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Finite Differences and Terminating Hypergeometric Series

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Abstract. By means of finite difference method, new proofs are presented for the binomial convolution formulae of Abel, Chu–Vandermonde and Hagen–Rothe. The same approach is illustrated also for the summation theorems of classical hypergeometric series due to Dixon, Pfaff–Saalschütz, Stanton and Minton (1970).

Finite differences are very useful in numerical mathematics. In this paper, we shall illustrate how to employ them to evaluate binomial sums and terminating hypergeometric series. The approach consists of the following three steps:

- First for a given a binomial identity, identifying a parameter x as a variable and expressing the binomial sum in terms of finite differences.
- Then evaluating the binomial sum for particular values of x with the help of properties of finite differences.
- Finally confirming the binomial identity via the fundamental theorem of algebra, i.e., two polynomials of degrees $\leq n$ are identical if they have the same values at $n+1$ distinct points.

New proofs will be presented for the binomial convolution formulae of Abel, Chu–Vandermonde and Hagen–Rothe. As further examples of classical hypergeometric series, we examine also Pfaff–Saalschütz summation theorem, Dixon's formula, Stanton's extension [23] of

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Andrews' ${}_5F_4$–sum [1] and Minton's seminal theorem [18] on the series with integer parameter differences.

Following Bailey [2, §2.1], we shall use, the notation below for the classical hypergeometric series

$$ {}_{1+p}F_p\left[\begin{matrix} a_0, a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_p \end{matrix} \,\Big|\, z\right] = \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k (a_2)_k \cdots (a_p)_k}{k!(b_1)_k (b_2)_k \cdots (b_p)_k} \, z^k $$

where the shifted factorial is defined by

$$ (\lambda)_0 = 1 \quad \text{and} \quad (\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1) \quad \text{for} \quad n = 1, 2, \cdots $$

with its multi–parameter form being abbreviated as

$$ \left[\begin{matrix} \alpha, \beta, \cdots, \gamma \\ A, B, \cdots, C \end{matrix}\right]_n = \frac{(\alpha)_n (\beta)_n \cdots (\gamma)_n}{(A)_n (B)_n \cdots (C)_n}. $$

Throughout the paper, our attention will focus only on the terminating series, i.e., one of the numerator parameters $\{a_i\}_{i=0}^p$ results in a nonpositive integer.

1. Finite Differences

The finite difference operator Δ with unit increment is defined by

$$ \Delta^0 f(x) := f(x) \quad \text{and} \quad \Delta f(x) := f(1+x) - f(x). $$

For a natural number n, the differences of order n is given by

$$ \Delta^n f(x) := \Delta\big\{\Delta^{n-1} f(x)\big\} $$

which is expressed by the following Newton–Gregory formula (cf. [21, Chapter 1])

$$ \Delta^n f(x) = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} f(x+k). \tag{1} $$

In particular, when $p_m(x)$ is a polynomial of degree $m \leq n$ with the leading coefficient c_m, the following properties are quite useful:

$$ \Delta^n p_m(x) = n! c_n \chi(m = n) \quad \text{and} \quad \Delta^n \frac{p_m(x)}{x - \lambda} = (-1)^n \frac{n! p_m(\lambda)}{(x-\lambda)_{n+1}} $$

where χ stands for the usual logical function with $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$.

The former equality is well–known. The latter can be justified easily as follows. First when $p_m(x) \equiv 1$, it is trivial to check it by the induction principle. Observing that $\frac{p_m(x)-p_m(\lambda)}{x-\lambda}$ is a polynomial of degree $m-1$ with the nth differences equal to zero, we have immediately

$$\Delta^n \frac{p_m(x)}{x-\lambda} = \Delta^n \frac{p_m(\lambda)}{x-\lambda} = (-1)^n \frac{n!p_m(\lambda)}{(x-\lambda)_{n+1}}.$$

In addition, we shall use $\Delta_c^n f(x) = \Delta^n f(x)_{|x=c}$ for the differences starting at $x = c$.

