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CONTENTS

EDITORIAL

The new exchange material announced in the last two issues has now been allocated by the IMS Committee following receipt of offers to house it, as follows:

Iranian Journal of Mathematical Chemistry $\to$ NUIM.
Note di Matematica (Salento) $\to$ NUIM.
International J. Group Theory $\to$ NUIG.
Transaction on Combinatorics $\to$ NUIM.

The Committee decided that the data on the current storage locations of exchange material should be provided to members. This information is in a file which may be downloaded from
`http://www.maths.nuim.ie/documents/uploads/IMSWebsiteList.pdf`

Organisers of scientific meetings supported by the Society are encouraged to provide accounts for the Bulletin in time for the Winter issue.

Ph.D. students completing their programme are reminded to submit their thesis abstract as soon as the thesis has been approved. Supervisors are asked to encourage students to do this.

Members may find the Newsletter of the EMS interesting and useful. The EMS Newsletter is the journal of record of the European Mathematical Society and is one of the most widely read periodicals in Europe dealing with matters of interest to the mathematical community. The Newsletter features announcements about meetings and conferences, articles outlining current trends in scientific development, reports on member societies, and many other informational items. It may be read online at:
`http://www.ems-ph.org/journals/journal.php?jrn=news`

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WHY WE NEED MATHEMATICS IN THE RPE ERA

RICHARD M. TIMONEY

Abstract. A thriving scientific eco-system at the national level in Ireland requires many kinds of interaction between many kinds of scientists, engineers, economic analysts and industry, and in this scenario Mathematics has a vital rôle to play. The M for Mathematics in the acronym STEM must include the new knowledge that comes from ongoing research.

1. Background

The Research Prioritisation Steering Group which met between October 2010 and September 2011 has produced a series of recommendations that are now being implemented as part of national policy. In particular 14 themes have been identified in the report as priorities for public investment.

The report [1, Table 1] states that the underlying axioms (or high level criteria) used for the choices were as follows:

> The Four High Level Criteria for Assessment of Priority Areas
>
> *(1) The priority area is associated with a large global market or markets in which Irish-based enterprise already compete or can realistically compete*
> *(2) Publicly performed R&D in Ireland is required to exploit the priority area and will complement private sector research and innovation in Ireland*
> *(3) Ireland has built or is building (objectively measured) strengths in research disciplines relevant to the priority area*
> *(4) The priority area represents an appropriate approach to a recognised national challenge and/or a global challenge to which Ireland should respond*

Received on 26-6-2013.
This note has benefitted substantially from suggestions made by S. Buckley (NUIM), J. Carroll (DCU), J. Grannell (UCC) and P. Lynch (UCD)..

For ease of reference, we list the priority areas in appendix A.

2. The importance of Mathematics

A recent UK report [3] (compiled for the EPSRC) is instructive in that it attempts to quantify (in accounting terms) the benefits accruing to the UK economy from Mathematics and graduates in Mathematics.

In that regard it acknowledges being influenced by a website [4] of the IMA (Institute of Mathematics and its Applications) which in turn identifies some mathematical applications that are already realised and some that are aspirations. Headings include "Fighting Infections with Symmetry" (priority area G), "Keeping Hearts Pumping" (priority area E), "Networking for the Future" (priority area A), "Taking decisions, not risks" and "Hydrogen: the fuel in water". Each of the (more than 30) headings is fleshed out in a two page flyer detailing the ideas behind the catchy titles, but we will not discuss them systematically here.

The report [3] comes up with a working definition of Mathematical Sciences Research (MSR), which is worth noting. Here is an abbreviated quote.

> *Mathematical science includes such diverse areas as algebra and analysis, dynamical systems, mathematical physics, operational research, probability and statistics: areas which touch on all aspects of everyday life.*
>
> *For the purposes of this study we use the working definition of MSR as being high-end research in mathematical sciences carried out in academic institutions, research centres, businesses, individuals and Government that adds to the store of accumulated mathematical knowledge.*

3. Commentary

The conclusions reached in [1] are broadly in line with ideas adopted in other countries and fundamentally fail to recognise the essential ingredient of inspiration provided by highly motivated individuals, and the long-established record of surprises.

The M of STEM is not explicitly treated in the report, despite its manifest involvement in almost all modern developments. The

EPSRC in the UK does include Mathematics as a theme with the explanation

> *Research in the mathematical sciences is a key element for the advancement of all areas of science and technology, as well as being a vital area of science in itself. Our aim is to sustain core research capability, while promoting transformative and cross-disciplinary research that has the potential for significant impact.*

(`http://goo.gl/TDbzg` or `http://www.epsrc.ac.uk/` (May 2013).)

In the Irish context we have quite a few notable developments that were certainly not anticipated by many — for example John Boyd Dunlop and his commercialisation of the pneumatic tyre, Harry Ferguson of tractor and four-wheel drive fame, George Boole's logic (mid 19th century) which now pervades the digital era, William Rowan Hamilton's theories that now form the essential building blocks for many aspects of Physics and are used in computer games and graphics (see the successful Irish company Havok, `www.havok.com`, an Intel company now), and the large deviation theory which John T. Lewis used as the basis for the successful start-up Corvil (`http://www.corvil.com/`). The CTO and the Chief Scientist at Corvil hold PhDs in Mathematics. (The work of Corvil has a mathematical base and is included under priority areas A, B and N. The video games industry worldwide was valued at 65 billion US dollars in 2011, and at 248 million Euro in Ireland in 2012.)

Professor Arieh Iserles, a recent winner of the IMA-LMS David Crighton Medal, discusses (March 2013) the track record of predictions for technological advances over the past 60 years, comparing 1953 projections to 2013 reality, with striking lack of overlap between the lists. (See report at [2, p. 15].) First he discusses some examples including FFT (Fast Fourier Transform, Gauss 1805) and RSA encryption (named after a patent by Rivest, Shamir and Adelman in 1977), two further examples of the essential rôle of Mathematics. He uses these to argue against strategic choices, but says "a strategy for mathematics could be implemented at a higher level: research groups, departments/institutions and national/international levels".

He proceeds to guess that "the main driving force has shifted from physics to information over the past 50 years – the information age is here", something that would support the notion that Mathematics might be vital for items A, B and C in the RPE list (see appendix A). In assertions that would perhaps make the late David Crighton, a computational fluid dynamicist who worked on aeroacoustics, rise from his grave in indignation, he is quoted as stating:

> *Information is the future and is a multidisciplinary mathematical enterprise, for example machine learning currently needs input from Approximation Theory, Bayesian Statistics, Computer Science, Functional Analysis, Graph Theory, Non-Parametric Statistics, Optimisation, Random Matrix Theory.*

4. Case studies

As mentioned above, the web site [4] has quite a few well-documented examples of the importance of Mathematics in relation to recent and prospective developments.

In the Irish context, Peter Lynch (of UCD and formerly of Met Éireann) has written a series of short pieces aimed at the general public which are published in the Irish Times (fortnightly since July 2012) and he also has a blog [5] which contains links to his Irish Times articles.

For instance, his Irish Times column on March 7th 2013 was entitled *X-ray vision: How CT scans changed medicine* and discusses the mathematical basis for a CT scanner installed recently in Tallaght Hospital. In his blog (at `http://thatsmaths.com/2013/03/07/ct-scans-and-the-radon-transform/`) he goes into more mathematical detail and gives some recent bibliographic references. This is a topic where mathematical progress could well result in improved medical outcomes for patients from the next generation of scanners or from an update of their software. The particular topics to be addressed include clearer imaging via removal of artifacts (which might involve improved mathematical treatment of the raw data as well as changes to the design of the machine) and the possibility of lower power (resulting in lower exposure of patients to electromagnetic radiation) which can be expected to result from improvements in the underlying theory (but which would also be likely to involve hardware redesign). This should be included in priority areas E and F.

The website [6] also discusses scanning (in an item headed *How Math Can Save Your Life: Tomography* by Dr. Chris Budd and

Dr. Cathryn Mitchell; both are at the University of Bath, Budd is a mathematician and Mitchell an engineer).

While he was at Met Éireann, Peter Lynch was deeply involved in the development of HIRLAM (acronym for the High Resolution Limited Area Model), a Numerical Weather Prediction forecast system developed by an international group. This model involves physics, mathematics and computer programming techniques. It was especially necessary to improve forecasts for Ireland, a small area where the weather can vary dramatically over short distances, and HIRLAM works together with lower resolution European area forecasts to produce refinements that are useful here. The man in the street is easily convinced of the need for greater accuracy in forecasts and indeed there are economic benefits in accurate predictions of extreme events such as heavy snow or severe storms. Food production (involved in priority area I) can certainly benefit from accuracy in weather prediction, as could perhaps energy production (wind farms or marine energy — area J), and there is certainly scope to continue to make improvements. A key goal is to be able to make long term projections with confidence, but there are mathematical no-go theorems which show that the current approach cannot be extended to forecasts beyond several days into the future. It is an area where new ideas could produce a revolutionary change.

