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INFINITELY MANY POSITIVE INTEGER SOLUTIONS OF THE QUADRATIC DIOPHANTINE EQUATIONS $x^2 - 8B_nxy - 2y^2 = \pm 2^r$

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Abstract. In this study, we consider the quadratic Diophantine equations given in the title and determine when these equations have positive integer solutions. Moreover, we find all positive integer solutions of them in terms of Balancing numbers B_n, Pell and Pell-Lucas numbers, and the terms of the sequence $\{v_n\}$, where $\{v_n\}$ is defined by $v_0 = 2$, $v_1 = 6$, and $v_{n+1} = 6v_n - v_{n-1}$ for $n \geq 1$.

1. Introduction

A Diophantine equation is an equation in which only integer solutions are allowed. The name "Diophantine" comes from Diophantos, an Alexandrian mathematician of the third century A. D., who proposed many Diophantine problems; but such equations have a very long history, extending back to ancient Egypt, Babylonia, and Greece. In general, a quadratic Diophantine equation is an equation in the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \tag{1}$$

where $a, b, c, d, e,$ and f are fixed integers. There has been much interest in determining all integer solutions to Diophantine equations among mathematicians. In particular, several papers [30, 6, 29, 2, 3, 4, 17, 33, 7, 12, 10] deal with such equations. The principal question when studying a given Diophantine equation is whether a solution exists; and in the case they exist, how many solutions there are and whether there is a general form for the solutions. For more details on Diophantine equations, see [21, 31, 23, 8, 14, 32, 20].

In [11], Keskin and Yosma considered the Diophantine equations

$$x^2 - L_nxy + (-1)^n y^2 = \pm 5^r, \tag{2}$$

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where $n > 0$, $r > 1$, and L_n denotes the n^{th} Lucas number. The authors determined when (2) have positive integer solutions. Later, applying some properties of Fibonacci and Lucas numbers, they gave all positive integer solutions of (2) in terms of Fibonacci and Lucas numbers. In [13], Keskin, Karaatlı, and Şiar determined when the equations

$$x^2 - 5F_nxy - 5(-1)^ny^2 = \pm 5^r, \tag{3}$$

where F_n denotes the n^{th} Fibonacci number, have positive integer solutions under some assumptions that $n \geq 1$, $r \geq 0$, using some basic properties of Fibonacci and Lucas sequences and also some cases in which Fibonacci and Lucas sequences have square terms. Then the authors found all positive integer solutions of (3). In this study, we are interested in determining explicitly all positive integer solutions (x, y) of the equations

$$x^2 - 8B_nxy - 2y^2 = \pm 2^r, \tag{4}$$

where B_n denotes the n^{th} balancing number, in terms of balancing numbers, Pell and Pell-Lucas numbers, and the terms of the sequence $\{v_n\}$.

2. Close Relations Between Balancing Sequence, Pell and Pell-Lucas Sequences, and the sequence $\{v_n\}$

Before we can explain about the sequences mentioned in the title above, we need to recall the generalized Fibonacci and Lucas sequences.

Let P and Q be non-zero integers. We consider the generalized Fibonacci sequence $\{U_n\}$

$$U_0 = 0,\ U_1 = 1,\ U_{n+1} = PU_n - QU_{n-1} \text{ for } n \geq 1 \tag{5}$$

and the generalized Lucas sequence $\{V_n\}$

$$V_0 = 2,\ V_1 = P,\ V_{n+1} = PV_n - QV_{n-1} \text{ for } n \geq 1. \tag{6}$$

The numbers U_n and V_n are called the n^{th} generalized Fibonacci and Lucas numbers, respectively. Moreover, generalized Fibonacci and Lucas numbers for negative subscripts are defined as

$$U_{-n} = \frac{-U_n}{Q^n} \text{ and } V_{-n} = \frac{V_n}{Q^n} \tag{7}$$

with $n \geq 1$. If $\alpha = (P + \sqrt{P^2 - 4Q})/2$ and $\beta = (P - \sqrt{P^2 - 4Q})/2$, assuming $P^2 - 4Q \neq 0$, are zeros of $x^2 - Px + Q$, then we have the well known Binet formulas

$$U_n = (\alpha^n - \beta^n)/(\alpha - \beta) \text{ and } V_n = \alpha^n + \beta^n \tag{8}$$

for all $n \in \mathbb{Z}$. When $P = 1$ and $Q = -1$, $\{U_n\} = \{F_n\}$ and $\{V_n\} = \{L_n\}$ are the familiar sequences of Fibonacci and Lucas numbers, respectively. For $P = 2$ and $Q = -1$, $\{U_n\}$ and $\{V_n\}$ are the familiar Pell sequence $\{P_n\}$ and Pell-Lucas sequence $\{Q_n\}$, respectively. Furthermore, when $Q = 1$, we represent $\{U_n\}$ and $\{V_n\}$ by $\{u_n\}$ and $\{v_n\}$. It clearly follows from (3) that

$$u_{-n} = -u_n \text{ and } v_{-n} = v_n \tag{9}$$

for all $n \geq 1$. For further details on generalized Fibonacci and Lucas sequences, we refer the reader to [9, 22, 25, 26].

The terms of a sequence $\{U_n\}$ may be partitioned into disjoint classes by means of the following equivalence relation:

$U_m \sim U_n$ if and only if there exist non-zero integers x and y satisfying $x^2 U_m = y^2 U_n$, or equivalently $U_m U_n$ is a square. If $U_m \sim U_n$, then U_m and U_n are said to be in the same square-class. A square-class containing more than one term of the sequence is called non-trivial. Similarly, we can define the square-class of $\{V_n\}$.