2. Chu–Vandermonde Convolution

As a warm-up, we illustrate the method first by showing the Chu–Vandermonde convolution formula (cf. Bailey [2, §1.3]

$${}_2F_1\left[\begin{matrix}-n, \ x \\ y\end{matrix}\,\middle|\, 1\right] = \frac{(y-x)_n}{(y)_n} \tag{2}$$

which is often stated equivalently as the following binomial identity:

$$\sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k} = \binom{x+y}{n}.$$

Rewrite (2) equivalently as

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(x)_k}{(y)_k} = \frac{(y-x)_n}{(y)_n}. \tag{3}$$

Denote by $P(x)$ the above binomial sum, which is a polynomial of degree n in x. Keeping in mind of the relation

$$\frac{(y+m)_k}{(y)_k} = \frac{(y+k)_m}{(y)_m}$$

we can reformulate $P(x)$ at $x = y + m$ as

$$\begin{aligned} P(y+m) &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(y+m)_k}{(y)_k} \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(y+k)_m}{(y)_m}. \end{aligned}$$

Observing that the binomial sum just displayed results in the nth differences of the polynomial $\frac{(x+y)_m}{(y)_m}$ with degree m, we deduce that

$$P(y+m) = (-1)^n \frac{n!}{(y)_n} \chi(m=n) \quad \text{for} \quad 0 \le m \le n.$$

Therefore the polynomial $P(x)$ has the same values at the $n+1$ distinct points $\{y+m\}_{m=0}^n$ as $(y-x)_n/(y)_n$ with the same degree n. According to the fundamental theorem of algebra, they are identical. This proves (3) and so the Chu–Vandermonde identity (2).

3. Pfaff–Saalschütz Summation Theorem

In classical hypergeometric series, the Pfaff–Saalschütz summation theorem is fundamental (cf. Bailey [2, §2.2] and Chu [4])

$$ {}_3F_2\left[\begin{matrix} -n, & x, & y \\ 1+z, & x+y-z-n \end{matrix} \,\middle|\, 1\right] = \frac{(1+z-x)_n(1+z-y)_n}{(1+z)_n(1+z-x-y)_n} \tag{4}$$

which can be reproduced, as the following binomial sum (cf. Gould [14, Entry 17.3; P.71]):

$$\sum_{k=0}^n \binom{n}{k} \frac{\binom{x}{k}\binom{y}{k}}{\binom{x+y+z+n}{k}\binom{z+k}{k}} = \frac{\binom{x+z+n}{n}\binom{y+z+n}{n}}{\binom{x+y+z+n}{n}\binom{z+n}{n}}.$$

Firstly, rewrite the equality (4) equivalently as

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(x)_k(y)_k}{(1+z)_k(x+y-z-n)_k} = \frac{(1+z-x)_n(1+z-y)_n}{(1+z)_n(1+z-x-y)_n}. \tag{5}$$

For the polynomial given by $(1+z-x-y)_n = (-1)^n(x+y-z-n)_n$, if multiplying by this across the last equation, we would get an identity between two polynomials of degree n in x. In order to prove it, it suffices to check the equality (5) for $n+1$ distinct values of x.

Let $R(x)$ be the sum displayed in (5). In view of the relation

$$\frac{(z+m)_k(y)_k}{(1+z)_k(y+m-n)_k} = \frac{(1+z+k)_{m-1}(1-y-k)_{n-m}}{(1+z)_{m-1}(1-y)_{n-m}}$$

we have the following expression

$$\begin{aligned} R(z+m) &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(z+m)_k(y)_k}{(1+z)_k(y+m-n)_k} \\ &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(1+z+k)_{m-1}(1-y-k)_{n-m}}{(1+z)_{m-1}(1-y)_{n-m}} \end{aligned}$$

which vanishes for $m = 1, 2, \cdots, n$ because it results in the nth differences of the following polynomial $(1+x+z)_{m-1}(1-x-y)_{n-m}$ of degree $n-1$.

Taking into account of $R(0) = 1$ besides, we conclude that equality (5) is valid for the $n+1$ distinct values $\{0\} \cup \{z+m\}_{m=0}^{n-1}$ of x. This confirms (5) and so the Pfaff–Saalschütz summation formula (4).

It should be pointed out that the proof presented here resembles much the one found recently by Gessel [13], but with the difference that our proof is based on the polynomial $R(x)$ of degree n while Gessel's on another polynomial of degree $2n$ together with its symmetric property.

