = \frac{2n \pm \sqrt{(-2n)^2 - 4(2nq-q^2)}}{2} = n \pm (n-q) \in \{2n-q, q\}.$$

This implies that $\mathfrak{so}(p, 2n-p) \cong \mathfrak{so}(q, 2n-q)$. We conclude that Theorem 1.1(b) gives all the cases of non-isomorphic real forms of $\mathfrak{so}(2n, \mathbb{C})$ with equal character. $\square$

Since non-isomorphic real forms with equal character may occur only in types A and D [2, p.517], this completes the proof of Theorem 1.1.

References

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