**Irish Math. Soc. Bulletin**
Number 72, Winter 2013, 101–105
ISSN 0791-5578

# PROBLEMS

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## Problems

The first problem, posed by Niall Ryan of the University of Limerick, relates to the theory of complete elliptic integrals.

**Problem 72.1.** For each integer $n \geqslant 0$, let

$$S_n = \sum_{m=0}^{\infty} \frac{K_{2m}}{2m+n+1} + \sum_{\substack{m=0 \\ 2m\neq n}}^{\infty} \frac{K_{2m}}{2m-n},$$

where

$$K_{2m} = \left[\frac{(2m)!}{2^{2m}(m!)^2}\right]^2.$$

Prove that

$$S_n = \begin{cases} 0, & n \text{ odd}, \\ 2K_n\left(\log 2 - \displaystyle\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\right), & n \text{ even}. \end{cases}$$

The second problem was contributed by Finbarr Holland of University College Cork.

**Problem 72.2.** Prove that the integral

$$\int_0^{\infty} \frac{x\sin x}{2+2\cos x - 2x\sin x + x^2}\,dx$$

exists as a Riemann integral, but not as a Lebesgue integral, and determine its value as a Riemann integral.

The third problem was posed by Tom Moore of Bridgewater State University, USA.

Received on 10-12-2013.

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**Problem 72.3.** For $n = 1, 2, \ldots,$ the triangular numbers $T_n$ and square numbers $S_n$ are given by the formulas

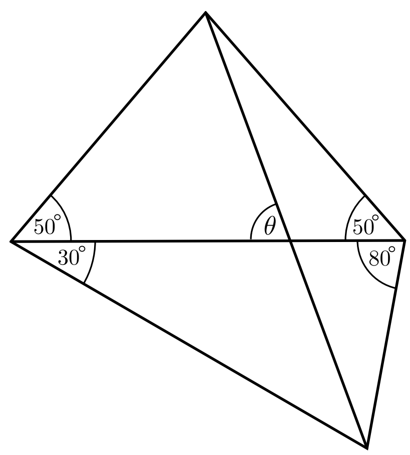
$$T_n = \frac{n(n+1)}{2} \quad \text{and} \quad S_n = n^2.$$

It is well known that every even perfect number is a triangular number. Prove that every even perfect number greater than 6 can be expressed as the sum of a triangular number and a square number.

## Solutions

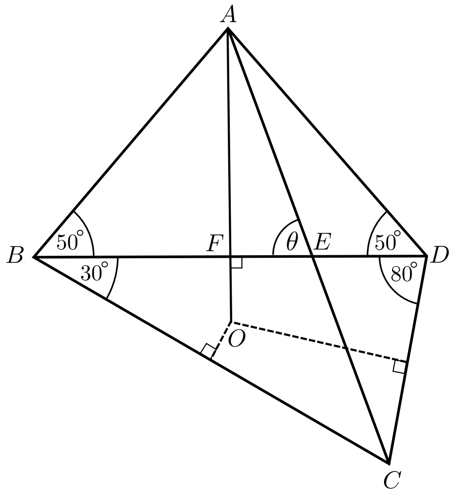
Here are solutions to the problems from *Bulletin* Number 70. The first solution was contributed by Prithwijit De of the Homi Bhabha Centre for Science Education, Mumbai, India. It was also solved by the North Kildare Mathematics Problem Club, by applying the sine rule for triangles.

*Problem 70.1.* Find $\theta$.




*Solution 70.1.* Label the vertices of the quadrilateral $A$, $B$, $C$, and $D$, as shown below, and let $E = AC \cap BD$. In triangle $ABD$, draw the altitude at $A$, and extend it beyond the point $F$ of intersection of this line with $BD$. Since the altitude is a perpendicular bisector of the isosceles triangle $ABD$, it follows that any point on the line $AF$ is equidistant from $B$ and $D$. In particular, the circumcentre $O$ of the triangle $BCD$ lies on the line $AF$. As $BCD$ is an acute-angled triangle, the point $O$ lies within $BCD$.

Observe that $\angle BOD = 2\angle BCD = 140°$. Observe also that $OB$, $OC$, and $OD$ are equal in length. Therefore $\angle OBD = \angle ODB = 20°$, so $\angle OBA = \angle ODA = 70°$. Since the triangles $OBF$ and $ODF$ are congruent, we see that $\angle FOB = \angle FOD = 70°$. Hence