4. Convolution Formulae of Hagen–Rothe

More general convolutions of binomial coefficients are evaluated by Hagen and Rothe (cf. Comtet [11, §3.1] and Mohanty [19, §4.2])

$$\sum_{k=0}^{n}\frac{x}{x+ky}\binom{x+ky}{k}\binom{z-ky}{n-k} = \binom{x+z}{n}, \tag{6a}$$

$$\sum_{k=0}^{n}\frac{x}{x+ky}\binom{x+ky}{k}\frac{z-ny}{z-ky}\binom{z-ky}{n-k} = \frac{x+z-ny}{x+z}\binom{x+z}{n}. \tag{6b}$$

There are many different proofs. Some of them can be found in [8, 10, 15, 24]. Denote by $\mathcal{P}(x)$ the binomial sum in (6a), which is obviously a polynomial of degree n. Its value at $x = m - z$ can be manipulated as

$$\begin{aligned}\mathcal{P}(m-z) &= \sum_{k=0}^{n} \frac{m-z}{m-z+ky}\binom{m-z+ky}{k}\binom{z-ky}{n-k}\\ &=\frac{m-z}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(1+m-k+ky-z)_{k-1}(ky-z)_{n-k}\\ &=\frac{m-z}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(ky-z)_{m}(1+m-k+ky-z)_{n-m-1}.\end{aligned}$$

For $0 \leq m < n$, we assert that $\mathcal{P}(m-z)$ vanishes because it results in the nth differences of the following polynomial

$$(xy-z)_m(1+m-x-z+xy)_{n-m-1} \quad \text{of degree} \quad n-1.$$

When $m = n$, we can evaluate

$$\begin{aligned}\mathcal{P}(n-z) &= \frac{n-z}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{(ky-z)_n}{n-k-z+ky}\\ &= \frac{n-z}{n!}\Delta_0^n\frac{(xy-z)_n}{n-x-z+xy}\\ &= \frac{n-z}{n!(y-1)}\Delta_0^n\frac{(\frac{z-ny}{y-1})_n}{x-\frac{z-n}{y-1}}\\ &= (-1)^n\frac{n-z}{y-1}\frac{(\frac{z-ny}{y-1})_n}{(-\frac{z-n}{y-1})_{n+1}} = 1.\end{aligned}$$

Therefore, $\mathcal{P}(x)$ is a polynomial with the same values at the $n+1$ distinct points $\{m-z\}_{m=0}^{n}$ as another polynomial $\binom{x+z}{n}$ of degree n. This shows that both polynomials are identical which proves (6a).

Analogously, let $\mathcal{Q}(x)$ be the binomial sum in (6b), which is again a polynomial of degree n. Its value at $x = m - z$ reads as

$$\mathcal{Q}(m-z) = \sum_{k=0}^{n} \frac{m-z}{m-z+ky}\binom{m-z+ky}{k}\frac{z-ny}{z-ky}\binom{z-ky}{n-k}$$
$$=\frac{(z-m)(z-ny)}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(1+m-k+ky-z)_{k-1}(1+ky-z)_{n-k-1}$$
$$=\frac{(z-m)(z-ny)}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(1+ky-z)_{m-1}(1+m-k+ky-z)_{n-m-1}.$$

For $1 \leq m < n$, it is clear that $\mathcal{Q}(m-z)$ vanishes because it results in the nth differences of the following polynomial

$$(1-z+xy)_{m-1}(1+m-x-z+xy)_{n-m-1} \quad \text{of degree} \quad n-2.$$

In addition, we can evaluate

$$\begin{aligned}
\mathcal{Q}(0-z) &= \frac{z(z-ny)}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{(1-k-z+ky)_{n-1}}{ky-z}. \\
&= \frac{z(z-ny)}{n!}\Delta_0^n\frac{(1-x+xy-z)_{n-1}}{xy-z} \\
&= \frac{z(z-ny)}{n!y}\Delta^n\frac{(1-z/y)_{n-1}}{x-z/y}\Big|_{x=0} \\
&= (-1)^n\frac{z(z-ny)}{y}\frac{(1-z/y)_{n-1}}{(-z/y)_{n+1}} = (-1)^n y
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{Q}(n-z) &= \frac{(z-n)(z-ny)}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{(1+ky-z)_{n-1}}{n-k+ky-z} \\
&= \frac{(z-n)(z-ny)}{n!}\Delta_0^n\frac{(1+xy-z)_{n-1}}{n-x+xy-z} \\
&= \frac{(z-n)(z-ny)}{n!(y-1)}\Delta^n\frac{(1+\frac{z-ny}{y-1})_{n-1}}{x-\frac{z-n}{y-1}}\Big|_{x=0} \\
&= (-1)^n\frac{(z-n)(z-ny)}{y-1}\frac{(1+\frac{z-ny}{y-1})_{n-1}}{(-\frac{z-n}{y-1})_{n+1}} = 1-y.
\end{aligned}$$