Returning to medical devices (area E) the enhancement of cochlear implants is discussed at
`http://www.whydomath.org/node/hearing/WhatNext.html`
(retrieved May 2013, a page of [6]), where it is asserted that the limitations in terms of refined sound appreciation by patients cannot be overcome by physical design changes alone. Rather a call is made for mathematical progress.

In another direction, the possibility of viable quantum computers compromising the security of the widely used RSA-based cryptographic systems is a cause for concern. In [7] an approach is discussed where the complexity of solving number theoretic problems (at the heart of RSA) is replaced by more mathematically sophisticated ideas (concerned with abstract algebra and combinatorics). This approach seems to promise the benefit of being provably difficult to break and not easy for a hypothetical quantum computer, whereas the problem of factorisation of large numbers might be found to be less difficult than currently believed even without a quantum computer. One might speculate that a new version of the former publicly listed company Baltimore Technologies might emerge from research in this direction (covered under area B), perhaps even based in Ireland again.

In a nutshell then, Mathematics pervades modern life even though the public and policy-makers may not fully appreciate its place. The Mathematics that is and will be needed includes the very considerable body of mathematical knowledge, theory and technique that has already been developed and is an increasing challenge to convey to new generations. However it is also certain that new mathematical discoveries and insights will be vital for the near and more distant future. The sources [4] and [6] provide well-presented evidence of some aspects of the ubiquity of mathematics today, while the blog [5] provides a local Irish perspective.

Appendix A. List of the priority areas

A. Future Networks & Communications
B. Data Analytics, Management, Security & Privacy
C. Digital Platforms, Content & Applications
D. Connected Health and Independent Living
E. Medical Devices
F. Diagnostics
G. Therapeutics: Synthesis, Formulation, Processing and Drug Delivery
H. Food for Health
I. Sustainable Food Production and Processing
J. Marine Renewable Energy
K. Smart Grids & Smart Cities
L. Manufacturing Competitiveness
M. Processing Technologies and Novel Materials
N. Innovation in Services and Business Processes

References

[1] Report of the Research Prioritisation Steering Group - Department of Jobs, Enterprise and Innovation (2012) http://www.djei.ie/publications/science/2012/research_prioritisation.pdf

[2] Report on the David Crighton Lectures and Medal Presentation 2013, London Mathematical Society Newsletter No. 425 (May 2013). http://newsletter.lms.ac.uk/425/425.pdf

[3] *Measuring the Economic Benefits of Mathematical Science Research in the UK*, Report commissioned by the Engineering and Physical Sciences Research Council (EPSRC), November 2012, available at http://goo.gl/z4wNK

[4] *Mathematics Matters*, Website of the The Institute of Mathematics and its Applications (IMA), http://www.ima.org.uk/i_love_maths/mathematics_matters.cfm (retrieved 16/5/2013)

[5] Peter Lynch, *ThatsMaths*, a Blog at http://thatsmaths.com/

[6] WhyDoMath, http://www.whydomath.org/index.html (retrieved 18/5/2013)

[7] T. Güneysu, V. Lyubashevsky, and T. Pöppelmann, *Practical Lattice-Based Cryptography: A Signature Scheme for Embedded Systems*, in Cryptographic Hardware and Embedded Systems – CHES 2012, E. Prouff and P. Schaumont (Eds.), Lecture Notes in Computer Science **7428**, 530-547, Springer 2012.

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Conjugate deficiency in finite groups

STEPHEN M. BUCKLEY AND DESMOND MACHALE

Abstract. We consider the function $r(G) = |G| - k(G)$, where the group G has exactly $k(G)$ conjugacy classes. We find all G where $r(G)$ is small and pose a number of relevant questions.

1. Introduction

Let G be a finite group and let G have exactly $k(G)$ conjugacy classes of elements. One of the most startling results in finite group theory is the following beautiful theorem of Burnside [3, p.295].

Theorem A. *If* $|G|$ *is odd, then* $|G| - k(G) \equiv 0 \pmod{16}$.

We note that no such result can hold if $|G|$ is even. For example, if S_3 is the symmetric group of order 6 and A_4 is the alternating group of order 12, then $k(S_3) = 3$, $k(A_4) = 4$, so that $r(S_3) = 3$, $r(A_4) = 8$, and $\gcd(3, 8) = 1$.

Burnside proved Theorem A using matrix representation theory, but later authors such as Hirsch [5] and Poland [7] proved Burnside's result by elementary means and in fact generalized it. Theorem A has some immediate consequences which are pretty and useful enough to impress students taking a first course in group theory.

Consequence B. *Groups of orders 3, 5, 7, 9, 11, 13, 15, and 17 are all abelian.*

Consequence C. *A non-abelian group of order 21 has exactly 5 conjugacy classes.*

The form of Theorem A suggests that it would be worthwhile to consider the function $r(G) := |G| - k(G)$, which we call the *conjugate deficiency* of a finite group G. In this note, we prove a number of results about $r(G)$ including the following.

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Theorem 1. *There are only finitely many groups G with a given value of $r(G) > 0$.*

We note the obvious fact that there are infinitely many finite groups with $r(G) = 0$, and these are precisely the abelian groups. In what follows, we disregard these groups, so that throughout G will denote a finite non-abelian group.

We use the following notation for some families of groups: C_n is the cyclic group of order n; S_n is the symmetric group of order $n!$; A_n is the alternating group of order $n!/2$; D_n is the dihedral group of order $2n$, $n > 2$; and Q_n is the dicyclic group of order $4n$, $n > 1$ (in particular, Q_2 is the quaternion group).

Theorem 2. *There is no G with $r(G) = 1$, 2, 4, 5, or 7.*

Theorem 3. *The groups with $r(G) = 3$ are S_3, D_4, and Q_2.*

Theorem 4. *There are exactly nine groups with $r(G) = 6$.*

Theorem 5. *The only group with $r(G) = 8$ is A_4.*

This example A_4 knocks on the head the conjecture that $r(G) \equiv 0 \pmod 3$ if $|G|$ is even. However Hirsch [5] shows that if $|G|$ is even and $3 \nmid |G|$, then $r(G) \equiv 0 \pmod 3$. Also if $|G|$ is odd and $3 \nmid |G|$, then $r(G) \equiv 0 \pmod{48}$.

Theorem 6. *The odd order groups which satisfy $r(G) = 16$ are one group of order 21 and two groups of order 27.*

Theorem 7. *The only odd order group which satisfies $r(G) = 32$ is the non-abelian group of order 39.*

Theorem 8. *There are exactly six odd order groups satisfying $r(G) = 48$.*

We begin with the following elementary lemma which, combined with a knowledge of groups of small order, yields all the above results.

Lemma 9. *Suppose G is a non-abelian group. Let p be the least prime dividing $|G|$, and suppose $(G : Z(G)) \geq n$, where $Z(G)$ is the centre of G. Then*

$$k(G) \leq \frac{n+p-1}{pn} \cdot |G| \,.$$

In particular,

$$k(G) \le \frac{p^2+p-1}{p^3} \cdot |G|\,.$$

Proof. The number of single element conjugacy classes in G equals $|Z(G)|$, and so is at most $|G|/n$. Since the size of a conjugacy class is a divisor of $|G|$, any other class has at least p elements, so

$$k(G) \le \frac{1}{n}|G| + \frac{1}{p}\left(1-\frac{1}{n}\right)|G| = \frac{n+p-1}{pn} \cdot |G|\,.$$

Since G is non-abelian, $G/Z(G)$ is not cyclic. Thus we can take $n = p^2$ to get the second estimate. □

We remark that this result is best possible, being attained for the non-abelian groups of order p^3, both for $p = 2$ and p an odd prime. It follows from Lemma 9 that

$$r(G) = |G| - k(G) \ge |G|\left(1 - \frac{n+p-1}{np}\right) = \frac{(n-1)(p-1)}{np}|G|\,.$$

Thus

$$|G| \le \frac{np \cdot r(G)}{(n-1)(p-1)}\,, \tag{1}$$

where p is the least prime dividing $|G|$ and $n \le (G : Z(G))$. Using the second estimate in Lemma 9, we get

$$|G| \le \frac{p^3 \cdot r(G)}{(p^2-1)(p-1)}\,, \tag{2}$$

Since $p^3/(p^2-1)(p-1)$ obviously decreases as p increases, we have the following:

$$|G| \le \frac{8r(G)}{3}\,, \qquad \text{for all finite non-abelian groups } G. \tag{3}$$

$$|G| \le \frac{27r(G)}{16}\,, \qquad \text{for all finite non-abelian groups } G \text{ of odd order.} \tag{4}$$

Moreover, we have equality in (3) if and only if $(G : Z(G)) = 4$, and equality in (4) if and only if $(G : Z(G)) = 9$. By (3), there is an upper bound on $|G|$ for any given $r(G) > 0$. Theorem 1 now follows since there are only finitely many finite groups whose order does not exceed a given number.