$\angle AOD = \angle ADO = 70°$. As $OC$ and $OD$ are equal in length and $\angle ODC = 60°$, triangle $OCD$ is equilateral. Therefore triangles $AOC$ and $ADC$ are congruent. Therefore

$$\angle OAC = \angle DAC = \frac{1}{2}\angle OAD = 20°.$$

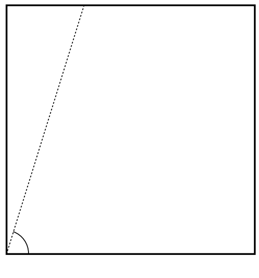
Finally, we see that

$$\theta = \angle AEF = 90° - \angle FAE = 70°.$$

□

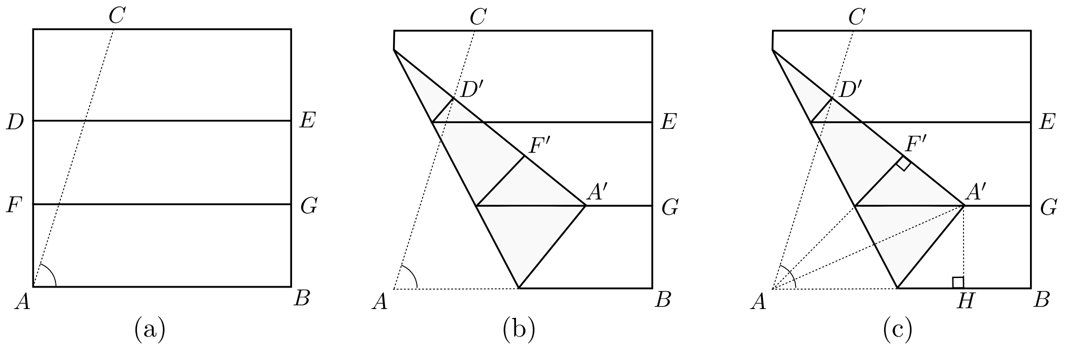
The second problem was solved by the North Kildare Mathematics Problem Club and the proposer. As well as giving an explicit solution, the North Kildare Mathematics Problem Club refer to page 266 of volume 1 of Thomas Heath's annotated translation of *Euclid's Elements* (Dover Publications, 1956) where there is a discussion of angle-trisection methods of the Ancient Greeks. The solution described in brief below is attributed by George Martin (page 155 of *Geometric Constructions*, Springer-Verlag, 1998) to Hisashi Abe, who published his method in Japanese. The construction can be found on many websites about paper folding.

*Problem 70.2*. Fold a square piece of paper to create an angle, as shown below.



Using a sequence of folds, trisect this angle.

*Solution 70.2.* Label points $A$, $B$, and $C$ as shown in figure (a) below. Make a horizontal fold $DE$. Make another horizontal fold $FG$




such that $AF$ and $FD$ are equal in length, as shown in figure (a). Now fold the corner $A$ of the paper so that the point $D$ touches the line $AC$ at a point $D'$ and point $A$ touches the line $FG$ at a point $A'$, as shown in figure (b). Let $F'$ be the image point of $F$ under this fold.

The line $AF'$ is the image of the line $A'F$ under reflection in the crease of the fold just performed. Therefore $AF'$ is perpendicular to $A'F'$. It is now straightforward to check that the three right-angled triangles $AF'D'$, $AA'F'$, and $AHA'$ are congruent. In particular, $\angle A'AH = \frac{1}{3}\angle CAB$.

□

The solution to the third problem is an amalgamation of solutions submitted by Niall Ryan of the University of Limerick and the North Kildare Mathematics Problem Club. The problem was also solved by the proposer.

*Problem 70.3.* Let $n$ be a positive integer. Prove that

$$\sum_{m=-\infty}^{\infty} \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in|x|} e^{-imx}\, dx \right| = 1 + \frac{4}{\pi} \sum_{k=1}^{n} \frac{1}{2k-1}\,.$$

*Solution 70.3.* For each integer $m$ we let

$$c_n(m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in|x|} e^{-imx} dx.$$

Evaluating the integral, we find that

$$|c_n(m)| = \begin{cases} \dfrac{2n}{|n^2 - m^2|\pi}, & m - n \text{ odd}, \\ \frac{1}{2}, & m = \pm n, \\ 0, & \text{otherwise.} \end{cases}$$

Since $|c_n(n)| + |c_n(-n)| = 1$, it remains to show that

$$\sum_{\substack{m=-\infty \\ m \neq \pm n}}^{\infty} |c_n(m)| = \frac{4}{\pi} \sum_{k=1}^{n} \frac{1}{2k-1}.$$

Let

$$S_1 = \sum_{m<-n} |c_n(m)|, \quad S_2 = \sum_{-n<m<n} |c_n(m)|, \quad S_3 = \sum_{m>n} |c_n(m)|.$$

Using the change of variable $l = -m$, we see that $S_1 = S_3$. We can evaluate $S_3$ by writing it as a telescoping sum:

$$S_3 = \frac{1}{\pi} \sum_{\substack{m>n \\ m-n \text{ odd}}} \left( \frac{1}{m-n} - \frac{1}{m+n} \right) = \frac{1}{\pi} \sum_{k=1}^{n} \frac{1}{2k-1}.$$

Also,

$$S_2 = \frac{1}{\pi} \sum_{\substack{-n<m<n \\ m-n \text{ odd}}} \left( \frac{1}{n-m} + \frac{1}{n+m} \right) = \frac{2}{\pi} \sum_{k=1}^{n} \frac{1}{2k-1}.$$

Therefore

$$S_1 + S_2 + S_3 = \frac{4}{\pi} \sum_{k=1}^{n} \frac{1}{2k-1},$$

as required. □

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com`.

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