Irish Math. Soc. Bulletin
Number 72, Winter 2013, 31–32
ISSN 0791-5578

ON (m,p)-ISOMETRIC OPERATORS AND OPERATOR TUPLES ON NORMED SPACES

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This is an abstract of the PhD thesis *On (m,p)-isometric operators and operator tuples on normed spaces* written by Philipp Hoffmann under the supervision of Dr Michael Mackey and Dr Mícheál Ó Searcóid at the School of Mathematical Sciences, UCD and submitted in May 2013.

The thesis deals with two kinds of tuples of commuting, bounded linear operators $(T_1, ..., T_d) =: T \in B(X)^d$ on a normed (real or complex) vector space X.

The first kind are so-called *(m,p)-isometric tuples*, which, given $m \in \mathbb{N}_0$ and $p \in (0,\infty)$, are defined by satisfying the following:

$$\sum_{k=0}^{m}(-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!}\|T^\alpha x\|^p = 0, \quad \forall x \in X.$$

Here, $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}_0^d$ is a multi-index, $|\alpha|$ the sum of its entries, $\frac{k!}{\alpha!} = \frac{k!}{\alpha_1!\cdots\alpha_d!}$ a multinomial coefficient and $T^\alpha := T_1^{\alpha_1}\cdots T_d^{\alpha_d}$.

In the case of $d = 1$, these objects are called *(m,p)-isometric operators* and have been introduced by Agler [1] on Hilbert spaces (with $p = 2$) and Bayart [3] on Banach spaces. For general $d \geq 1$, these tuples have been introduced on Hilbert spaces (with $p = 2$) by Gleason and Richter [5]. The first main result is as follows:

Theorem 1. *$T \in B(X)^d$ is an (m,p)-isometric tuple if, and only if, there exists a (necessarily unique) family of polynomials $f_x : \mathbb{R} \to \mathbb{R}$, $x \in X$, of degree $\leq m-1$ with $f_x|_{\mathbb{N}_0} = \left(\sum_{|\alpha|=n} \frac{n!}{\alpha!}\|T^\alpha x\|^p\right)_{n\in\mathbb{N}_0}$.*

2010 *Mathematics Subject Classification.* 47B99, 05A10.
Key words and phrases. normed space, Banach space, operator tuple, m-isometry, (m,p)-isometry, (m,∞)-isometry, generalised isometry.
Received on 7-10-2013.
Support from University College Dublin through a Research Demonstratorship and a bursary is gratefully acknowledged.
I would further like to thank Dr Michael Mackey and Dr Mícheál Ó Searcóid for their supervision and support.

This extends earlier results by Agler and Stankus [2], Gleason and Richter [5], Bermúdez, Martinón and Negrín [4] and Bayart [3].

The second main result follows as a corollary from Theorem 1:

Theorem 2. *Let $m \geq 2$ and let $T \in B(X)$ be an (m,p)-isometric operator and not an $(m-1,p)$-isometric operator. Then there exist $m_0 \geq 2$ and $p_0 \in (0,\infty)$ such that T is a (μ,q)-isometric operator (and not a $(\mu-1,q)$-isometric operator) if, and only if, $(\mu,q) = (k(m_0-1)+1, kp_0)$ for some $k \in \mathbb{N}_0$ with $k \geq 1$.*

The second kind of objects studied, are so-called (m,∞)-*isometric tuples*, which, given $m \in \mathbb{N}_0$ with $m \geq 1$, are defined by satisfying

$$\max_{\substack{|\alpha|=0,\dots,m \\ |\alpha| \text{ even}}} \|T^\alpha x\| = \max_{\substack{|\alpha|=0,\dots,m \\ |\alpha| \text{ odd}}} \|T^\alpha x\|, \quad \forall x \in X.$$

The main result on these operator tuples is as follows:

Theorem 3. *Let $T \in B(X)^d$ be an (m,∞)-isometric tuple. Then T is a $(1,\infty)$-isometric tuple under the equivalent norm $|.|_\infty$, given by $|x|_\infty = \max_{\alpha \in \mathbb{N}_0^d} \|T^\alpha x\| = \max_{|\alpha|=0,\dots,m-1} \|T^\alpha x\|$, for all $x \in X$.*

Finally, we prove some statements on operator tuples (or operators) which are both, (m,p)-isometric and (μ,∞)-isometric:

Theorem 4. *Let $T = (T_1,\dots,T_d) \in B(X)^d$ be an (m,p)-isometric and a (μ,∞)-isometric tuple.*

(i) *If $d = 1$, then $T \in B(X)$ is an isometry.*
(ii) *If $m = 1$ or $\mu = 1$ or $d = m = \mu = 2$, one operator T_{j_0} is an isometry and all other operators satisfy $T_j^m = 0$ for $j \neq j_0$.*

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