Balancing numbers were introduced by Behera and Panda [1], by considering natural numbers b and r satisfying the equation

$$1 + 2 + ... + (b - 1) = (b + 1) + (b + 2) + ... + (b + r). \tag{10}$$

Here, r is the *balancer* corresponding to the *balancing number* b. The n^{th} balancing number is denoted by B_n and the balancing numbers B_n for $n \geq 2$ are obtained from the recurrence relation

$$B_0 = 0,\ B_1 = 1,\ B_{n+1} = 6B_n - B_{n-1} \text{ for } n \geq 1. \tag{11}$$

Actually, substituting $P = 6$ and $Q = 1$ into (5) and (6) gives that $u_n = B_n$ and the sequence $\{v_n\}$, which is mentioned in the title of this section. This means that both balancing sequence and the sequence $\{v_n\}$ are special cases of the generalized Fibonacci and Lucas sequences for the case when $P = 6$ and $Q = 1$. Now we state some well known definition, theorems, and identities regarding the sequences $\{P_n\}, \{Q_n\}, \{B_n\}$, and $\{v_n\}$ that will be needed later.

Definition 2.1. Let a and b be integers, at least one of which is not zero. The greatest common divisor of a and b, denoted by (a, b), is the largest integer which divides both a and b.

Theorem 2.2. *Let γ and δ be the roots of the equation $x^2 - 2x - 1 = 0$. Then we have $P_n = \dfrac{\gamma^n - \delta^n}{2\sqrt{2}}$ and $Q_n = \gamma^n + \delta^n$ for all $n \in \mathbb{Z}$.*

Theorem 2.3. *Let α and β be roots of the characteristic equation $x^2 - 6x + 1 = 0$. Then $u_n = \dfrac{\alpha^n - \beta^n}{4\sqrt{2}}$ and $v_n = \alpha^n + \beta^n$ for all $n \in \mathbb{Z}$.*

From the theorems above, it is easily seen that $B_n = P_{2n}/2$ and $v_n = Q_{2n}$ for $n \geq 0$. We assume from this point on that $P = 6$ for the sequence $\{v_n\}$.

Most of the properties below for $\{P_n\}, \{Q_n\}, \{B_n\}$, and $\{v_n\}$ are well known (see, for example [28]). Properties (17) to (22) can be easily obtained by using Binet's formulas. Properties (23) to (26) can be found in [26, 27]. Properties (31) and (32) were proved in [18]. The proofs of the others are easy and will be omitted.

$$Q_n^2 - 8P_n^2 = 4(-1)^n, \tag{12}$$

$$v_n^2 - 32u_n^2 = 4, \tag{13}$$

$$P_{2n} = P_n Q_n \text{ and } B_{2n} = B_n v_n, \tag{14}$$

$$v_n = B_{n+1} - B_{n-1}, \tag{15}$$

$$Q_{2n} = Q_n^2 - 2(-1)^n, \tag{16}$$

$$v_m v_n + 32 B_m B_n = 2 v_{m+n}, \tag{17}$$

$$v_m v_n - 32 B_m B_n = 2 v_{m-n}, \tag{18}$$

$$B_m v_n + B_n v_m = 2 B_{m+n}, \tag{19}$$

$$B_m v_n - B_n v_m = 2 B_{m-n}, \tag{20}$$

$$v_{m+n}^2 - 32 B_m B_n v_{n+m} - 32 B_m^2 = v_n^2, \tag{21}$$

$$32 B_{m+n}^2 - 32 B_n B_{m+n} v_m - v_m^2 = -v_n^2, \tag{22}$$

$$(B_n, v_n) = 1 \text{ or } 2, \tag{23}$$

$$B_m | B_n \Leftrightarrow m | n, \tag{24}$$

$$v_m | v_n \Leftrightarrow m | n \text{ and } n/m \text{ is odd}, \tag{25}$$

$$v_m | B_n \Leftrightarrow m | n \text{ and } n/m \text{ is even}, \tag{26}$$

$$2 | B_n \Leftrightarrow 2 | n \Leftrightarrow 2 | P_n, \tag{27}$$

$$2 \nmid B_n \Leftrightarrow 2 \nmid n \Leftrightarrow 2 \nmid P_n, \tag{28}$$

$$2 | Q_n \text{ and } 2 | v_n. \tag{29}$$

Moreover, from (12) and (13), it is clear that

$$4 \nmid Q_n \text{ and } 4 \nmid v_n, \tag{30}$$

respectively.

If $d = (m, n)$, then

$$\begin{cases} (P_m, Q_n) = Q_d \text{ if } m/d \text{ is even} \\ (P_m, Q_n) = 1 \text{ otherwise.} \end{cases} \tag{31}$$

Let $m = 2^a m'$, $n = 2^b n'$, m', n' are odd, $a, b \geq 0$, and $d = (m, n)$. Then

$$(V_m, V_n) = \begin{cases} V_d \text{ if } a = b \\ 1 \text{ or } 2 \text{ if } a \neq b. \end{cases} \tag{32}$$

3. Some Theorems and Lemmas

In this section, we shall need some new theorems, lemmas, and corollaries. The following theorem gives us some information about the sum of the squares of balancing numbers. Since it is readily proved by using Binet's formulas, we omit the details.

Theorem 3.1. *Let B_k denotes the k^{th} balancing number. Then*

$$\sum_{k=1}^{n} B_k^2 = \frac{1}{32}(B_{2n+1} - (2n + 1)) \tag{33}$$

Hence, we have the following immediate corollary.

Corollary 3.2. *Let n be an odd positive integer. Then*

$$B_n \equiv n(\text{mod} 32). \tag{34}$$

Since the proof of the following lemma is straightfoward induction, we omit the details.

