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PROBLEMS

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Problems

The first problem was contributed by Finbarr Holland of University College Cork.

Problem 77.1. Suppose that $f : [0,1] \to \mathbb{R}$ is a convex function and $\int_0^1 f(t)\,dt = 0$. Prove that

$$\int_0^1 t(1-t)f(t)\,dt \leqslant 0,$$

with equality if and only if $f(t) = a(2t-1)$ for some real number a.

I learned the second problem from Tony Barnard of Kings College London some years ago.

Problem 77.2. Each member of a group of n people writes his or her name on a slip of paper, and places the slip in a hat. One by one the members of the group then draw a slip from the hat, without looking. What is the probability that they all end up with a different person's name?

For the third problem, interpret 'evaluate' to mean that you should express the given quantity in a simple formula in terms of integers and known constants using standard functions and the usual operations of arithmetic.

Problem 76.3. Evaluate

$$1 + \cfrac{1^2}{1 + \cfrac{2^2}{1 + \cfrac{3^2}{1 + \cdots}}}.$$

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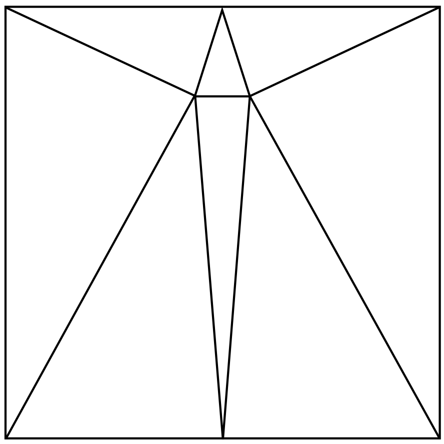
Solutions

Here are solutions to the problems from *Bulletin* Number 75.

The well-known picture in the solution to the first problem was supplied by both Ángel Plaza and the North Kildare Mathematics Problem Club. The argument that there must be at least 8 triangles was supplied by Henry Ricardo of the New York Math Circle, USA. It was also supplied as a sketch by the North Kildare Mathematics Problem Club. The club point out that this proof can be found in work of Charles Cassidy and Graham Lord (J. Rec. Math. 13, 1980/81). Cassidy and Lord attribute the problem of finding a tessellation using 8 triangles to Martin Gardner (Sci. Amer. 202, 1960), and refer to an earlier proof that 8 is minimal by H. Lindgren (Austral. Math. Teacher 18, 1962).

Problem 75.1. What is the least positive integer n for which a square can be tessellated by n acute-angled triangles?

Solution 75.1. The figure below shows that a square can be tessellated by 8 acute-angled triangles.



The following sequence of observations proves (in brief) that any tessellation comprises at least 8 triangles.

(i) Any vertex interior to the square must be incident to at least 5 triangles because the sum of 4 acute angles is less than 2π.

(ii) Any vertex on a side of the square must be incident to at least 3 triangles as the sum of 2 acute angles is less than π.

(iii) Each corner of the square must be incident to at least 2 triangles because $\pi/2$ is not acute.

(iv) Suppose there are no interior vertices. Choose a corner of the square. By (iii), there is an edge from this corner to a side of the square (or possibly the opposite corner). This edge 'traps' another corner, preventing it from being connected to a side of the square (or its opposite corner) by an edge, in contradiction with (iii).

(v) Suppose there is a unique interior vertex u. By a similar argument to (iv), u is connected by 4 edges to each of the 4 corners of the square. By (i), there must be another edge of the tessellation incident to u; by assumption, this edge must be incident to a vertex v on a side ℓ of the square. By (ii), there must be another edge of the tessellation that is incident to v and another vertex w, where w does not lie on ℓ or one of the corners incident to ℓ. The vertex w cannot lie in the interior of the square, so it must lie on one of the remaining sides or corners of the square. However, this is impossible without two edges intersecting.

(vi) By (v), there are at least two interior vertices. Each interior vertex must be incident to at least 5 triangles. Any 2 interior vertices can be incident to at most 2 common triangles, as they share at most 1 edge. Therefore, the tessellation has at least $5 + 5 - 2 = 8$ triangles. $\square$

The second problem was solved by the North Kildare Mathematics Problem Club and the proposer, Finbarr Holland.