Using (3), we see that $|G| \le 16/3$ if $r(G) \le 2$, and no such non-abelian group exists. If $r(G) = 3$, then $|G| \le 8$. There are exactly 3 non-abelian groups of order at most 8, namely S_3, D_4 and Q_2, and $r(G) = 3$ in all three cases.

Using (3), we see that $|G| \le 16$ if $r(G) \le 6$, so to understand how $4 \le r(G) \le 6$ can arise, we need to examine all non-abelian groups of orders between 9 and 16 inclusive. There are fourteen such groups, and for nine of these we have $k(G) = 6$, namely D_5; Q_3; $D_6 = S_3 \times C_2$; and the six groups of order 16 with $(G : Z(G)) = 4$, namely $D_4 \times C_2$, $Q_2 \times C_2$, and 16/8, 16/9, 16/10, and 16/11, in the notation of [8]. The five remaining non-abelian groups with orders between 9 and 16 inclusive have larger deficiencies: $k(A_4) = 8$ and $k(D_7) = k(D_8) = k(Q_4) = k(SD_{16}) = 9$, where SD_{16} is the semidihedral group of order 16. Thus there are no groups with $r(G) \in \{4, 5\}$, and nine groups with $r(G) = 6$.

Using (3), we see that $|G| \le 64/3$ if $r(G) \le 8$, so to understand how $7 \le r(G) \le 8$ can arise, we need to examine the five non-abelian groups with order at most 16 and $r(G) > 6$, plus groups of order between 17 and 21 inclusive. Of the five with order at most 16 and $k(G) > 6$, the only one with $r(G) \le 8$ is A_4 giving $r(A_4) = 8$.

As for the groups of larger order between 17 and 21, we need only check the even order groups, since (4) tells us that $|G| \le 27/2 < 16$ if $|G|$ is odd and $r(G) \le 8$. It remains to check $|G| \in \{18, 20\}$, and there are six such groups: three of order 18 (D_9, $S_3 \times C_3$, and a semidirect product of $C_3 \times C_3$ by C_2) and three of order 20 (D_{10}, Q_5, and the general affine group of degree 1 over GF_5). In each case, $r(G) > 8$. This establishes Theorems 2, 3, 4, and 5.

We now turn to the case where G has odd order, as suggested by Theorem A. If $|G|$ is odd and $r(G) = 16$, then by (4), $|G| \le 27$, and just three groups emerge: the non-abelian group of order 21, and two groups of order 27. Again for $|G|$ odd and $r(G) = 32$, we must have $|G| \le 54$ and just one group emerges, namely the non-abelian group of order 39. For $|G|$ odd and $r(G) = 48$, we must have $|G| \le 81$, and we get 10 groups: one of order 55, one of order 57, two of order 63, and six of order 81. This establishes Theorems 6, 7, and 8.

Now let $t(n)$ be the number of groups which satisfy $r(G) = n$. Here is a table listing the values of $t(n)$ for $n \le 30$, obtained by the above methods.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$t(n)$	0	0	3	0	0	9	0	1	7	0	0	23	0	0	10

n	16	17	18	19	20	21	22	23	24	25	26	27	28	28	30
$t(n)$	4	1	31	1	0	12	0	0	49	0	0	15	0	0	32

The dihedral groups alone suffice to get $r(G)$ equal to any multiple of 3. In fact for $n > 1$, it is well known that $k(D(2n-1)) = n+1$ and $k(D(2n)) = n+5$, and so $r(D(2n-1)) = r(D(2n)) = 3(n-1)$.

It seems difficult to predict the values of $t(n)$, but it is easy to see that

$$r(A \times G) = |A| r(G)$$

whenever A is a finite abelian group. Since there are abelian groups of all orders, it follows that if a given number n is a value of $r(G)$, then so is mn for all $m \in \mathbb{N}$. Moreover $t(mn) \geq t(n)$ for all $m, n \in \mathbb{N}$. This suggests that it would be important to consider prime numbers p for which $t(p) > 0$.

We note that for each prime p, there is a group of order p^3 with $p^2 + p - 1$ classes, so that $r(G) = (p^2-1)(p-1)$ is always possible. In addition, if p and q are primes with $2 < p < q$, where $p \mid (q-1)$, then the nonabelian group of order pq has $p + (q-1)/p$ conjugacy classes, and so

$$r(G) = \frac{(q-1)(p^2-1)}{p}.$$

We close with a number of related problems, some of which could prove difficult to solve.

Problem 1. *Give a realistic upper bound for $t(n)$ for each n.*

Problem 2. *Characterize the numbers n for which $t(n) = 0$.*

With the help of [8] and GAP [4], we see that the numbers in the above problem begin

1, 2, 4, 5, 7, 10, 11, 13, 14, 20, 22, 23, 25, 26, 28, 29,
31, 37, 41, 43, 46, 47, 49, 50, 52, 53, 58, 59, 61, 62, ...

Are there infinitely many such numbers?

Problem 3. *Are there infinitely many primes p for which $t(p) > 0$?*

The primes less than 199 for which $t(p) > 0$ are as follows:

$$3, 17, 19, 83, 97, 107, 113, 137, 149,$$
$$151, 157, 167, 173, 179, 181, 193, 197.$$

These values were found using the Small Groups Library of GAP ([4], [1]) by searching through groups of order at most 511.

Problem 4. *Are there infinitely many pairs $(n, n+1)$ where $3 \nmid n$ and $3 \nmid (n+1)$ such that $t(n) = t(n+1) = 0$?*

Problem 5. *For each $k \geq 4$, is there an odd order group G with $r(G) = 2^k$?*

If the answer to this last problem is positive, then we can find a group of odd order with $r(G) = 16l$ for all $l \in \mathbb{N}$ by taking direct products as previously described. The answer is indeed positive for $4 \leq k \leq 12$, because of groups of order 21, 39, 75, 147, 291, 579, 1161, 2307, 4221; the largest three of these orders were found with the help of GAP. The desired group is given in all except two cases by a semidirect product $C_n \rtimes C_3$, for $n = |G|/3$. The two exceptional cases are $|G| = 75$ in which case $G = C_5^2 \rtimes C_3$, and $|G| = 4221$ in which case G is of type $(C_7 \rtimes C_3) \times (C_{67} \rtimes C_3)$. There does not seem to be a clear enough pattern to these examples to justify a conjecture that the answer is always positive.

Problem 6. *Is the function $t(n)$ onto $\mathbb{N}$? Is there, for example, an n with $t(n) = 2$?*

Problem 7. *For n odd and $n > 3$, do there exist primes p and q with $2 < p < q$ where $p \mid (q-1)$, such that $n = p + (q-1)/p$?*

Computer results [2] show that this result is true for all n, $3 < n < 10\,000\,001$. If it is true in general, then it provides an answer to the following question posed by the second author in [6].

> *For each odd $k > 3$, does there exist an odd order non-abelian group with exactly k conjugacy classes?*

Of particular interest is $r(S_n) = n! - p(n)$, where $p(n)$ is the number of partitions of n. This purely arithmetic function is of some interest in its own right, so we ask:

Problem 8. *What is the range of values of $r(S_n)$?*

We say that n is primitive if $t(n) \neq 0$, but $t(d) = 0$ for each proper divisor d of n. For example, 3, 8, 17, and 19 are primitive.

Problem 9. *Are there infinitely many primitive values of n?*

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EXPANDER FAMILIES, GROUP STRUCTURE, AND SEMIDIRECT PRODUCTS

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Abstract. Expander families are, essentially, sequences of large, sparse, pseudorandom graphs. Many such families have been constructed as Cayley graphs. It is an interesting and open question to determine which groups yield Cayley graphs that form expander families. In this paper, we give a brief survey of expander families, with an emphasis on known results pertaining to that question. One minor new result is a necessary but insufficient condition for a sequence of finite solvable groups, each constructed by iterating semidirect products, to yield an expander family as a sequence of Cayley graphs.