Lemma 3.3. *Let n be an even positive integer. Then*

$$B_n \equiv 3n(\text{mod} 32). \tag{35}$$

Now we can give the similar property for v_n as a result of Corollary 3.2, Lemma 3.3, and the identity (15).

Corollary 3.4. *Let n be a nonnegative integer. Then*

$$v_n \equiv \begin{cases} 2(\text{mod } 32) \textit{ if } n \textit{ is even} \\ 6(\text{mod } 32) \textit{ if } n \textit{ is odd} \end{cases}. \tag{36}$$

In the equations $x^2 - 8B_nxy - y^2 = \pm 2^r$, replacing x by x^2 and y by y^2, we come across the square terms of balancing sequence, Pell and Pell-Lucas sequences, and the sequence $\{v_n\}$. So, we must state some theorems concerning the square terms of these sequences.

The following theorem is given by Ljunggren [15] and also by Cohn [5].

Theorem 3.5. *If $n \geq 1$, then the equation $P_n = x^2$ has positive solutions $(n, x) = (1, 1)$ or $(7, 13)$.*

We state the following theorem from [24].

Theorem 3.6. *Let $P > 0$ and $Q = -1$. If $U_n = wx^2$ with $w \in \{1, 2, 3, 6\}$, then $n \leq 2$ except when $(P, n, w) = (2, 4, 3)$, $(2, 7, 1)$, $(4, 4, 2)$, $(1, 12, 1)$, $(1, 3, 2)$, $(1, 4, 3)$, $(1, 6, 2)$, or $(24, 4, 3)$.*

The following theorem is given by [19].

Theorem 3.7. *Let $P > 2$ and $Q = 1$. If $u_n = cx^2$ with $c \in \{1, 2, 3, 6\}$ and $n > 3$, then $(P, n, c) = (338, 4, 1)$ or $(3, 6, 1)$.*

The proof of the following theorem can be obtained from Theorem 3.7, but we here give a different proof.

Theorem 3.8. *Let n be a positive integer. There is no balancing number except 1 satisfying the equation $B_n = x^2$.*

Proof. Assume that $B_n = x^2$ for some $x > 0$. Suppose n is even. Then $n = 2k$ for some $k > 0$. By (14), it follows that

$$B_n = B_{2k} = B_k v_k = x^2. \tag{37}$$

Firstly, let k be odd. Then by Corollaries 3.2 and 3.4, it is seen that $B_k \equiv k \pmod{32}$ and $v_k \equiv 6 \pmod{32}$. Substituting these into (37) gives $x^2 \equiv 6k \pmod{32}$, implying that $x^2 \equiv 6k \pmod 8$. Since k is odd, $k \equiv 1, 3, 5, 7 \pmod 8$. Hence, we immediately have

$$x^2 \equiv 6k \equiv 2, 6 \pmod 8, \tag{38}$$

which is impossible since $x^2 \equiv 0, 1, 4 \pmod 8$. Secondly, let k be even. Then by (27), (29), and (23), it is clear that $(B_k, v_k) = 2$. Thus, x is even. Taking $B_k = 2a$ and $v_k = 2b$ with $(a, b) = 1$, we get $x^2 = B_k v_k = 4ab$, implying that $ab = (x/2)^2$. Then $a = u^2$ and $b = v^2$ for some $u, v > 0$. Hence, we have $B_k = 2a^2$. Using the fact that $B_k = P_{2k}/2$ gives $P_{2k} = (2u)^2$. By Theorem 3.5, we obtain $2k = 1$ or 7. But both of them are impossible in integers. Now suppose n is odd. Since $B_n = P_{2n}/2$, we have $P_{2n} = 2x^2$. By (14), it is clear that $P_nQ_n = 2x^2$. Furthermore, by the help of (12), (28), and (29), it can be seen that $(P_n, Q_n) = 1$. Then either

$$P_n = u^2,\ Q_n = 2v^2 \tag{39}$$

or

$$P_n = 2u^2,\ Q_n = v^2 \tag{40}$$

for some $u, v > 0$.

If (39) holds, then by Theorem 3.5, we obtain $n = 1$ or 7. When $n = 1$, $B_1 = 1 = x^2$ and therefore $x = 1$ is a solution. When $n = 7$, there is no integer x such that $B_7 = 40391 = x^2$. If (40) holds, then from (29), we see that v is even. This implies that $4|Q_n$, which is impossible by (30). This completes the proof. □

Theorem 3.9. *There is no positive integer x such that $v_n = x^2$.*

Proof. Assume that $v_n = x^2$ for some $x > 0$. By Corollary 3.4, it follows that $v_n \equiv 2, 6 (\text{mod } 8)$. Hence, $x^2 \equiv 2, 6 (\text{mod } 8)$, which is impossible. This completes the proof. □

Theorem 3.10. *If $n \geq 0$ and $x > 0$ are integers such that $v_n = 2x^2$, then $(n, x) = (0, 1)$.*

Proof. Assume that $v_n = 2x^2$ for some $x > 0$. Clearly, n is not odd, if it were then by Corollary 3.4, we get $2x^2 \equiv 6 (\text{mod } 8)$, which is impossible. So, n is even. Also, by Theorems 2.2 and 2.3, we see that $v_n = Q_{2n}$ and by (16), $Q_{2n} = Q_n^2 - 2$. Hence, we get $v_n = Q_n^2 - 2$. On the other hand, by (12), since $Q_n^2 - 8P_n^2 = 4$, we immediately have $v_n = 8P_n^2 + 2 = 2x^2$, implying that $4P_n^2 + 1 = x^2$. This shows that $x^2 - (2P_n)^2 = 1$. Solving this equation gives $x = 1$. Thus, $n = 0$. This completes the proof. □