Therefore, $\mathcal{Q}(x)$ is a polynomial with the same values at the $n+1$ distinct points $\{m-z\}_{m=0}^{n}$ as another polynomial $\frac{x+z-ny}{x+z}\binom{x+z}{n}$ of degree n. This implies that both polynomials are identical which proves (6b).

5. Dixon's Terminating Summation Formula

One of the terminating forms of Dixon's summation theorem is (cf. Bailey [3])

$$ {}_3F_2\left[\begin{matrix} -n, & x, & y \\ 1-x-n, & 1-y-n \end{matrix}\,\middle|\, 1\right] = \frac{(1+\ell)_\ell(x+y+\ell)_\ell}{(x+\ell)_\ell(y+\ell)_\ell}\chi(n=2\ell). \quad (7) $$

A well–known particular case of it is the following alternating sum of cubic binomial coefficients (cf. Gould [14, Entry 6.6; P.51]):

$$ \sum_{k=0}^{n}(-1)^k\binom{n}{k}^3 = (-1)^\ell\frac{(3\ell)!}{(\ell!)^3}\chi(n=2\ell). $$

For a real number x, denote by $\lfloor x\rfloor$ its integer part. Rewrite (7) equivalently as

$$ \begin{aligned} &\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(x)_k(y)_k}{(1-x-n)_k(1-y-n)_k} \\ &= \frac{(1+\ell)_\ell(x+y+\ell)_\ell}{(x+\ell)_\ell(y+\ell)_\ell}\chi(n=2\ell). \end{aligned} \quad (8) $$

Multiplying this equation by $(x+n-\lfloor\frac{n}{2}\rfloor)_{\lfloor\frac{n}{2}\rfloor}$, we would get a polynomial identity of degree $\le\lfloor\frac{n}{2}\rfloor$. This can be justified by combining the relation

$$ \frac{(x)_k(x+n-\lfloor\frac{n}{2}\rfloor)_{\lfloor\frac{n}{2}\rfloor}}{(1-x-n)_k} = (-1)^k\frac{(x)_k(x+n-\lfloor\frac{n}{2}\rfloor)_{\lfloor\frac{n}{2}\rfloor}}{(x+n-k)_k} $$

with

$$ \frac{(x)_k(x+n-\lfloor\frac{n}{2}\rfloor)_{\lfloor\frac{n}{2}\rfloor}}{(x+n-k)_k} = \begin{cases} (x)_k(x+n-k)_{\lfloor\frac{n}{2}-k\rfloor}, & k\le\lfloor\frac{n}{2}\rfloor; \\ (x)_k/(x+n-k)_{k-\lfloor\frac{n}{2}\rfloor}, & k>\lfloor\frac{n}{2}\rfloor. \end{cases} $$

In order to prove (8), we need only to validate it for $1+\lfloor\frac{n}{2}\rfloor$ distinct values of x. Let $S(x)$ be the finite sum displayed in (8). In view of the equation

$$\frac{(y)_k(1-y+m-n)_k}{(y-m)_k(1-y-n)_k}=\frac{(y-m+k)_m(1-y-n+k)_m}{(y-m)_m(1-y-n)_m}$$

we have the following expression

$$\begin{aligned}S(1-y+m-n)&=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(y)_k(1-y+m-n)_k}{(y-m)_k(1-y-n)_k}\\&=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(y-m+k)_m(1-y-n+k)_m}{(y-m)_m(1-y-n)_m}\end{aligned}$$

which vanishes for $0\le m<n/2$ because it results in the nth differences of the polynomial $(x+y-m)_m(1+x-y-n)_m$ of degree $2m<n$. When n is odd, we are done because (8) is valid for the $1+\lfloor\frac{n}{2}\rfloor$ distinct values $\{1-y+m-n\}_{m=0}^{\lfloor\frac{n}{2}\rfloor}$ of x.