1. Introduction

Roughly speaking, expander families model large, fast, cheap, and reliable communication networks. Alternatively, one can view them as large, sparse, pseudorandom regular graphs. In §2, we give the precise definitions. Expander families have a multitude of real-world applications (especially in computer science) as well as connections to many other branches of mathematics. In part for these reasons, a great deal of research has been done on them recently. In §3, we provide a short survey of known results and open problems. For an elementary introduction to the subject, we refer to [9]; for an advanced discussion, see [10].

One common method for forming expander families is the Cayley graph construction. It is an open problem to find necessary and sufficient conditions for a sequence of finite groups to admit an expander family as a sequence of Cayley graphs. The class of

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nonabelian simple groups was resolved (in the affirmative) only recently, after several decades' work; the final case was proved in 2011 by Emmanuel Breuillard, Ben Green, and Terence Tao [4]. In §4, we discuss in more detail what is known about the relationship between group structure and expansion.

Section 5 is guided by the principle that one ought to begin as simply and as generally as possible. It is known that many classes of groups, including abelian groups, do not yield expander families. We begin the section by surveying several previously established positive results concerning solvable groups and iterated semidirect products, as our view is that the family of groups which is easiest to analyze but is not yet excluded is the family of groups constructed by recursively forming semidirect products with cyclic groups. A new result (Theorem 5.1) provides a necessary but insufficient condition for a sequence of groups so constructed to yield an expander family.

2. Basic definitions

Definition 2.1. Let X be a finite graph with vertex set V. Let S be a set of vertices of X. We define the *boundary* of S, denoted ∂S, to be the set of edges in X incident to both a vertex in S and a vertex not in S. We define the *isoperimetric constant* of X, denoted $h(X)$, to be the minimum, over all nonempty subsets S of V containing no more than half the vertices of X, of $|\partial S|/|S|$, where $|A|$ denotes the cardinality of the set A.

Definition 2.2. Let (X_n) be a sequence of finite graphs. We say (X_n) is an *expander family* if

(1) each X_n is regular, each with the same degree, and
(2) $|V_n| \to \infty$, where V_n is the vertex set of X_n, and
(3) there exists a real number $\epsilon > 0$ such that $h(X_n) \geq \epsilon$ for all n.

Let G be a group, and let Γ be a symmetric subset of G. (Recall that to say Γ is *symmetric* means that if $\gamma \in \Gamma$, then $\gamma^{-1} \in \Gamma$.) Recall that the *Cayley graph* $\mathrm{Cay}(G, \Gamma)$ is the graph with vertex set G so that two vertices x and y are adjacent if and only if $xy^{-1} \in \Gamma$. Note that $\mathrm{Cay}(G, \Gamma)$ is regular with degree $|\Gamma|$.

Definition 2.3. Let (G_n) be a sequence of finite groups. We say that (G_n) *yields an expander family* if there exists a positive integer

d and symmetric subsets $\Gamma_n \subset G_n$ with $|\Gamma_n| = d$ for all n such that $(\mathrm{Cay}(G_n, \Gamma_n))$ is an expander family.

3. Expander families: A brief overview

There are three main approaches to determining whether a sequence of regular graphs forms an expander family: combinatorial, probabilistic, and (in the case of Cayley graphs and related constructions) representation-theoretic.

3.1. **Combinatorial methods.**

The definition of the isoperimetric constant is combinatorial in nature. However, of all the existing constructions of expander families, together with the proofs they they are just that, only the one in the paper [2] uses the definition directly to prove the result.

Looking closely at the definition, one can see why it is difficult to work with. It is a minimum that ranges over the collection of all subsets of the vertex set containing no more than half the vertices. The number of possible subsets grows exponentially with the order of the graph. Consequently, most proofs have come at the problem indirectly, tying the isoperimetric constant to other graph invariants that are easier to work with.

3.2. **Probabilistic methods: Random walk theory.**

The eigenvalues of the adjacency operator of a graph encode a great deal of information, though not complete information, about its structure. The book [6] provides a thorough overview of many of the connections between graph eigenvalues and other graph invariants. In the study of expander families, the most important such connection is the following double inequality, attributed to Alon, Milman, Tanner, and Dodziuk. Recall that the eigenvalues of the adjacency operator of a finite d-regular graph are all real and that for any such eigenvalue λ, we have $|\lambda| \leq d$.

Theorem 3.1. *Let X be a finite d-regular graph with isoperimetric constant h and second-largest eigenvalue λ_1. Then*

$$\frac{d - \lambda_1}{2} \leq h \leq \sqrt{2d(d - \lambda_1)}.$$

See [9] for a proof of Theorem 3.1.

The significance of Theorem 3.1 is that a sequence of d-regular graphs is an expander family if and only if λ_1 is uniformly bounded away from d. Tools from linear algebra, such as the Rayleigh-Ritz theorem, can then be brought to bear. The landmark paper [15], which introduces the zig-zag product of graphs as well as other graph constructions, uses this approach to prove that iterating zig-zag products in an appropriate way will yield expander families.

The theory of random walks sheds some light on why we might expect λ_1 to be related to h. A large isoperimetric constant indicates that a graph is "all mixed up," that it is somewhat pseudo-random. Too much structure will cause a graph to have large sets of vertices with small boundary. Even cycle graphs furnish an illustrative example; in a $2n$-cycle, the "bottom half" of the graph forms an isoperimetric set with just two boundary edges, giving us $h = 2/n$, which vanishes as $n \to \infty$.

From the viewpoint of random walks, being "all mixed up" means that a random walker on the graph will get lost quickly on it. For a connected nonbipartite regular graph, it is known that any initial probability distribution will converge to the uniform distribution as one repeatedly takes random steps. Regarding the random walk as a Markov process on the graph, one can see quickly by diagonalizing and taking powers of the adjacency matrix that λ_1 controls the rate of this convergence.

Roughly speaking, Theorem 3.1 tells us that h is large if and only if λ_1 is small. For a d-regular graph with adjacency operator A, the Rayleigh-Ritz theorem asserts that λ_1 equals the maximum, over all unit vectors v orthogonal to the constant vector, of $\langle Av, v\rangle$. (To see why this holds, diagonalize and recall that the largest eigenvalue is d.) So to continue our rough analysis, h is large if and only if $\langle Av, v\rangle$ is small for all unit vectors v orthogonal to the constant vector. But the inner product $\langle Av, v\rangle$ is small if and only if the angle between Av and v is large, which in turn holds if and only if Av is relatively far from v. This fact—the fact that adjacency operators of graphs with large isoperimetric constant move most unit vectors a long distance—conforms with our intuition that graphs in expander families scramble everything up.

For good expansion, then, we want λ_1 to be as small as possible. However, there is an asymptotic lower bound, due to Alon and Boppona, on how small λ_1 can be. More precisely, we have the following theorem.

Theorem 3.2. *Let d be a fixed positive integer, and let (X_n) be a sequence of finite d-regular graphs whose orders approach infinity. Then* $\liminf \lambda_1(X_n) \geq 2\sqrt{d-1}$.

One can prove Theorem 3.2 by obtaining a lower bound for the number of closed walks with a given fixed point in the universal cover of a d-regular graph, that is, a d-regular tree. Alternatively, one can use the Rayleigh-Ritz theorem. Both proofs are presented in [9].

Let X be a finite regular graph with n vertices. Let $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ be the eigenvalues of the adjacency operator of X, listed in nondecreasing order. Define

$$\lambda := \begin{cases} \max\{|\lambda_1|, |\lambda_{n-1}|\} & \text{if } X \text{ is nonbipartite} \\ \max\{|\lambda_1|, |\lambda_{n-2}|\} & \text{if } X \text{ is bipartite.} \end{cases}$$

Motivated in part by Theorem 3.2, we define a d-regular graph X to be *Ramanujan* if $\lambda \leq 2\sqrt{d-1}$. For an integer $d \geq 3$, a short computation shows that a family of d-regular Ramanujan graphs will necessarily be an expander family. Indeed, Ramanujan graphs are in some sense optimal expanders.

If $d \geq 3$ is an integer such that $d-1$ is a prime power, then there exists a family of d-regular Ramanujan graphs [5, 11, 13, 14]. For every $d \geq 3$ not of that form, it is an open problem as to whether a family of d-regular Ramanujan graphs exists.