Theorem 3.11. *There is no positive integer x such that $B_n = v_m x^2$.*

Proof. Assume that $B_n = v_m x^2$ for some $x > 0$. Since $v_m | B_n$, it follows from (26) that $m|n$ and $n = 2km$ for some $k > 0$. This implies by (14) that

$$B_n = B_{2km} = B_{km} v_{km} = v_m x^2. \tag{41}$$

Let k be odd. Then $B_{km} \frac{v_{km}}{v_m} = x^2$. Clearly, from (23), $d = \left(B_{km}, \frac{v_{km}}{v_m}\right) = 1$ or 2. If $d = 1$, then $B_{km} = a^2$, $v_{km} = v_m b^2$ for some $a, b > 0$. By Theorem 3.8, we have $km = 1$. This yields that $k = 1$, $m = 1$, and therefore $n = 2$. Hence, we conclude that $B_2 = v_1 x^2$, i.e., $6 = 2x^2$, which is impossible in integers. If $d = 2$, then $B_{km} = 2a^2$, $v_{km} = 2v_m b^2$ for some $a, b > 0$. From (29) and (30), since v_m is even and $4 \nmid v_m$, it is seen that $v_{km} = 2v_m b^2$ is impossible.

Now let k be even. Then from (41), we have $\frac{B_{km}}{v_m} v_{km} = x^2$. Using the fact that $\left(\frac{B_{km}}{v_m}, v_{km}\right) = 1$ or 2, we get

$$B_{km} = v_m a^2, \ v_{km} = b^2 \tag{42}$$

or

$$B_{km} = 2v_m a^2, \ v_{km} = 2b^2 \tag{43}$$

for some $a, b > 0$. Assume (42) is satisfied. Since k is even, it follows from Corollary 3.4 that $v_{km} = b^2 \equiv 2(\text{mod } 8)$, which is impossible. Assume (43) is satisfied. Then by Theorem 3.10, we get $km = 0$, implying that $n = 0$, which is impossible. This completes the proof. □

Theorem 3.12. *There is no positive integer x such that $B_n = 2v_m x^2$.*

Proof. Assume that $B_n = 2v_m x^2$ for some $x > 0$. Since $v_m | B_n$, it follows from (26) that $m|n$ and $n = 2km$ for some $k > 0$. This implies from (14) that

$$B_n = B_{2km} = B_{km} v_{km} = 2v_m x^2. \tag{44}$$

Let k be even. Then $\frac{B_{km}}{v_m} v_{km} = 2x^2$. Clearly, from (25), $d = \left(\frac{B_{km}}{v_m}, v_{km}\right) = 1$ or 2. If $d = 1$, then either

$$B_{km} = v_m a^2, \ v_{km} = 2b^2 \tag{45}$$

or

$$B_{km} = 2v_m a^2, \ v_{km} = b^2 \tag{46}$$

for some $a, b > 0$. It is obvious by Theorems 3.11 and 3.9 that both (45) and (46) are impossible. If $d = 2$, then either

$$B_{km} = 2v_m a^2, \ v_{km} = (2b)^2 \tag{47}$$

or

$$B_{km} = v_m (2a)^2, \ v_{km} = 2b^2 \tag{48}$$

for some $a, b > 0$. From (30), since $4 \nmid v_m$, it is seen that (47) is impossible. It is obvious by Theorem 3.11 that (48) is impossible.

Now let k be odd. Then from (44), we have $B_{km}\frac{v_{km}}{v_m} = 2x^2$. Clearly, from (25), $d = \left(B_{km}, \frac{v_{km}}{v_m}\right) = 1$ or 2. If $d = 1$, then

$$B_{km} = a^2,\ v_{km} = 2v_m b^2 \tag{49}$$

or

$$B_{km} = 2a^2,\ v_{km} = v_m b^2 \tag{50}$$

for some $a, b > 0$. If (49) holds, then by Theorem 3.8, it follows that $km = 1$, i.e., $k = 1$ and $m = 1$. This implies that $v_1 = 2v_1b^2$, which is impossible in integers. It is clear from Theorem 3.7 that (50) is impossible.

If $d = 2$, then

$$B_{km} = 2a^2,\ v_{km} = v_m(2b)^2 \tag{51}$$

or

$$B_{km} = (2a)^2,\ v_{km} = 2v_m b^2 \tag{52}$$

for some $a, b > 0$. (51) is impossible by Theorem 3.7. If (52) holds, then by Theorem 3.8, it follows that $km = 1$, i.e., $B_1 = 1 = (2a)^2$, which is impossible. This completes the proof. □

Theorem 3.13. *(Theorem 2 of* [16]*) The generalized Lucas sequence has at most one non-trivial square-class. Furthermore, if* $P \equiv 2 \pmod 4$, *then we have not non-trivial square-class except* $(v_1, v_2) = (338, 114242)$ *when* $P = 338$, $Q = 1$. *If* $P \equiv 0 \pmod 4$, *then we have not non-trivial square-classes when* $2 \nmid mn$ *or* $2|(m, n)$.

4. Positive Integer Solutions of the Equations $x^2 - 8B_nxy - 2y^2 = \pm 2^r$ in Terms of Balancing Numbers, Pell and Pell-Lucas Numbers, and the Terms of the Sequence $\{v_n\}$

In this section, we determine when the equations $x^2 - 8B_nxy - 2y^2 = \pm 2^r$, $x^2 - 8B_nxy^2 - 2y^4 = \pm 2^r$, and $x^4 - 8B_nx^2y - 2y^2 = \pm 2^r$ have positive integer solutions under the assumptions that $n, r \geq 0$. Moreover, we give all positive integer solutions of the equations above.