When $n=2\ell$ is even, we have found that the polynomial $S(x)$ has ℓ zeros $\{1-y+m-2\ell\}_{m=0}^{\ell-1}$. In addition, we have to compute, for $m=\ell$, the following extreme value

$$\begin{aligned}S(1-y-\ell)&=\sum_{k=0}^{2\ell}(-1)^k\binom{2\ell}{k}\frac{(y+k-\ell)_\ell(1-y-2\ell+k)_\ell}{(y-\ell)_\ell(1-y-2\ell)_\ell}\\&=\frac{(2\ell)!}{(y-\ell)_\ell(1-y-2\ell)_\ell}=\frac{(2\ell)!}{(1-y)_\ell(y+\ell)_\ell}\end{aligned}$$

which coincides with the right member of equation (8) specified with $x=1-y-\ell$. Therefore, we have validated the equality (8) for the $1+n/2$ distinct values $\{1-y+m-n/2\}_{m=0}^{n/2}$ of x, also when n is even.

This completes the proof of (8) and also the terminating summation formula (7).

6. Stanton's Extension of Andrews' ${}_5F_4$-sum

Recently, Gessel [13] found an ingenious proof for the following summation formula

$$ {}_5F_4\left[\begin{matrix}-1-2n, & 1+x+n, & x, & z, & \frac{1}{2}+x-z\\ & \frac{x-n}{2}, & \frac{1+x-n}{2}, & 2z, & 1+2x-2z\end{matrix}\,\middle|\,1\right]\equiv 0. \quad (9)$$

It was discovered by Andrews [1, Eq 1.6] in determinant evaluation connected to plane partitions. For different proofs of (9), refer to [9, 12, 23, 25].

Following Gessel's approach, we present a similar proof for the extended formula below which is due to Stanton [23, Eq.A.2] (cf. Chu [9, Eq.2.22] also):

$$
{}_6F_5\left[\begin{matrix}-1-2n, & 1+\lambda, & x+n, & x, & z, & \frac12+x-z\\ & \lambda, & \frac{x-n}{2}, & \frac{1+x-n}{2}, & 2z, & 1+2x-2z\end{matrix}\,\middle|\,1\right]
= \frac{\lambda-x-n}{\lambda(1+x+3n)}\begin{bmatrix}\frac32,\ 1+x-2z,\ 2z-x\\ 1-x, \frac12+z, 1+x-z\end{bmatrix}_n. \tag{10}
$$

Rewrite the last formula as a binomial equality

$$
\sum_{k=0}^{2n+1}(-1)^k\binom{2n+1}{k}\frac{\lambda+k}{\lambda}\frac{(x)_k(x+n)_k}{(\frac{x-n}{2})_k(\frac{1+x-n}{2})_k}\frac{(z)_k(\frac12+x-z)_k}{(2z)_k(1+2x-2z)_k}
= \frac{\lambda-x-n}{\lambda(1+x+3n)}\begin{bmatrix}\frac32,\ 1+x-2z,\ 2z-x\\ 1-x, \frac12+z, 1+x-z\end{bmatrix}_n. \tag{11}
$$

Multiplying across the last equation by $(\frac12+z)_n(1+x-z)_n$, we would get a polynomial identity of degree $2n$ in z, if we can show that the following expression results in a polynomial of degree $2n$ in z:

$$
\frac{(z)_k(\frac12+z)_n}{(2z)_k}\times\frac{(\frac12+x-z)_k(1+x-z)_n}{(1+2x-2z)_k}.
$$

Because the second fraction becomes the first one under the substitution $z\to\frac12+x-z$, it is sufficient to prove that the first fraction is a polynomial of degree n in z. This is indeed the case in view of the following expression:

$$
\frac{(z)_k(\frac12+z)_n}{(2z)_k}=\begin{cases}\frac{(z)_k(\frac12+z)_k(\frac12+z+k)_{n-k}}{(2z)_k}=\frac{(2z)_{2k}(\frac12+z+k)_{n-k}}{4^k(2z)_k}, & k\le n;\\[2mm] \frac{(z)_n(z+n)_{k-n}(\frac12+z)_n}{(2z)_k}=\frac{(2z)_{2n}(z+n)_{k-n}}{4^n(2z)_k}, & k>n.\end{cases}
$$