3.3. **Representation-theoretic methods.** The vast majority of expander families have been constructed via algebraic methods, especially the Cayley graph construction. For such graphs, one can take advantage of the underlying group structure to attack the question of expansion. In particular, the adjacency operator of a finite Cayley graph enjoys a natural direct sum decomposition indexed by the irreducible linear representations of the group—see [3], for example, for a discussion of this useful fact.

In this direct sum, the trivial representation corresponds to the space of constant vectors; the nontrivial representations index the summands in its orthogonal complement. So for Cayley graphs, we expect the isoperimetric constant to be large if and only if the restriction of A to each such summand moves unit vectors far. That motivates the following definition.

Definition 3.3. Let G be a finite group. Let Γ be a subset of G. Define the *Kazhdan constant* $\kappa(G, \Gamma)$ to be the minimum value of $||\pi(\gamma)v - v||$, where γ ranges over Γ; π ranges over all nontrivial irreducible unitary representations of G; and v ranges all unit vectors in the underlying representation space of π.

(Remark: Compactness of unit spheres shows that a minimum is achieved.)

Theorem 3.4. *Let G be a finite group, and let Γ be a symmetric subset of G. Let d, h, and λ_1 be the degree, isoperimetric constant, and second-largest eigenvalue, respectively, of the Cayley graph of G with respect to Γ. Then*

$$2\sqrt{dh} \geq \kappa(G, \Gamma) \geq \sqrt{\frac{2(d-\lambda_1)}{d}}.$$

A proof of Theorem 3.4 can be found in [9] or [12].

It follows immediately from Theorems 3.4 and 3.1 that a sequence of d-regular Cayley graphs is an expander family if and only if the corresponding Kazhdan constants are uniformly bounded away from zero. Many proofs that certain families of groups yield expander families rely primarily on this fact. The point is that for many families of groups, we can use the detailed information we have about their irreducible representations to come up with a lower bound for the Kazhdan constant.

4. Group structure and expansion

Given a sequence of finite groups whose orders approach infinity, does it yield an expander family? Stated in full generality, this question remains open. However, several partial results are known.

Theorem 4.1. *Any sequence of finite nonabelian simple groups whose orders approach infinity yields an expander family.*

Several authors over several decades joined forces to prove Theorem 4.1. The proof relies on the classification of finite simple groups. The survey [8] discusses all cases except Suzuki groups, which had not yet been finished at that time. The case of alternating groups, which is dealt with [7], required special attention. The proof was completed in [4] by showing that Suzuki groups yield expander families.

Although the answer is positive for nonabelian simple groups, not so for perfect groups. The n-fold product G_n of the alternating group on 5 letters provides a counterexample; indeed, given any positive integer d, then for sufficiently large n no set of d elements will generate G_n, so the associated Cayley graphs will be disconnected and therefore have vanishing isoperimetric constant.

We now turn our attention to negative results.

Lemma 4.2. *No sequence of abelian groups yields an expander family.*

The idea of the proof is that expander families have logarithmic diameter (as a function of the number of vertices), whereas for Cayley graphs on abelian groups, the diameter grows at least as fast as a root function. For details, see [9, Prop. 4.25].

Lemma 4.3. *Let (G_n) and (Q_n) be sequences of finite groups such that each Q_n is a homomorphic image of G_n. Suppose that $|Q_n| \to \infty$ and that (Q_n) does not yield an expander family. Then (G_n) does not yield an expander family.*

Proof. The idea of the proof is to project down from a Cayley graph on G_n to a corresponding Cayley graph on Q_n, then take the inverse image of a subset of Q_n that achieves the minimum in the definition of isoperimetric constant. For details, see [9, Prop. 2.20]. □

Lemma 4.4. *Let (G_n) be a sequence of finite groups with $|G_n| \to \infty$. For each n, let H_n be a subgroup of G_n. Suppose that the sequence $[G_n : H_n]$ of indices is bounded. If (H_n) does not yield an expander family, then G_n does not yield an expander family.*

The idea of the proof of Lemma 4.4 is to use Schreier generators to transfer from G_n to H_n. For details, see [9, Prop. 2.46].

The paper [12] also discusses other restrictions to expansion, from a function-analytic point of view.

5. Expanders and semidirect products

Cyclic groups are nearly always the easiest family of groups to work with. However, Lemma 4.2 shows that no sequence of cyclic groups yields an expander family. Next, one might consider dihedral groups, which are in some sense next-easiest. Lemmas 4.2 and 4.4 together imply, though, that the dihedral groups also do not yield an expander family. More generally, a sequence $(H_n \rtimes K_n)$ of semidirect products of cyclic groups cannot yield an expander family, for either the sequence (K_n) of quotients is unbounded, or else the sequence (H_n) of subgroups has bounded index. Proceeding inductively, we see that no sequence of groups, each constructed by iterating semidirect products of cyclic groups k times for some fixed positive integer k, can yield an expander family.

With that in mind, we consider sequences of groups constructed recursively as follows. Let (K_n) be a sequence of cyclic groups. Let $G_1 = K_1$. Let $G_{n+1} = G_n \rtimes K_n$. Can a sequence (G_n) so constructed yield an expander family?

We begin by discussing several known results relevant to this question, some of which suggest that this construction is not as unpromising as it first appears. We then conclude by proving a necessary condition for such a sequence of iterated semidirect products to yield an expander family and by giving an example to show that this condition is not sufficient.

5.1. **Known results.** Any group constructed by iterating semidirect products of cyclic groups will necessarily be solvable. Perhaps solvability precludes expansion? Almost, but not quite. Lemmas 4.2, 4.3, and 4.4 together imply that no sequence of solvable groups *with bounded derived length* can yield an expander family. Lubotzky and Weiss show in [12], however, that there exists a sequence of solvable groups (indeed, p-groups) that yields an expander family.

If (X_n) is an expander family and each graph X_n has r_n vertices, then $\operatorname{diam}(X_n) = O(\log r_n)$. In other words, expander families have logarithmic diameter. Let C_k denote the cyclic group of order k, and let G_n be the wreath product of C_2 with C_n. That is, $G_n = C_2^n \rtimes C_n$, where C_n acts on C_2^n by cyclically permuting the coordinates. Then (G_n) admits a sequence of 3-regular Cayley graphs (the *cube-connected cycle graphs*) with logarithmic diameter. But each G_n has derived length 2, so this sequence cannot be an expander family. (See [9] for more details of this example.) The point here is that a semidirect products of two abelian groups admit Cayley graphs that in a sense come close to being an expander family.

In [15], Reingold, Vadhan, and Wigderson defined a new graph operation called the zigzag product. They show that iterating zigzag products appropriately will yield an expander family. In [1], Alon, Lubotzky, and Wigderson show that under certain circumstances, the zigzag product of two Cayley graphs is a Cayley graph on the semidirect product of the two underlying groups. We note that one of the Reingold-Vadhan-Wigderson constructions, the base graph is a Cayley graph on an abelian group. In [16], Rozenman, Shalev, and Wigderson employ the results of [1] to construct expander families as Cayley graphs on iterated wreath products of alternating groups.

5.2. **Iterated semidirect products of cyclic groups.** In this subsection, we investigate the question of when groups constructed by iterating semidirect products of cyclic groups can yield expander families. We provide a necessary condition, and then give an example to show that it is not sufficient.

Theorem 5.1. *Suppose (C_n) is a sequence of nontrivial finite cyclic groups, where a_n generates C_n. Let $G_1 = C_1$. Suppose that for all $n \geq 2$, we have that $G_{n+1} = G_n \rtimes_{\theta_n} C_{n+1}$ for some homomorphism $\theta_n : C_{n+1} \to \mathrm{Aut}(G_n)$. If (G_n) yields an expander family, then $\theta_n(a_n)$ must be outer for infinitely many n.*

Proof. We first show that if G, H are groups, where H is cyclic with generator a, and $\theta : H \to \mathrm{Aut}(G)$ is a homomorphism such that $\theta(a)$ is inner, then $(G \rtimes_\theta H)' = G'$. Here we identify G and H as subgroups of $G \rtimes_\theta H$ via the embeddings $g \mapsto (g, 1)$ and $h \mapsto (1, h)$. The inclusion $G' \subset (G \rtimes_\theta H)'$ is immediate.