We omit the proof of the following theorem, as it is based a straightforward induction.

Theorem 4.1. *Let $k \geq 0$ be an integer. Then all nonnegative integer solutions of the equation $u^2 - 2v^2 = 2^k$ are given by*

$$(u,v) = \begin{cases} (2^{\frac{k-2}{2}} v_m,\ 2^{\frac{k+2}{2}} B_m) \text{ if } k \text{ is even} \\ (2^{\frac{k+1}{2}} P_{2m+1},\ 2^{\frac{k-3}{2}} Q_{2m+1}) \text{ if } k \text{ is odd} \end{cases}$$

with $m \geq 0$ and all nonnegative integer solutions of the equation $u^2 - 2v^2 = -2^k$ are given by

$$(u,v) = \begin{cases} (2^{\frac{k-2}{2}} Q_{2m+1},\ 2^{\frac{k}{2}} P_{2m+1}) \text{ if } k \text{ is even} \\ (2^{\frac{k+3}{2}} B_m,\ 2^{\frac{k-3}{2}} v_m) \text{ if } k \text{ is odd} \end{cases}$$

with $m \geq 0$.

Theorem 4.2. *If k is even, then all positive integer solutions of the equation $x^2 - 8B_n xy - 2y^2 = 2^k$ are given by $(x,y) = (2^{\frac{k}{2}} \frac{v_{n+m}}{v_n}, 2^{\frac{k+4}{2}} \frac{B_m}{v_n})$ with $m \geq 1$, $n|m$ and $m = 2rn$ for some $r > 0$. If k is odd, then the equation $x^2 - 8B_n xy - 2y^2 = 2^k$ has positive integer solutions only when $n = 0$ and the solutions are given by $(x,y) = (2^{\frac{k+1}{2}} P_{2m+1}, 2^{\frac{k-3}{2}} Q_{2m+1})$ with $m \geq 0$.*

Proof. Assume that $x^2 - 8B_n xy - 2y^2 = 2^k$ for some $x, y > 0$. Multiplying both sides of the equation by 4 and completing the square give $(2x - 8B_n y)^2 - (64B_n^2 + 8)y^2 = 2^{k+2}$. It is clear from (13) that $64B_n^2 + 8 = 2v_n^2$. Hence, the preceding equation becomes $(2x - 8B_n y)^2 - 2(v_n y)^2 = 2^{k+2}$. Let k be even. Then by Theorem 4.1, we obtain $|2x - 8B_n y| = 2^{\frac{k}{2}} v_m$ and $v_n y = 2^{\frac{k+4}{2}} B_m$. Since $4 \nmid v_n$ and v_n is even, it follows that $(4, v_n) = 2$ and therefore $\frac{v_n}{2} y = 2^{\frac{k+2}{2}} B_m$. It can be easily seen that $(\frac{v_n}{2}, 2^{\frac{k+2}{2}}) = 1$. Thus, we get $\frac{v_n}{2} | B_m$, that is $\frac{v_n}{2} | 2\frac{B_m}{2}$ for even m. Since $(\frac{v_n}{2}, 2) = 1$, $\frac{v_n}{2} | \frac{B_m}{2}$, implying that $v_n | B_m$. Therefore, we get from (26) that $n|m$ and $m = 2rn$ for some $r > 0$. Hence, we conclude that $y = 2^{\frac{k+4}{2}} \frac{B_m}{v_n}$. Suppose that $2x - 8B_n y = 2^{\frac{k}{2}} v_m$. Substituting the value of y into the preceding equation gives $x = 2^{\frac{k}{2}} \frac{v_m v_n + 32 B_m B_n}{2v_n}$. This implies from (17) that $x = 2^{\frac{k}{2}} \frac{v_{m+n}}{v_n}$. Now suppose that $2x - 8B_n y = -2^{\frac{k}{2}} v_m$. In a similar manner, we readily obtain $x = 2^{\frac{k}{2}} \frac{32 B_m B_n - v_m v_n}{2v_n}$. This gives from (18) $x = -2^{\frac{k}{2}} \frac{v_{m-n}}{v_n}$. But in this case, x is negative and so we omit it. As a consequence, we get $(x,y) = (2^{\frac{k}{2}} \frac{v_{n+m}}{v_n}, 2^{\frac{k+4}{2}} \frac{B_m}{v_n})$. Now let k be odd. Then by Theorem 4.1, we have $v_n y = 2^{\frac{k-1}{2}} Q_{2m+1}$. Since $v_n = Q_{2n}$ and $v_n | v_n y$, it follows that $Q_{2n} | 2^{\frac{k-1}{2}} Q_{2m+1}$, implying that $Q_{2n} | 2^{\frac{k+1}{2}} \frac{Q_{2m+1}}{2}$.

It can be easily seen from (32) that $(Q_{2n},\ Q_{2m+1}/2) = 1$. Hence, $Q_{2n}|2^{\frac{k+1}{2}}$ and this is possible only when $n = 0$. Thus, the main equation $x^2 - 8B_nxy - 2y^2 = 2^k$ turns into the equation $x^2 - 2y^2 = 2^k$, whose solutions are $(x, y) = (2^{\frac{k+1}{2}}P_{2m+1},\ 2^{\frac{k-3}{2}}Q_{2m+1})$ by Theorem 4.1.