In order to prove the identity (11), it is enough to validate it for $2n+1$ distinct values of z. Denote by $T(z)$ the binomial sum displayed on the left of (11). We are going to evaluate

$$T(\tfrac{x-m}{2}) \;=\; \sum_{k=0}^{2n+1}(-1)^k\binom{2n+1}{k}\frac{\lambda+k}{\lambda}P(k)Q(k)$$

where

$$P(k) := \frac{(\frac{x-m}{2})_k(\frac{1+x+m}{2})_k}{(\frac{x-n}{2})_k(\frac{1+x-n}{2})_k} \quad\text{and}\quad Q(k) :=: \frac{(x)_k(x+n)_k}{(x-m)_k(1+x+m)_k}.$$

According to the expressions

$$P(k) = \begin{cases} \dfrac{(\frac{x-n}{2}+k)_{\frac{n-m}{2}}(\frac{1+x-n}{2}+k)_{\frac{m+n}{2}}}{(\frac{x-n}{2})_{\frac{n-m}{2}}(\frac{1+x-n}{2})_{\frac{m+n}{2}}}, & m = n \pmod 2; \\[2ex] \dfrac{(\frac{x-n}{2}+k)_{\frac{m+n+1}{2}}(\frac{1+x-n}{2}+k)_{\frac{n-m-1}{2}}}{(\frac{x-n}{2})_{\frac{m+n+1}{2}}(\frac{1+x-n}{2})_{\frac{n-m-1}{2}}}, & m \neq n \pmod 2; \end{cases}$$

and

$$Q(k) = \begin{cases} \dfrac{(x-m+k)_m(1+x+m+k)_{n-m-1}}{(x-m)_m(1+x+m)_{n-m-1}}, & m \geq 0; \\[2ex] \dfrac{(x-m+k)_{m+n}(1+x+m+k)_{-m-1}}{(x-m)_{m+n}(1+x+m)_{-m-1}}, & m < 0; \end{cases}$$

we can see that for $-n \leq m < n$, both $P(k)$ and $Q(k)$ are polynomials of k with degrees n and $n-1$, respectively. Therefore $T(\frac{x-m}{2})$ vanishes for $-n \leq m < n$ because it is essentially the $(2n+1)$th differences of a polynomial of degree $2n$. Furthermore, we can also compute the following extreme value:

$$\begin{aligned} T(\tfrac{x-n}{2}) &= \sum_{k=0}^{2n+1}(-1)^k\binom{2n+1}{k}\frac{\lambda+k}{\lambda}\frac{(x)_k(x+n)_k}{(x-n)_k(1+x+n)_k}\frac{(\frac{1+x+n}{2})_k}{(\frac{1+x-n}{2})_k} \\ &= \sum_{k=0}^{2n+1}(-1)^k\binom{2n+1}{k}\frac{(x+n)(\lambda+k)}{\lambda(x-n)_n}\frac{(x-n+k)_n(\frac{1+x-n}{2}+k)_n}{(k+x+n)(\frac{1+x-n}{2})_n} \\ &= \frac{(x+n)(\lambda-x-n)(2n+1)!(-2n)_n(\frac{1-x-3n}{2})_n}{\lambda(x-n)_n(x+n)_{2n+2}(\frac{1+x-n}{2})_n} \\ &= \frac{\lambda-x-n}{\lambda(1+x+3n)}\left[\begin{matrix} \frac{3}{2},\ 1+n,\ -n \\ 1-x,\ \frac{1+x-n}{2},\ 2+x+n \end{matrix}\right]_n \end{aligned}$$

which coincides with the right member (11) at $z = \frac{x-n}{2}$. In conclusion, we have checked the equality (11) for the $2n+1$ distinct values $z = \frac{x-m}{2}$ with $-n \leq m \leq n$. This completes the proof of (11) and so Stanton's summation formula (10).

7. Convolution Identities of Abel

Instead of the fundamental theorem of algebra on polynomials, the Lagrange interpolation can also be utilized to justify the final passage in the proving process of binomial identities for the examples hitherto illustrated.

