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David Bleecker and Bernhelm Booss-Bavnbek: Index Theory with Applications to Mathematics and Physics, International Press, October 2013. ISBN:978-1-57146-264-0, USD 95.00, 766+xxii pp.

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1. Introduction

"In his [Bonn] Arbeitstagung lecture given 16 July 1962 Atiyah formulated the problem of expressing the index of elliptic operators in terms of topological invariants associated to their symbol and stated the fundamental conjecture for the Dirac operator ... A few months later, in Feburary 1963, Atiyah and Singer announced the general index formula for elliptic operators on closed manifolds and indicated the main steps of a proof ... K-theory which gave the essential framework for the statement of the index theorem had been introduced by Atiyah and Hirzebruch following Grothendieck's lead in their 1959 paper.... The *central and deep point* in this new cohomology theory was the Bott isomorphism", recalls Brieskorn (1936-2013) in [7] (see [1], [4], [3]). Both Brieskorn and Booß-Bavnbeck received their doctorates in Bonn under Friedrich Hirzebruch (1927-2012). In its draft version from 2012 [11], the book under review has been dedicated to Hirzebruch and Bleecker's PhD supervisor Chern.

The history of the book reflects this ancestry. It started as a German language textbook [6] from 1977, which was translated and somewhat extended by Bleecker in 1985. The book grew further to a 766 pages hardcover volume, more than twice that of the original textbook, or to a weight of 1.470kg (two pints, that is, and it may make you equally giddy).

2. Content of the book

The book is organised into

I: *Operators with Index and Homotopy Theory*, (Chapters 1-4, 132 pages),

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II : *Analyis on Manifolds*, (Chapters 5-9, 118 pages),
III: *The Atiyah-Singer Index Formula*, (Chapters 10-13, 112 pages),
IV: *Index Theory in Physics and the Local Index Theorem* (Chapters 14-18, 342 pages).

The book provides two appendices, the first devoted to *Fourier Series and Integrals - Fundamental Principles*, the second to *Vector Bundles*.

A bounded linear operator T acting in a (complex) Hilbert space is *Fredholm*, written $T \in \mathcal{F}$, if $\ker T$ and $\mathrm{coker} T$ are finite dimensional. The *index* of T is given by

$$\mathrm{index}\, T = \dim \ker T - \dim \mathrm{coker}\, T\ . \qquad (1)$$

The index is invariant under small perturbations and in fact a homotopy invariant. Thus it generalises to continuous families of Fredholm operators over a compact parameter space X, $T : X \to \mathcal{F}$, mapping $x \mapsto T_x$, for $x \in X$. If the kernel of T_x and T_x^*, respectively, has constant dimension, it defines an isomorphism class of complex vector bundles in $\mathrm{Vec}(X)$. In order to make sense of the difference in the semi-group $\mathrm{Vec}(X)$, one introduces the Grothendieck group $K(X)$. Specifically, we have $\mathrm{index}\, T \in K(X)$ in eq. (1). The determinant line bundle generalises the index bundle with interesting links to recent developments in physics (zeta function regularisation, multiplicative anomaly), which are hardly discussed in this book though a reference to the work of Charles Nash is provided [10] (best wishes on the occasion of your retirement!).

Elliptic operators on sections of complex vector bundles provide a primary source of Fredholm operators (Part II, Chapters 5 and 6). Chapter 7 is a *Crash Course* on *Sobolev spaces*. In Chapter 8, elliptic *Pseudo-Differential Operators* are introduced. Let $P = \sum_{|\alpha|\leq m} A^\alpha(x)D^\alpha$ be a differential operator on $X = \mathbb{R}^n$, acting on smooth functions u with compact support. Then

$$(Pu)(x) = \int e^{i\langle k,x\rangle} p(x,k)\hat{u}(k)dk\ ,$$

where $\hat{u}$ is the Fourier transform of u and the polynomial $p(x,k) = \sum_{|\alpha|\leq m} A^\alpha(x)k^\alpha$ is the amplitude (or total symbol) of P. For more general C^∞ functions p of x and k, P defines a pseudo-differential operator, subject to a growth condition in the variable k. There is a notion of principal symbol for pseudo-differential operators. Unlike the amplitude p, the principal symbol turns out to have a geometric meaning, and for elliptic operators, its knowledge is sufficient for computing the index. Using charts, the definitions carry over to operators $P : C_0^\infty(X, E) \to C^\infty(X, F)$ between smooth sections of vector bundles E, F over a manifold X. Elliptic operators of this kind yield a space $\mathrm{Ell}(E, F)$. There is a way to construct a global amplitude [5] by considering the pull-back of E, F along the projection $\pi : T^*X \to X$. (Here T^*X is the cotangent bundle.) This gives rise to a one-to-one map $C^\infty(T^*X, \mathrm{Hom}(\pi^*E, \pi^*F)) \to \mathrm{Ell}(E, F)$, $p \mapsto P$, up to small perturbations of P that do not affect the index.