For the converse, we compute that

$$\begin{aligned}
& (g_1a^r)(g_2a^s)(g_1a^r)^{-1}(g_2a^s)^{-1} \\
= \ & (g_1a^r)(g_2a^s)(x^{-r}g_1^{-1}x^ra^{-r})(x^{-s}g_2^{-1}x^sa^{-s}) \\
= \ & (g_1x^rg_2x^{-r}a^{r+s})(x^{-r}g_1^{-1}x^{-s}g_2^{-1}x^{r+s}a^{-r-s}) \\
= \ & g_1x^rg_2x^{s-r}g_1^{-1}x^{-s}g_2^{-1}x^{-s} \\
= \ & (g_1x^r)(g_2x^s)(g_1x^r)^{-1}(g_2x^s)^{-1}
\end{aligned}$$

where $\theta(a)$ is the inner automorphism $g \mapsto xgx^{-1}$. Therefore $(G \rtimes_\theta H)' = G'$.

Therefore, if only finitely many $\theta(a_n)$ are outer, then $|G_n/G_n'| \to \infty$. Observe, however, that together Lemmas 4.2 and 4.3 imply that if $(|G_n/G_n'|)$ is unbounded, then (G_n) does not yield an expander family. The theorem follows. ☐

We now show that the converse of Theorem 5.1 fails; that is, we exhibit an example of a sequence (G_n) of groups constructed as in

Theorem 5.1 but with infinitely many (indeed, all but finitely many) of the θ_n outer such that (G_n) does not yield an expander family.

We construct (G_n) as follows. Let $G_1 = \mathbb{Z}_2$, the group of integers modulo 2 under addition. Let $G_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$. (Recall that the direct product is a special case of the semidirect product.) Define $\theta_2 : \mathbb{Z}_2 \to \mathrm{Aut}(G_2)$ by $\theta_2(1) : (a, b) \mapsto (b, a)$. Let $G_3 = G_2 \rtimes_{\theta_2} \mathbb{Z}_2$. Observe that $G_3 \cong D_4$, the dihedral group of order 8. Define the dihedral group of order $2n$ by $D_n := \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle$. Define $\tau : \mathbb{Z}_2 \to \mathrm{Aut}(D_n)$ by $\tau(1) : r \mapsto r^{-1}$ and $s \mapsto rs$. For $n \geq 4$, let $G_n = D_{2^{n-2}} \rtimes_\tau \mathbb{Z}_2$. Observe that $D_{2^{n-1}} \cong G_n$ by the isomorphism $r \mapsto (s, 1), s \mapsto (1, 1)$. So the sequence (G_n) is indeed constructed by iterating semidirect products with cyclic groups.

When n is even, the commutator subgroup of D_n is generated by r^2 and so has order $n/2$. Hence, for $n \geq 4$, we have

$$|(D_{2^{n-2}} \rtimes_\tau \mathbb{Z}_2)'| = |D'_{2^{n-1}}| = 2^{n-2},$$

whereas $|D'_{2^{n-2}}| = 2^{n-3}$. From the first half of the proof of Theorem 5.1, then, it follows that τ is outer.

It remains to be shown that (G_n) does not yield an expander family. First observe that for all n, the group G_n admits $\mathbb{Z}_{2^{n-2}}$ as a subgroup of index 2. From Lemma 4.2, we know that $(\mathbb{Z}_{2^{n-2}})$ does not yield an expander family. It then follows from Lemma 4.4 that (G_n) does not yield an expander family.

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THE TENSOR PRODUCT OF A CSL AND AN ABSL

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Abstract. We study the question that asks whether the tensor product of two reflexive subspace lattices is reflexive. In particular, we study the tensor product of a commutative subspace lattice $\mathcal{L}$ and an atomic boolean subspace lattice $\mathcal{M}$ and we prove that it is equal to the extended tensor product of the two subspace lattices. Furthermore, we give a description of the subspace lattice $\mathcal{L} \otimes \mathcal{M}$ and with the help of a result of Harrison in [3] we prove that it is reflexive. We also show that the lattice tensor product formula holds for any Arveson algebra of $\mathcal{L}$ and $\operatorname{alg} \mathcal{M}$.

1. Introduction

In this paper we consider every Hilbert space to be separable. If H is a Hilbert space, then we set $\mathcal{B}(H)$ to be the set of all bounded operators acting on H and $\mathcal{P}(H)$ to be the set of all orthogonal projections acting on H. If $P, Q \in \mathcal{P}(H)$ then we define $P \vee Q$ to be the projection with range $PH \vee QH$ and $P \wedge Q$ the projection with range $PH \wedge QH$. It is clear that $\mathcal{P}(H)$ is a lattice with respect to the binary operations of the intersection $\wedge$ and the closed linear span $\vee$. A strongly closed sublattice of $\mathcal{P}(H)$ (with respect to the binary operations of the intersection and the linear span) that contains 0 and the identity operator I is called a subspace lattice.

If $\mathcal{A}$ is an operator algebra acting on a Hilbert space H then we define

$$\operatorname{lat} \mathcal{A} = \{P \in \mathcal{P}(H) : PTP = TP \text{ for each } T \in \mathcal{A}\}.$$

Obviously $\operatorname{lat} \mathcal{A}$ is a subspace lattice. Subspace lattices of this form are called reflexive. Similarly, if $\mathcal{L}$ is a subspace lattice then we

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define

$$\operatorname{alg}\mathcal{L} = \{T \in \mathcal{B}(H) : L^{\perp}TL = 0, \text{ for each } L \in \mathcal{L}\}.$$

Clearly $\operatorname{alg}\mathcal{L}$ is a weakly closed unital subalgebra of $\mathcal{B}(H)$.

Let T_i be a bounded operator acting on a Hilbert space H_i for $i = 1, 2$. We define $T_1 \otimes T_2$ be the bounded operator acting on the Hilbert space $H_1 \otimes H_2$ such that, if $x_1 \in H_1$ and $x_2 \in H_2$, then $T_1 \otimes T_2(x_1 \otimes x_2) = T_1x_1 \otimes T_2x_2$.

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are ultraweakly closed algebras, then we denote by $\mathcal{A}_1 \otimes \mathcal{A}_2$ the ultraweakly closed algebra generated by the elementary tensors $A_1 \otimes A_2$ where $A_i \in \mathcal{A}_i$, $i = 1, 2$. Similarly, if $\mathcal{L}_1$ and $\mathcal{L}_2$ are subspace lattices we denote by $\mathcal{L}_1 \otimes \mathcal{L}_2$ the smallest subspace lattice containing all elementary tensors $L_1 \otimes L_2$ where $L_i \in \mathcal{L}_i$, $i = 1, 2$.

Given two subspace lattices $\mathcal{L}$ and $\mathcal{M}$, the algebra tensor product formula (ATPF) holds for $\mathcal{L}$ and $\mathcal{M}$ if $\operatorname{alg}(\mathcal{L} \otimes \mathcal{M}) = \operatorname{alg}\mathcal{L} \otimes \operatorname{alg}\mathcal{M}$. Analogously, the lattice tensor product formula (LTPF) holds for two operator algebras $\mathcal{A}$ and $\mathcal{B}$ if $\operatorname{lat}(\mathcal{A} \otimes \mathcal{B}) = \operatorname{lat}\mathcal{A} \otimes \operatorname{lat}\mathcal{B}$. The LTPF was first introduced by Hopenwasser in [4].

Let Y be a compact metric space and ν a finite regular Borel measure on Y. If $A \subseteq Y$ is measurable, then we denote by M_A the map that sends $\psi \in L^2(Y, \nu)$ to $\psi\chi_A$ where χ_A is the characteristic function on A.

Let $\mathcal{L}$ be a commutative subspace lattice (CSL). It follows from Arveson [1] that there exists a compact metric space X, a standard preorder $\leq$ on X and a finite regular Borel measure μ on X such that $\mathcal{L}$ is unitarily equivalent to

$$\mathcal{L}(X, \mu, \leq) = \{M_B : B \subseteq X \text{ measurable and almost increasing}\}.$$

A subset $B \subseteq X$ is almost increasing if there exists a null subset $\Gamma \subseteq X$ such that $B \backslash \Gamma$ is increasing. If $\mathcal{M}$ be a subspace lattice acting on a Hilbert space K, then a function $\phi : X \to \mathcal{M}$ is almost increasing if there exists a null subset $\gamma \subseteq X$ such that ϕ is increasing on $X \backslash \gamma$. Also, the function ϕ is measurable if the map $x \to (\phi(x)\xi, \eta)$ is measurable for all $\xi, \eta \in K$. We define $L^\infty(X, \mu, \leq, \mathcal{M})$ to be the space of all essentially bounded, $\mathcal{M}$-valued, almost increasing and measurable functions on X. If $\phi \in L^\infty(X, \mu, \leq, \mathcal{M})$, then we denote by M_ϕ the map from $L^2(X, \mu, K)$ to $L^2(X, \mu, K)$ such that $(M_\phi f)(x) = \phi(x)f(x)$ for all $f \in L^2(X, \mu, K)$ and for all $x \in X$. The **extended tensor product** of $\mathcal{L}$ and $\mathcal{M}$ is defined to be the space

$$\{M_\phi : \phi \in L^\infty(X, \mu, \leq, \mathcal{M})\}$$

and it is denoted by $\mathcal{L} \otimes_{ext} \mathcal{M}$. In many occasions it is easier to identify $\mathcal{L} \otimes_{ext} \mathcal{M}$ with $L^\infty(X, \mu, \leq, \mathcal{M})$ through the map that sends ϕ to M_ϕ for all $\phi \in L^\infty(X, \mu, \leq, \mathcal{M})$. Also, if $B \subseteq X$ is almost increasing and measurable and $L \in \mathcal{M}$, then through the map that sends $M_B \otimes L$ to the function $x \to \chi_B(x)L$, where $x \in X$, we identify $\mathcal{L} \otimes \mathcal{M}$ with a subset of $\mathcal{L} \otimes_{ext} \mathcal{M}$ and we consider $\mathcal{L} \otimes \mathcal{M} \subseteq \mathcal{L} \otimes_{ext} \mathcal{M}$.

