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PROBLEMS

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Problems

The first problem was contributed by Finbarr Holland of University College Cork.

Problem 73.1. Let U_n denote the Chebyshev polynomial of the second kind of degree n, which is the unique polynomial that satisfies the equation $U_n(\cos\theta) = \sin((n+1)\theta)/\sin\theta$. The polynomial U_{2n} satisfies $U_{2n}(t) = p_n(4t^2)$, where

$$p_n(z) = \sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}z^{n-k}.$$

Prove that p_n is irreducible over the integers when $2n+1$ is a prime number.

The second and third problems were passed on to me by Tony Barnard of King's College London.

Problem 73.2. Find all positive integers a, b, and c such that

$$\begin{aligned} bc &\equiv 1 \pmod{a} \\ ca &\equiv 1 \pmod{b} \\ ab &\equiv 1 \pmod{c}. \end{aligned}$$

Problem 73.3. Prove that

$$\frac{1}{10\sqrt{2}} < \frac{1}{2}\times\frac{3}{4}\times\frac{5}{6}\times\cdots\times\frac{99}{100} < \frac{1}{10}.$$

Solutions

Here are solutions to the problems from *Bulletin* Number 71. The first problem was solved by J.P. McCarthy of University College Cork and also by the North Kildare Mathematics Problem Club.

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We give McCarthy's solution to (a) and the NKMPC's solution to (b). The problem was also solved by the proposer.

Problem 71.1. For $n = 0, 1, 2, \ldots,$ the triangular numbers T_n and Jacobsthal numbers J_n are given by the formulas

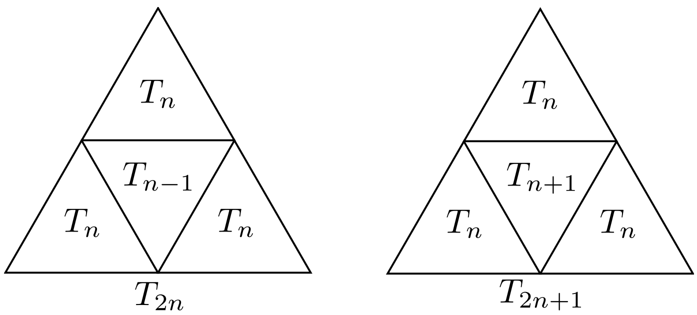
$$T_n = \frac{n(n+1)}{2} \quad \text{and} \quad J_n = \frac{2^n - (-1)^n}{3}.$$

(a) Prove that for each integer $n \geqslant 3$ there exist positive integers a, b, and c such that $T_n = T_a + T_b T_c$.
(b) Prove that infinitely many square numbers can be expressed in the form $J_a J_b + J_c J_d$ for positive integers a, b, c, and d.

Solution 71.1. (a) It is straightforward to check that

$$T_{2n} = T_{n-1} + 3T_n$$
$$T_{2n+1} = T_{n+1} + 3T_n.$$

These equations are illustrated in the figure below.



Since $T_2 = 3$, the result follows immediately.
(b) We have

$$J_{2n} + J_{2n+1} = \frac{2^{2n} - 1}{3} + \frac{2^{2n+1} + 1}{3} = 2^{2n}.$$

Therefore the square of each positive integer power of 2 can be written as a sum of two Jacobsthal numbers, and since $J_1 = 1$ the result follows immediately. □

The second problem was solved separately by Niall Ryan of the University of Limerick, the North Kildare Mathematics Problem Club, and the proposer. All solutions were in the same spirit, and we present a solution based on that of the proposer.

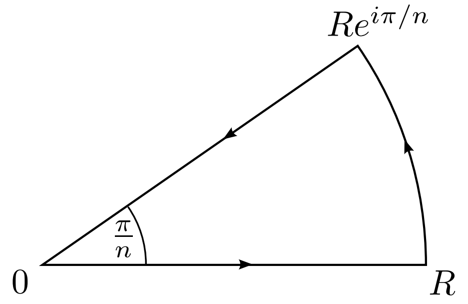
Problem 71.2. Prove that for each integer $n \geqslant 3$,

$$\int_0^\infty \frac{x-1}{x^n - 1}\, dx = \frac{\pi}{n \sin(2\pi/n)}.$$

Solution 71.2. Let

$$f(z) = \frac{1}{z^{n-1} + \cdots + z + 1}.$$

This function is analytic on a region containing the closed contour shown below.



Applying Cauchy's theorem to f, and letting $R \to \infty$, we obtain

$$\int_0^\infty \frac{x-1}{x^n-1}\,dx + e^{i\pi/n}\int_0^\infty \frac{e^{i\pi/n}x-1}{x^n+1}\,dx = 0.$$

Therefore

$$\begin{aligned}\int_0^\infty \frac{x-1}{x^n-1}\,dx &= -\mathrm{Re}\left[e^{i\pi/n}\int_0^\infty \frac{e^{i\pi/n}x-1}{x^n+1}\,dx\right]\\ &= \cos\left(\frac{\pi}{n}\right)K(n,0) - \cos\left(\frac{2\pi}{n}\right)K(n,1), \qquad (1)\end{aligned}$$

where, for non-negative integers m and n,

$$K(n,m) = \int_0^\infty \frac{x^m}{x^n+1}\,dx.$$

Using a similar contour to above but with angle $2\pi/n$ rather than π/n, one can obtain the well-known formula

$$K(n,m) = \frac{\pi}{n\sin((m+1)\pi/n)}, \quad n > m+1.$$

Substituting the expressions for $K(n,0)$ and $K(n,1)$ into (1) gives the required result. □

Problem 71.3 remains unsolved!

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com`.

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