For the sake of "simplicity, accessibility and transparency", in Part III, the authors decide to "develop a larger portion of algebraic topology by means of a theorem of Raoul Bott concerning the topology of $\mathrm{GL}(N, \mathbb{C})$, i.e. on the basis of linear algebra, rather than on the basis of the theory of simplicial complexes and their homology and cohomology."(p. 252). *Winding numbers* (Chapter 10) play a central role in questions about stability of planetary orbits in celestial mechanics. In keeping visible the political sympathies of the authors, the book mentions challenging engineering tasks related to "the unmanned soft landing of the lunar module Luna 9 on February 3, 1966" (resp. Luna 1 in the previous versions). A very careful discussion of winding numbers follows, from a geometric, a combinatorial, a calculus, an algebraic and a functional analytic view point. The *index theorem* relates two of the possible generalisations to higher dimension: a local one (the topological index) and a global one (the analytic index). For example, the Euler characteristic $\chi(X)$ of a compact oriented differentiable surface X can be described as

- the number of isolated zeros of a tangent vector field on X, counted with proper multiplicity (the local index of the vector field at that point);
- the degree of the map $S^1 \to S^1$ (a winding number);
- the alternating sum of the number of vertices, edges and faces (for polyhedra).

The third approach generalises to higher dimension as an alternating sum of dimensions of cohomology groups of TX. Closely related is the description of $\chi(X)$ as

- the index of the elliptic differential operator $(d+d^*) : \Omega^{\text{even}}(X) \to \Omega^{\text{odd}}(X)$ based on the de Rahm operator d.

More generally, "one considers cycles where a given number of vector fields become dependent" (p. 262, citing Atiyah), linking topology to $GL(n, \mathbb{C})$. The mapping degree $\pi_{n-1}(\mathrm{GL}(N, \mathbb{C})) \to \mathbb{Z}$ (n even) is the first step towards Bott's Periodicity Theorem. If P is a pseudo-differential operator acting on $C_0^\infty(\mathbb{R}^n \times \mathbb{C}^N)$, for fixed $x \in \mathbb{R}^n$, its principal symbol defines a continuous map $\sigma(P)(x, .) : S^{n-1} \to \mathrm{GL}(N, \mathbb{C})$ (Chapter 11). Chapter 12 deals with Hermitian vector bundles E, F over a closed manifold X. Let $P \in \mathrm{Ell}(E, F)$ have principal symbol $\sigma(P)$. The restriction of $\sigma(P) : T^*X \to \mathrm{Hom}(\pi^*E, \pi^*F)$ to the sphere bundle $SX \subset T^*X$ defines isomorphisms, so there is a naturally associated element

$$[\sigma(P)] = [\pi^*E, \pi^*F; \sigma(P)|_{SX}] \in K(BX, SX) \cong K(T^*X)\,,$$

represented by the difference bundle obtained by gluing $\pi^*E|_{BX}$ and $\pi^*F|_{BX}$ (where $BX \subset T^*X$ is the ball bundle) on $BX \cup_{SX} BX$ along SX using $\sigma(P)|_{SX}$. It turns out that index P defined by eq. (1) depends only on the equivalence class $[\sigma(P)] \in K(TX)$. On the other hand, there is a notion of *topological index*, and the *Atiyah Singer Index Theorem* states that these two are equal,

$$\text{analytic index} = \text{topological index}$$

as group homomorphisms $K(TX) \to \mathbb{Z}$. While the analytic index is easy to define, it is hard to compute. In contrast, the topological index can be explicitely calculated in many cases, but its definition is too involved to be reproduced here.

A particular feature is the workout of the embedding proof of the Atiyah-Singer Index formula for non-trivial normal bundle of X. The crucial step is the *multiplicative property* of the index. The authors follow a suggestion made in ([9], p. 188) and apply the Bokobza-Haggiag formalism [5] to simplify this partial discussion. Eventually the cobordism proof is discussed shortly and compared to the embedding and the heat equation proof (see below).

Part IV gives a crash course in *Classical Field Theory* and in *Quantum Theory* (Chapter 14) and treats the *Geometric Preliminaries* like principal fiber bundles, connections and curvature, and characteristic classes (Chapter 15), which in part have been used already in Chapter 12. Chapter 16 on *Gauge Theoretic Instantons* investigates an application of the index theorem which is important for both mathematicians and physicists, namely the computation of the dimension of the moduli space of self-dual connections (instantons) on a principal G-bundle, where G is a compact semi-simple Lie group, on a compact oriented Riemannian 4-manifold. In the massive Chapter 17 (130 pages) it is shown that the classical geometric operators such as the signature operator, the de Rham operator, the Dolbeault operator and the Yang-Mills operator can be locally expressed in terms of twisted Dirac operators, so that *The Local Index Theorem for twisted Dirac operators* applies. In Section 4 of Chapter 17, an asymptotic expansion for the heat kernel is presented in great and useful detail over 27 pages, using the geometric concepts introduced previously in Chapter 15. The last Chapter in the book, Chapter 18, is devoted to the *Theory* of the *Seiberg-Witten* equation (1994). The authors don't try to keep up with recent developments but aim at a "digestible presentation of the main results" (p. 643). Sketches and some details of the proofs are given.