The extended tensor product was firstly introduced by Harrison in [3]. One of the main results obtained in that paper is that the extended tensor product of a completely distributive CSL $\mathcal{L}$ and any subspace lattice $\mathcal{M}$ is equal to their tensor product. An interesting question emerging from this result is whether the equality between the tensor product and the extended tensor product still holds, if we remove the property of complete distributivity from the subspace lattice $\mathcal{L}$ to the subspace lattice $\mathcal{M}$. The main result of this paper answers the previous question positively in the case where the subspace lattice $\mathcal{M}$ is an atomic Boolean subspace lattice (ABSL). Also, if $\mathcal{M}$ is an ABSL, it follows that the tensor product of $\mathcal{L}$ and $\mathcal{M}$ is reflexive and that the LTPF holds for every Arveson algebra of $\mathcal{L}$ and $\operatorname{alg} \mathcal{M}$. Furthermore, we have a description of $\mathcal{L} \otimes \mathcal{M}$ in terms of the elements of $\mathcal{L}$ and the atoms of $\mathcal{M}$.

2. The main results

Recall at this point of the paper that a subspace lattice $\mathcal{M}$ is an ABSL if it is distributive, complemented (i.e. for every $P \in \mathcal{M}$ there exists an element $P' \in \mathcal{M}$ such that $P \wedge P' = 0$ and $P \vee P' = I$) and there exists a subset $\mathcal{K} \subset \mathcal{M}$ of non-zero elements (called the atoms) such that (i) if $M \in \mathcal{M}$, $K \in \mathcal{K}$ and $0 \subseteq M \subseteq K$, then either $M = 0$ or $M = K$ and (ii) if $M \in \mathcal{M}$ then M is equal to the closed span of the atoms that it majorises.

Lemma 2.1. *If $\mathcal{M}$ is an ABSL acting on a separable Hilbert space H, then $\mathcal{M}$ has at most a countable number of atoms.*

Proof. Let $(E_j)_{j \in J}$ be the set of atoms of $\mathcal{M}$ and $(e_k^{(j)})_{k \in K_j}$ be an orthonormal basis of $E_j H$, $j \in J$. Since H is separable, it follows that K_j is at most countable for all $j \in J$. It is also clear that

$\vee_{j\in J}(\vee_{k\in K_j} e_k^{(j)}) = H$. The class of subsets of $\mathcal{T} = \cup_{j\in J}\{e_k^{(j)} : k \in K_j\}$ whose closed span equals H is not empty as $\mathcal{T}$ is such a set itself. Also, if there exists $\mathcal{S} \subseteq \mathcal{T}$ and $i \in J$ such that $\mathcal{S} \cap E_i = \emptyset$, then $\vee\{s : s \in \mathcal{S}\} \subseteq \vee_{j\neq i} E_j \neq H$. Hence, for every subset $\mathcal{S}$ of $\mathcal{T}$ whose closed span is equal to H and for every $i \in J$ we have that $\mathcal{S}\cap E_i \neq \emptyset$ and thus the cardinality of each of those sets is bigger or equal to the cardinality of J.

Let $(f_n)_{n\in\mathbb{N}}$ be an orthonormal basis of H. Obviously, for each $n \in \mathbb{N}$, there exists a sequence $(a_l^{(n)})_{l\in\mathbb{N}}$ such that

$$a_l^{(n)} = \sum_{i=1}^{k_l^{(n)}} \mu_i^{(l,n)} e_i^{(l,n)}, \mu_i^{(l,n)} \in \mathbb{C},$$

where $k_l^{(n)} \in \mathbb{N}$, $e_i^{(l,n)} \in \cup_{j\in J}\{e_k^{(j)} : k \in K_j\}$ for all $1 \leq i \leq k_l^{(n)}$ and all $l \in \mathbb{N}$, and $f_n = \lim_{l\to\infty} a_l^{(n)}$. Let $\mathcal{V} = \{e_i^{(l,n)} : 1 \leq i \leq k_l^{(n)} \text{ and } l, n \in \mathbb{N}\}$. Then $\mathcal{V}$ is at most countable and the closed span of its elements is equal to H. Since $\mathcal{V} \subseteq \mathcal{T}$, it follows from our previous observation that the cardinality of J is smaller than the cardinality of $\mathcal{V}$ and thus it is at most countable proving our lemma. □

Theorem 2.2. *Let X be a compact metric space, μ be a finite Borel measure, $\leq$ be a standard preorder, $\mathcal{L} = \mathcal{L}(X, \mu, \leq)$ be a CSL acting on the Hilbert space $K = L^2(X, \mu)$ and $\mathcal{M}$ be an ABSL acting on a Hilbert space H with atoms $\{E_j : j \in J\}$. Then every element of $\mathcal{L}\otimes_{ext}\mathcal{M}$ can be written in the form $\vee_{j\in\mathbb{N}}(M_{\alpha_j} \otimes E_j)$ where $\alpha_j \subseteq X$ is almost increasing and measurable and thus $\mathcal{L}\otimes_{ext}\mathcal{M} = \mathcal{L} \otimes \mathcal{M}$.*

Proof. In the proof of this theorem we identify $L^\infty(X, \mu, \leq, \mathcal{M})$ with $\mathcal{L}\otimes_{ext}\mathcal{M}$. By Lemma 2.1, $\mathcal{M}$ has at most a countable number of atoms. Since $\mathcal{L}\otimes\mathcal{M} \subseteq \mathcal{L}\otimes_{ext}\mathcal{M}$, we only need to prove the opposite inclusion. Fix $P \in \mathcal{L}\otimes_{ext}\mathcal{M}$ and let $\beta_j = \{x \in X : P(x) \geq E_j\}$ for all $j \in J$. Let $j \in J$ and $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of E_j. It follows that

$$\begin{aligned}\beta_j &= \{x \in X : P(x) \geq E_j\} \\ &= \{x \in X : (P(x)e_k, e_k) = (E_j e_k, e_k),\ k \in \mathbb{N}\} \\ &= \bigcap_{k \in \mathbb{N}} \{x \in X : (P(x)e_k, e_k) = 1\}. \qquad (1)\end{aligned}$$

The function $x \to (P(x)\xi, \eta)$ is measurable by definition for all $\xi, \eta \in H$ and thus the set $\{x \in X : (P(x)e_k, e_k) = 1\}$ is measurable for all $k \in \mathbb{N}$. It follows from (1) that β_j is measurable for all $j \in J$. Also, P is almost increasing and thus there exists a null set A such that P is increasing on $X \backslash A$. If for some $j \in J$, $x \in \beta_j \backslash A$, $y \in X \backslash A$ and $x \leq y$, then by the definition of β_j, we have that $P(x) \geq E_j$. Since $x \leq y$ and P is increasing on $X \backslash A$, it follows that $P(y) \geq P(x) \geq E_j$. By the definition of β_j, we have that $y \in \beta_j$. Hence, β_j is increasing in $X \backslash A$ and thus it is almost increasing. Since $j \in J$ is arbitrary, β_j is almost increasing for all $j \in J$.