Conversely, if k is even and

$$(x, y) = (2^{\frac{k}{2}}\frac{v_{n+m}}{v_n}, 2^{\frac{k+4}{2}}\frac{B_m}{v_n})$$

with $m \geq 1$, $n|m$ and $m = 2rn$ for some $r > 0$, then by (21), it follows that $x^2 - 8B_nxy - 2y^2 = 2^k$. And if k is odd and $(x, y) = (2^{\frac{k+1}{2}}P_{2m+1}, 2^{\frac{k-3}{2}}Q_{2m+1})$ with $m \geq 0$, then $x^2 - 8B_nxy - 2y^2 = 2^k$ with $n = 0$. This completes the proof. □

Theorem 4.3. *If k is even, then the equation $x^2 - 8B_nxy - 2y^2 = -2^k$ has positive integer solutions only when $n = 0$ and the solutions are given by $(x, y) = (2^{\frac{k-2}{2}}Q_{2m+1},\ 2^{\frac{k}{2}}P_{2m+1})$ with $m \geq 0$. If k is odd, then all positive integer solutions of the equation $x^2 - 8B_nxy - 2y^2 = -2^k$ are given by $(x, y) = (2^{\frac{k+5}{2}}\frac{B_{n+m}}{v_n},\ 2^{\frac{k-1}{2}}\frac{v_m}{v_n})$ with $m \geq 1$, $n|m$ and $m = (2r + 1)n$ for some $r > 0$.*

Proof. When k is even, the method is similar to that used in Theorem 4.2 for the case when k is odd. So, we immediately have $v_ny = 2^{\frac{k+2}{2}}P_{2m+1}$. Since $4 \nmid v_n$ by (30) and $v_ny = 2^{\frac{k+2}{2}}P_{2m+1}$, it clearly follows that $v_n|2^{\frac{k+2}{2}}P_{2m+1}$. Since $v_n = Q_{2n}$, it can be easily seen from (31) that $(v_n, P_{2m+1}) = 1$. So, $v_n|2^{\frac{k+2}{2}}$ and this is possible only when $n = 0$. Hence, the main equation $x^2 - 8B_nxy - 2y^2 = -2^k$ turns into the equation $x^2 - 2y^2 = -2^k$, whose solutions are $(x, y) = (2^{\frac{k-2}{2}}Q_{2m+1},\ 2^{\frac{k}{2}}P_{2m+1})$ by Theorem 4.1. Now let k be odd. Multiplying both sides of the equation $x^2 - 8B_nxy - 2y^2 = -2^k$ by 4 and completing the square give $(2x - 8B_ny)^2 - (64B_n^2 + 8)y^2 = -2^{k+2}$. Using the fact that $64B_n^2 + 8 = 2v_n^2$ by (13), the previous equation becomes $(2x - 8B_ny)^2 - 2(v_ny) = -2^{k+2}$. Then by Theorem 4.1, we obtain $|2x - 8B_ny| = 2^{\frac{k+5}{2}}B_m$ and $v_ny = 2^{\frac{k-1}{2}}v_m$. Since $4 \nmid v_n$ and v_n is even, it follows that $(4, v_n) = 2$ and therefore $\frac{v_n}{2}y = 2^{\frac{k-1}{2}}\frac{v_m}{2}$. It can be easily seen that $(\frac{v_n}{2}, 2^{\frac{k-1}{2}}) = 1$. Thus, we get $\frac{v_n}{2}|\frac{v_m}{2}$, that is $v_n|v_m$. This implies from (25) that $n|m$ and $m = (2r + 1)n$ for some $r > 0$. Hence, we get $y = 2^{\frac{k-1}{2}}\frac{v_m}{v_n}$. Assume first that $2x - 8B_ny = 2^{\frac{k+5}{2}}B_m$. Substituting the value of y into the previous equation, we have

$x = 2^{\frac{k+5}{2}} \frac{B_n v_m + B_m v_n}{2v_n}$. Then by (19), we conclude that $x = 2^{\frac{k+5}{2}} \frac{B_{m+n}}{v_n}$. Now assume that $2x - 8B_n y = -2^{\frac{k+5}{2}} B_m$. In a similar manner, we get $x = 2^{\frac{k+3}{2}} \frac{B_n v_m - B_m v_n}{2v_n}$. By (20), we conclude that $x = 2^{\frac{k+3}{2}} \frac{B_{n-m}}{v_n}$. But in this case since $n - m < 0$, it follows from (9) that $B_{n-m} < 0$ and therefore we see that x is negative. So, we omit it. Conversely, if k is even and $(x, y) = (2^{\frac{k-2}{2}} Q_{2m+1}, 2^{\frac{k}{2}} P_{2m+1})$ with $m \geq 0$, then by (12), $x^2 - 8B_n xy - 2y^2 = -2^k$ with $n = 0$. And if k is odd and $(x, y) = (2^{\frac{k}{2}} \frac{B_{n+m}}{v_n}, 2^{\frac{k+4}{2}} \frac{v_m}{v_n})$ with $m \geq 1$, $n|m$ and $m = (2r+1)n$ for some $r > 0$, then by (22), $x^2 - 8B_n xy - 2y^2 = -2^k$. This completes the proof. □

Now we consider the equations $x^2 - 8B_n xy^2 - 2y^4 = \pm 2^k$ and $x^4 - 8B_n x^2 y - 2y^2 = \pm 2^k$, respectively.

Theorem 4.4. *If* $k \equiv 0, 2, 3 \pmod 4$, *then the equation* $x^2 - 8B_n xy^2 - 2y^4 = 2^k$ *has no solutions* x *and* y. *If* $k \equiv 1 \pmod 4$, *then the equation* $x^2 - 8B_n xy^2 - 2y^4 = 2^k$ *has positive integer solutions only when* $n = 0$ *and the solution is given by* $(x, y) = (2^{\frac{k+1}{2}}, 2^{\frac{k-1}{4}})$.