3. Comments and conclusion

The book tries to draw a complete and comprehensible picture of the field. In particular, it usually includes sketches of proofs that it cannot work out fully. The reader is encouraged to question the meaning of the formulae in a guided manner, and numerous exercises are included, mostly backed by helpful hints. The book is unusual in its willingness to go much into detail, which it does very carefully. In view of the amount of material it covers, structuring is a major issue and overall the book does a truly admirable job here. Inside the text, cohesion is established using many references to related discussions in other parts of the book. Luckily these come not only with the number of the relevant chapter and section, but also with a page number, so that a jump to other parts of the book is quick and easy and does not feel like a disruption. This also allows one to step in to the book at any place, and to get to know the book as a whole without reading it linearly from the beginning. A determined reader is suggested to follow a logical path through the material to approach the subject in one of the directions labelled as follows:

(1) Index Theorem and Topological K-Theory,
(2) Index Theorem via Heat Equation,
(3) Gauge Theoretic Physics,

(4) Spectral Geometry, and
(5) Global and Micro-Local Analysis.

The book is a rich source of citations and references for further reading, making it half an encyclopaedia, as a colleague would name it. Though these are included in the normal text, the conversational style keeps the reasoning running and the text does not appear overloaded. All in all the book is sagely written and pleasant to read.

This said, there are issues with the book. The definition of the topological index (Part III, Chapter 12) relies on the K-theoretical Thom isomorphism

$$K(X) \overset{\cong}{\to} K(V) \tag{2}$$

whenever $V \to X$ is a complex vector bundle over a compact manifold. For $V = \mathbb{R}^{2n} \times X$, (2) is just the Bott Periodicity isomorphism (given by the outer tensor product with a power of the Bott class $\mathbf{b} \in K(\mathbb{R}^2)$). The general case "can be considered as a reformulation or generalization of Bott Periodicity", where "the Bott class $\mathbf{b}$ ... corresponds to the canonical exterior class λ_V" (p. 289). Though efforts are made to define the class λ_V and thus the map, (2) is not actually explained or proved. Instead, the reader is referred to an "independent" proof in [6]. It seems that λ_V does not reappear in any later discussion. A survey in the literature indicates, however, that a complete proof of (2) is out of reach for the dedication and space in the book, which is not primarily devoted to K-theory.

In Part IV, Chapter 16, the introduction (p. 460f) of an invariant inner product on simple Lie algebras is cumbersome and should have been omitted. The matrix trace $\operatorname{tr} AB$ is sufficient for the purpose.

The same is true for the crash course in physics (Chapter 14) which is beyond the realm of the book. The presentation falls out of shape: Maxwell's equations for the field F are written in terms of components of the electric and magnetic fields, which are irrelevant to the book. One might have written

$$F = dA\,,$$
$$d * F = j\,,$$

since the Hodge-star operator is introduced in Chapter 13. The book, however, avoids the use of the $*$ operator by introducing an extra letter δ which it takes two attempts to explain ("the codifferential (the formal adjoint of d)", p. 366). The use of the letters A and F for the one- and two-form, respectively, is standard convention but conflicts with the presentation a hundred pages later (Chapter 16, p. 463), where now F denotes an element in the gauge group (the role played by A before).

The comments about quantum electrodynamics (QED) are incomprehensible, since no quantised fermions are introduced. After a brief mention of Feynman integrals, we read "Contrary to popular misconceptions (even held by good physicists) a formal power series in α does not necessarily converge, even at $\alpha = 1/137$" (p. 383). This is a suspicious statement, even for the year 1977!

The main link between the mathematical content of the book and quantum field theory (QFT) is provided by *instantons*. These are absolute minima of the Yang-Mills (YM) functional with non-trivial winding number. The proper mathematical framework for quantum YM is lattice gauge theory. The notion of a winding number is not available on the lattice, however, and the reader would want to see at least an argument why instantons are relevant to this setting. Unfortunately, none of these issues is addressed throughout the more than 50 pages. The authors include the original construction of instantons by Atiyah, Drinfeld, Hitchin and Manin but state that a proof that this construction yields all instantons would take them too long (p. 482). They could have given a short proof by following Donaldson and Kronheimer [8] who use the simpler and more powerful approach by Nahm. The chapter culminates in the *Main Theorem* (Theorem 16.37 on page 511) which states that under rather strict conditions on the manifold, like being self-dual and having positive scalar curvature, the moduli space of self-dual connections has a manifold structure, and its dimension is specified. The authors are aware that work after 1982 (Kronheimer, Mrowka, Taubes, and Uhlenbeck) has removed the restrictions, but they only present the old argumentation.

We have already commented on Chapter 18 on *Seiberg Witten Theory*. The reader should note that the chapter is not about Seiberg-Witten QFT but about the classical equation. This is global analysis and only marginally involves index theory.

The authors cite Hilbert (p. 133): "Any true progress brings with it the discovery of more incisive tools and simpler methods which at the time facilitate the understanding of earlier theories and eliminate older more awkward developments." Unfortunately, the girth acquired by the book since 1977 does not pay any heed to this insight.

Though the book has considerable merits, the referee often felt a relief when she looked up the citations and read the short and clear expositions by Atiyah.

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