In this section, we prove, by means of Taylor polynomials, the following deep generalization for the binomial theorem discovered by Abel (cf. Graham *et al* [16, §5.4], Riordan [20, §1.5] and [6, 10, 22] for example):

$$x\sum_{k=0}^{n}\binom{n}{k}(x+ky)^{k-1}(z-ky)^{n-k} = (x+z)^n, \tag{12a}$$

$$x\sum_{k=0}^{n}\binom{n}{k}(x+ky)^{k-1}\frac{z-ny}{z-ky}(z-ky)^{n-k} = \frac{x+z-ny}{x+z}(x+z)^n. \tag{12b}$$

Denote by $\mathrm{P}(z)$ the binomial sum in (12a). Its mth derivative at $z=-x$ gives

$$\mathrm{P}^{(m)}(-x) = m!x\binom{n}{m}\sum_{k=0}^{n-m}(-1)^{n-m-k}\binom{n-m}{k}(x+ky)^{n-m-1}.$$

For $0 \leq m < n$, the last sum results in the $(n-m)$th differences of a polynomial of degree $n-m-1$. Therefore $\mathrm{P}^{(m)}(-x)$ is equal to zero for $0 \leq m < n$ and $\mathrm{P}^{(n)}(-x) = n!$.

Observing further that $\mathrm{P}(z)$ is a polynomial of degree n with its derivatives of orders $\{m\}_{m=0}^{n}$ at $z=-x$ equal to those of another polynomial $(x+z)^n$. Hence both polynomials are identical, which gives Abel's first identity (12a).

Analogously, let $\mathrm{Q}(z)$ be the binomial sum in (12b). Then it is not hard to compute its mth derivative at $z=-x$ by

$$\begin{aligned}\mathrm{Q}^{(m)}(-x) &= m!x(x-my+ny)\binom{n}{m}\\ &\times \sum_{k=0}^{n-m}(-1)^{n-m-k}\binom{n-m}{k}(x+ky)^{n-m-2}\end{aligned}$$

which vanishes for $0\le m\le n-2$ because the last sum results in the $(n-m)$th differences of a polynomial of degree $n-m-2$.

Taking into account of $\mathrm{Q}^{(n)}(-x)=n!$ and $\mathrm{Q}^{(n-1)}(-x)=n!(-y)$, we conclude that the polynomial $\mathrm{Q}(z)$ have the same derivatives of orders $\{m\}_{m=0}^{n}$ at $z=-x$ as those of another polynomial $(x+z-ny)(x+z)^{n-1}$. Hence both polynomials are identical, which confirms Abel's second identity (12b).

8. Minton's Summation Theorem

Finally, we examine a seminal result of Minton [18] in classical hypergeometric series. It reads as the following summation theorem

$${}_{\ell+2}F_{\ell+1}\left[\begin{matrix}-n, & \lambda, & \{a_i+m_i\}_{i=1}^{\ell}\\ & 1+\lambda, & \{a_i\}_{i=1}^{\ell}\end{matrix}\,\middle|\,1\right]=\frac{n!}{(1+\lambda)_n}\prod_{i=1}^{\ell}\frac{(a_i-\lambda)_{m_i}}{(a_i)_{m_i}} \tag{13}$$

provided that m_i and n are nonnegative integers with $n\ge\sum_{i=1}^{\ell}m_i$. Different proofs and extensions of this formula can be found in Karlsson [17] and Chu [5, 7]. However, we believe that the proof given here is the simplest.

According to the relation

$$\frac{(a_i+m_i)_k}{(a_i)_k}=\frac{(a_i+k)_{m_i}}{(a_i)_{m_i}}$$

we may express (13) equivalently as the following equality

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\prod_{i=1}^{\ell}(a_i+k)_{m_i}}{\lambda+k}=\frac{n!}{(\lambda)_{n+1}}\prod_{i=1}^{\ell}(a_i-\lambda)_{m_i}. \tag{14}$$

Writing the last binomial sum in terms of finite differences, we can evaluate it immediately as

$$(-1)^n \Delta^n \frac{\prod_{i=1}^{\ell}(a_i+x)_{m_i}}{\lambda+x}\Big|_{x=0} = \frac{n!}{(\lambda)_{n+1}} \prod_{i=1}^{\ell}(a_i-\lambda)_{m_i}$$

which confirms (14) and so Minton's summation formula (13).

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