Proof. Firstly, assume that k is even in $x^2 - 8B_n xy^2 - 2y^4 = 2^k$. Then by Theorem 4.2, it follows that $(x, y^2) = (2^{\frac{k}{2}} \frac{v_{n+m}}{v_n}, 2^{\frac{k+4}{2}} \frac{B_m}{v_n})$ with $m \geq 1$, $n|m$ and $m = 2rn$ for some $r > 0$. Hence, we obtain $y^2 = 2^{\frac{k+4}{2}} \frac{B_m}{v_n}$. Now we divide the proof into two cases.

Case 1 : Let $k \equiv 0 \pmod 4$. Then the equation $y^2 = 2^{\frac{k+4}{2}} \frac{B_m}{v_n}$ clearly follows that $B_m = v_n u^2$, which is impossible by Theorem 3.11.

Case 2 : Let $k \equiv 2 \pmod 4$. Then the equation $y^2 = 2^{\frac{k+4}{2}} \frac{B_m}{v_n}$ yields that $B_m = 2v_n u^2$, which is impossible by Theorem 3.12.

Secondly, assume that k is odd. Then by Theorem 4.2, it follows that $(x, y^2) = (2^{\frac{k+1}{2}} P_{2m+1}, 2^{\frac{k-3}{2}} Q_{2m+1})$ with $m \geq 0$. This shows that $y^2 = 2^{\frac{k-3}{2}} Q_{2m+1}$. Again dividing the remainder of the proof into two cases, we have

Case 1 : Let $k \equiv 3 \pmod 4$. Then we obtain $Q_{2m+1} = u^2$ for some $u > 0$. By (29), since Q_{2m+1} is even, it follows that u is even and therefore $4|Q_{2m+1}$, which is impossible by (30).

Case 2 : Let $k \equiv 1 \pmod 4$. Then we have $Q_{2m+1} = 2u^2$ for some $u > 0$. By Theorem 3.6, we get $m = 0$. Thus $y^2 = 2^{\frac{k-1}{2}}$, implying that $y = 2^{\frac{k-1}{4}}$ and $x = 2^{\frac{k+1}{2}}$. This completes the proof. □

Theorem 4.5. *If* $k \equiv 2, 3 \pmod 4$, *then the equation* $x^2 - 8B_n xy^2 - 2y^4 = -2^k$ *has no solutions* x *and* y. *If* $k \equiv 0 \pmod 4$, *then all* *positive integer solutions of the equation* $x^2-8B_nxy^2-2y^4=-2^k$ *are given by* $(x,y)=(2^{\frac{k}{2}},2^{\frac{k}{4}})$ *or* $(x,y)=(239\cdot 2^{\frac{k}{2}},13\cdot 2^{\frac{k}{4}})$. *If* $k\equiv 1(\text{mod } 4)$, *then there is only one positive integer solution of the equation* $x^2-8B_nxy^2-2y^4=-2^k$ *given by* $(x,y)=(2^{\frac{k+5}{2}}B_n,2^{\frac{k-1}{4}})$.

Proof. Assume that k is even in $x^2-8B_nxy^2-2y^4=-2^k$. Then by Theorem 4.3, it follows that $(x,y^2)=(2^{\frac{k-2}{2}}Q_{2m+1},2^{\frac{k}{2}}P_{2m+1})$ with $m\geq 0$. Hence, we obtain $y^2=2^{\frac{k}{2}}P_{2m+1}$. Dividing the proof into two cases, we have

Case 1 : Let $k\equiv 0(\text{mod } 4)$. Then from the equation $y^2=2^{\frac{k}{2}}P_{2m+1}$, we obtain $P_{2m+1}=u^2$ for some $u>0$. By Theorem 3.5, we get $2m+1=1$ or $2m+1=7$. This implies that $m=0$ or $m=3$. If $m=0$, then we immediately have $x=2^{\frac{k}{2}}$ and $y=2^{\frac{k}{4}}$. If $m=3$, then we obtain $x=239\cdot 2^{\frac{k}{2}}$ and $y=13\cdot 2^{\frac{k}{4}}$.

Case 2 : Let $k\equiv 2(\text{mod } 4)$. Then the equation $y^2=2^{\frac{k}{2}}P_{2m+1}$ becomes $P_{2m+1}=2u^2$, which is impossible since $2\nmid P_{2m+1}$ by (28).

Now assume that k is odd. Then by Theorem 4.3, we have

$$(x,y^2)=(2^{\frac{k+5}{2}}\frac{B_{n+m}}{v_n},2^{\frac{k-1}{2}}\frac{v_m}{v_n})$$

with $m\geq 1$, $n|m$ and $m=(2r+1)n$ for some $r>0$. Hence, we obtain $y^2=2^{\frac{k-1}{2}}\frac{v_m}{v_n}$. Now we divide the remainder of the proof into two cases.

Case 1 : Let $k\equiv 1(\text{mod } 4)$. Then the equation $y^2=2^{\frac{k-1}{2}}\frac{v_m}{v_n}$ implies that $v_m=v_nu^2$ for some $u>0$. By Theorem 3.13, this is possible only when $n=m$. Hence, we get $x=2^{\frac{k+5}{2}}\frac{B_{2n}}{v_n}$. Also using (14) for the value of x gives that $x=2^{\frac{k+5}{2}}B_n$. Thus, we conclude that $(x,y)=(2^{\frac{k+5}{2}}B_n,2^{\frac{k-1}{4}})$.