Problem 75.2. Let

$$s_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}, \quad n = 0, 1, 2, \ldots .$$

Suppose $0 < \alpha < 1$. Prove that when $n \geqslant 1$,

$$e^x \leqslant \frac{s_n(x) - \alpha x s_{n-1}(x)}{1 - \alpha x} \quad \text{for all } x \in [0, 1/\alpha)$$

if and only if $\alpha \geqslant 1/(n + 1)$.

Solution 75.2. Suppose that the first inequality involving e^x is true. Rearranging this inequality we obtain

$$\frac{e^x - s_n(x)}{x(e^x - s_{n-1}(x))} \leqslant \alpha,$$

for $x \in (0, 1/\alpha)$. Now

$$\lim_{x\to 0^+} \frac{e^x - s_n(x)}{x^{n+1}} = \frac{1}{(n+1)!} \quad \text{and} \quad \lim_{x\to 0^+} \frac{x(e^x - s_{n-1}(x))}{x^{n+1}} = \frac{1}{n!},$$

so

$$\lim_{x\to 0^+} \frac{e^x - s_n(x)}{x(e^x - s_{n-1}(x))} = \frac{1}{n+1}.$$

Therefore $\alpha \geqslant 1/(n+1)$.

Conversely, suppose that $\alpha \geqslant 1/(n+1)$. Then, if $m \geqslant n$, we have

$$\frac{1}{m!} \leqslant \frac{1}{n!(n+1)^{m-n}} \leqslant \frac{\alpha^{m-n}}{n!}.$$

Hence

$$\begin{aligned}
\frac{s_n(x) - \alpha x s_{n-1}(x)}{1-\alpha x} &= s_{n-1}(x) + \frac{x^n}{n!(1-\alpha x)} \\
&= s_{n-1}(x) + \frac{1}{n!}\sum_{k=0}^{\infty} \alpha^k x^{k+n} \\
&\geqslant s_{n-1}(x) + \sum_{k=0}^{\infty} \frac{1}{(n+k)!} x^{k+n} \\
&= e^x,
\end{aligned}$$

for $x \in [0, 1/\alpha)$, with equality if and only if $x = 0$. □

The third problem was solved by Ángel Plaza, Eugene Gath of the University of Limerick, Henry Ricardo, and the North Kildare Mathematics Problem Club. We present the solution of Henry Ricardo, which was similar to some of the others.

*Problem 75.3***.** Given positive real numbers a, b, and c, prove that

$$a + b + c \leqslant \sqrt[3]{abc}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right).$$

Solution 75.3. Using the AM–GM inequality, we have

$$\begin{aligned}
\frac{a+b+c}{\sqrt[3]{abc}} &= \sqrt[3]{\frac{a^2}{bc}} + \sqrt[3]{\frac{b^2}{ca}} + \sqrt[3]{\frac{c^2}{ab}} \\
&= \sqrt[3]{\frac{a\cdot a\cdot b}{b\cdot b\cdot c}} + \sqrt[3]{\frac{b\cdot b\cdot c}{c\cdot c\cdot a}} + \sqrt[3]{\frac{c\cdot c\cdot a}{a\cdot a\cdot b}} \\
&\leqslant \frac{1}{3}\left(\frac{a}{b} + \frac{a}{b} + \frac{b}{c}\right) + \frac{1}{3}\left(\frac{b}{c} + \frac{b}{c} + \frac{c}{a}\right) + \frac{1}{3}\left(\frac{c}{a} + \frac{c}{a} + \frac{a}{b}\right) \\
&= \frac{a}{b} + \frac{b}{c} + \frac{c}{a},
\end{aligned}$$

and the desired inequality follows immediately. □

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com` in any format (we prefer LaTeX). Submissions for the summer Bulletin should arrive before the end of April, and submissions for the winter Bulletin should arrive by October. The solution to a problem is published two issues after the issue in which the problem first appeared. Please include solutions to any problems you submit, if you have them.

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