Case 2 : Let $k\equiv 3(\text{mod } 4)$. So, we immediately have $v_{n+m}=2v_nu^2$ for some $u>0$. Since v_n is even by (29), it is clear that $4|v_{n+m}$. But this is impossible by (30). This completes the proof. □

Theorem 4.6. *If* $k\equiv 0,1,2(\text{mod } 4)$, *then the equation* $x^4-8B_nx^2y-2y^2=2^k$ *has no positive integer solutions* x *and* y. *If* $k\equiv 3(\text{mod } 4)$, *then all positive integer solutions of the equation* $x^4-8B_nx^2y-2y^2=2^k$ *are given by* $(x,y)=(2^{\frac{k+1}{4}},2^{\frac{k-1}{2}})$ *or* $(x,y)=(13\cdot 2^{\frac{k+1}{4}},239\cdot 2^{\frac{k-1}{2}})$.

Proof. Assume that $x^4-8B_nx^2y-2y^2=2^k$ for some positive integers x and y. If k is even, then by Theorem 4.2, we have $(x^2,y)=(2^{\frac{k}{2}}\frac{v_{n+m}}{v_n},2^{\frac{k+4}{2}}\frac{B_m}{v_n})$ with $m\geq 1$, $n|m$ and m/n is even. Hence, we get $x^2=2^{\frac{k}{2}}\frac{v_{n+m}}{v_n}$.

Case 1 : Let $k \equiv 0 \pmod 4$. We readily obtain from $x^2 = 2^{\frac{k}{2}} \frac{v_{n+m}}{v_n}$ that $v_{n+m} = v_n u^2$ for some $u > 0$. By Theorem 3.13, this is possible only when $n + m = n$, implying that $m = 0$, which contradicts the fact that $m \geq 1$.

Case 2 : Let $k \equiv 2 \pmod 4$. So, we immediately have $v_{n+m} = 2v_n u^2$ for some $u > 0$. Since v_n is even by (29), we see that $4|v_{n+m}$, which is impossible by (30).

If k is odd, then by Theorem 4.2, we have

$$(x^2, y) = (2^{\frac{k+1}{2}} P_{2m+1}, 2^{\frac{k-3}{2}} Q_{2m+1})$$

with $m \geq 0$. This implies that $x^2 = 2^{\frac{k+1}{2}} P_{2m+1}$.

Case 1 : Let $k \equiv 1 \pmod 4$. Then form $x^2 = 2^{\frac{k+1}{2}} P_{2m+1}$, we obtain $P_{2m+1} = 2u^2$, which is impossible since $2 \nmid P_{2m+1}$ by (28).

Case 2 : Let $k \equiv 3 \pmod 4$. Then the equation $x^2 = 2^{\frac{k+1}{2}} P_{2m+1}$ gives that $P_{2m+1} = u^2$ for some $u > 0$. By Theorem 3.5, we get $2m + 1 = 1$ or $2m + 1 = 7$, implying that $m = 0$ or $m = 3$. Substituting these values of m into $(x^2, y) = (2^{\frac{k+1}{2}} P_{2m+1}, 2^{\frac{k-3}{2}} Q_{2m+1})$, we conclude that $(x, y) = (2^{\frac{k+1}{4}}, 2^{\frac{k-1}{2}})$ or $(x, y) = (13 \cdot 2^{\frac{k+1}{4}}, 239 \cdot 2^{\frac{k-1}{2}})$. This completes the proof. □

Theorem 4.7. *If $k \equiv 1, 2, 3 \pmod 4$, then the equation $x^4 - 8B_n x^2 y - 2y^2 = -2^k$ has no positive integer solutions x and y. If $k \equiv 0 \pmod 4$, then there is only one positive integer solution of the equation $x^4 - 8B_n x^2 y - 2y^2 = -2^k$ given by $(x, y) = (2^{\frac{k}{4}}, 2^{\frac{k}{2}})$.*

Proof. Assume that $x^4 - 8B_n x^2 y - 2y^2 = -2^k$ for some positive integers x and y. If k is even, then by Theorem 4.3, it follows that $(x^2, y) = (2^{\frac{k-2}{2}} Q_{2m+1}, 2^{\frac{k}{2}} P_{2m+1})$ with $m \geq 0$. Hence, we get $x^2 = 2^{\frac{k-2}{2}} Q_{2m+1}$.

Case 1 : Let $k \equiv 0 \pmod 4$. Hence, we immediately have from $x^2 = 2^{\frac{k-2}{2}} Q_{2m+1}$ that $Q_{2m+1} = 2u^2$. By Theorem 3.6, we get $m = 0$. This yields that $(x, y) = (2^{\frac{k}{4}}, 2^{\frac{k}{2}})$.

Case 2 : Let $k \equiv 2 \pmod 4$. Hence, we readily obtain $Q_{2m+1} = u^2$ for some $u > 0$. Since Q_{2m+1} is even by (29), it is clear that u is even and therefore $4|Q_{2m+1}$, which is impossible by (30).

If k is odd, then by Theorem 4.3, it follows that
$(x^2, y) = (2^{\frac{k+5}{2}} \frac{B_{n+m}}{v_n}, 2^{\frac{k-1}{2}} \frac{v_m}{v_n})$ with $m \geq 1$, $n|m$ and $m = (2r + 1)n$ for some $r > 0$. This implies that $x^2 = 2^{\frac{k+5}{2}} \frac{B_{n+m}}{v_n}$.

Case 1 : Let $k \equiv 1 (\text{mod } 4)$. Then from $2^{\frac{k+5}{2}} \frac{B_{n+m}}{v_n}$, we obtain $B_{n+m} = 2v_n u^2$, which is impossible by Theorem 3.12.

Case 2 : Let $k \equiv 3 (\text{mod } 4)$. Then we have $B_{n+m} = v_n u^2$, which is also impossible by Theorem 3.11. This completes the proof. □

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