It now suffices to show that $P = \bigvee_{j \in J} (M_{\beta_j} \otimes E_j)$. It is well known that there exists a null subset $N \subseteq X$ such that $(\bigvee_{j \in J} (M_{\beta_j} \otimes E_j))(x) = \bigvee_{j \in J} ((M_{\beta_j} \otimes E_j)(x))$ for all $x \in X \backslash N$ (for a detailed proof of this statement see Proposition 1.5.2 in [5]). It is easy to see that $(M_{\beta_j} \otimes E_j)(x) \leq P(x)$ for all $x \in X \backslash N$ and for all $j \in J$, and thus $(\bigvee_{j \in J} (M_{\beta_j} \otimes E_j))(x) \leq P(x)$ for all $x \in X \backslash N$. Since $\mathcal{M}$ is an ABSL, $P(x)$ is equal to the span of the atoms that it contains for all $x \in X$. Hence, in order to prove the opposite inclusion it is enough to show that all atoms contained in $P(x)$ are also contained in $(\bigvee_{j \in J} (M_{\beta_j} \otimes E_j))(x)$ for all $x \in X \backslash N$. Fix $x \in X \backslash N$ and $i \in J$ such that $E_i \leq P(x)$. It follows from the definition of β_i that $x \in \beta_i$ and thus

$$E_i \leq \bigvee_{j \in J} ((M_{\beta_j} \otimes E_j)(x)) = (\bigvee_{j \in J} (M_{\beta_j} \otimes E_j))(x).$$

Since E_i is an arbitrary atom contained in $P(x)$, it follows that

$$P(x) = (\bigvee_{j \in J} (M_{\beta_j} \otimes E_j))(x).$$

Since N is null and $x \in X \backslash N$ is arbitrary, $P = \bigvee_{j \in J} (M_{\beta_j} \otimes E_j)$ and the theorem is proved. $\square$

Recall at this point from Arveson [1] that for every CSL $\mathcal{L}$ there exists a minimal weak* closed algebra $\mathcal{A}_{\min}$ such that i) $\mathcal{A}_{\min}$ contains a maximal abelian selfadjoint algebra (masa), and ii) $\operatorname{lat}\mathcal{A}_{\min} = \mathcal{L}$. Every weak* closed algebra satisfying those two conditions is called an Arveson algebra.

Corollary 2.3. *Let $\mathcal{L}$ be a CSL acting on a separable Hilbert space K, $\mathcal{B}$ be an Arveson algebra of $\mathcal{L}$, $\mathcal{M}$ be an ABSL and $\mathcal{A} = \operatorname{alg}\mathcal{M}$. Then the LTPF holds for $\mathcal{A}$ and $\mathcal{B}$ and $\mathcal{L} \otimes \mathcal{M}$ is reflexive.*

Proof. Recall from Section 1 that there exists a compact metric space X, a finite Borel measure μ and is a standard preorder $\leq$ acting on X such that $\mathcal{L}$ is unitarily equivalent to $\mathcal{N} = \mathcal{L}(X, \mu, \leq)$. By [3, Theorem 1], we have that

$$\mathcal{N} \otimes_{ext} \operatorname{lat}\mathcal{A} = \operatorname{lat}(\mathcal{A}_{\min} \otimes \mathcal{A})$$

where $\mathcal{A}_{\min}$ is the smallest ultraweakly closed algebra containing a masa for which $\mathcal{N} = \operatorname{lat}\mathcal{A}_{\min}$. Since every ABSL is reflexive [2], we have that

$$\mathcal{N} \otimes_{ext} \mathcal{M} = \operatorname{lat}(\mathcal{A}_{\min} \otimes \mathcal{A}).$$

It follows by Theorem 2.2 that $\mathcal{N} \otimes \mathcal{M} = \operatorname{lat}(\mathcal{A}_{\min} \otimes \mathcal{A})$ and thus $\mathcal{N} \otimes \mathcal{M}$ is reflexive. Hence $\mathcal{L} \otimes \mathcal{M}$ is reflexive and if $\mathcal{B}_{\min}$ is the smallest ultraweakly closed algebra containing a masa for which $\mathcal{L} = \operatorname{lat}\mathcal{B}_{\min}$, then

$$\mathcal{L} \otimes \mathcal{M} = \operatorname{lat}(\mathcal{B}_{\min} \otimes \mathcal{A}) \supseteq \operatorname{lat}(\mathcal{B} \otimes \mathcal{A}) \supseteq \operatorname{lat}\mathcal{B} \otimes \operatorname{lat}\mathcal{A} = \mathcal{L} \otimes \mathcal{M}$$

and the LTPF holds for $\mathcal{A}$ and $\mathcal{B}$. $\square$

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EXAMINING REASONING-AND-PROVING IN THE TREATMENT OF COMPLEX NUMBERS IN IRISH SECONDARY MATHEMATICS TEXTBOOKS

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Abstract. This study examined student tasks in the area of complex number operations in six Irish secondary mathematics textbooks and a Project Maths teaching and learning plan for reasoning-and-proving (RP) opportunities. At the ordinary level, 9.1% of 1274 student tasks were coded as RP. At the higher level, 13.3% of 1373 tasks were coded as RP. The majority of argument opportunities in ordinary level materials were the lowest form of argument - proof-writing exercises. At the higher level, the majority of argument opportunities were within argument-specific or argument-general categories. Less than 2% of the tasks at the ordinary or higher level involved pattern identification or conjecture development. Students were not asked to test conjectures, construct counterexamples, develop proof subcomponents, or formulate RP objects in any of the seven sets of materials. Only one RP task appearing across all seven sets of materials involved the use of technology. The implication of these results as well as how textbook materials could be redesigned using the RP framework are discussed.

1. Introduction

The construction of proofs and its related set of actions: identification of patterns, construction of conjectures [30], and reasoning [18] are important fundamental practices that mathematicians frequently use to construct mathematical ideas. Furthermore mathematicians [26], mathematics educators [1], as well as national reform documents [4, 22, 21] have pointed out that these practices should also be important components of school mathematics classrooms.

Research in mathematics classrooms around the world has established that textbooks are an important force in shaping teachers'

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classroom lessons [2, 21, 32, 33]. Consequently, as educational systems promote mathematics education reform through national level curriculum documents, textbooks are often redesigned to operationalize that change for teachers and their students. An important section of Ireland's reform of its secondary mathematics program, Project Maths, is synthesis and problem-solving skills. This component appears within each of the five mathematics content strands spanning the leaving certificate syllabus and expects students to explore patterns, formulate conjectures, explain findings, and justify conclusions [20]. These are all components of Sylianides' reasoning-and-proving (RP) framework [30]. The purpose of the study described here is to examine the nature of reasoning-and-proving in the area of complex numbers in six different Irish secondary mathematics textbooks including a teaching and learning plan developed by individuals associated with the national Project Maths reform effort.

2. Background

2.1. **Textbooks' Influence on Classroom Instruction.**

While there might not be an isomorphism between the mathematics content and how that content is presented in textbooks and the classroom lesson or enacted curriculum [27] the former plays a strong role in determining the latter. For example, a recent survey in the United States involving 7,752 science and mathematics teachers had the following findings: over 80% of mathematics teachers used a commercially published textbook; at least 67% of mathematics teachers covered 75-100% of the content of their textbooks; and nearly half of the mathematics teachers surveyed used their textbooks 75% of the class time [2]. This strong influence of textbooks on teachers is also prevalent in Ireland as more than 75% of the Irish secondary school teachers in one survey used one mathematics textbook daily [24].

2.2. **Examining Proof-Related Constructs in Textbooks.**

A number of themes appear within research involving the presence of proof related processes in textbooks around the world. First, students are provided with limited opportunities to engage in the development of proof-related processes within textbook exercise sections [6, 23, 30, 31]. For instance, only 6% of over 9000 tasks appearing across twenty U.S. secondary mathematics textbooks provided students with opportunities make conjectures, develop arguments, find counterexamples, or correct mistakes in arguments [31]. Second, there is variability in students' proof opportunities across different mathematics content areas. For instance, middle school students (ages 12-14) were provided with more opportunities to engage in the development of valid arguments or proofs in number than in algebra within a reform-oriented mathematics textbook in the United States [30]. Third, proof related processes differ by textbook. For example, within a polynomial functions unit a U.S. reform-oriented secondary mathematics textbook contained five times as many instances of reasoning-and-proving as a more conventional reform-oriented secondary mathematics textbook [6].

3. Framework

A framework adapted from the work of Stylianides [30] was used in this study. It consists of five main components: pattern identification; conjecture development; argument construction; technological tools; and reasoning-and-proving objects. These components are described briefly here, but the interested reader can get more details in [6]. Pattern identification instances within student tasks of the textbook occur when data is presented or students are asked to generate a set of data in a variety of different representational forms and locate regularities within them. Those regularities can be of two different forms: plausible and definite. Definite patterns are those which an expert can identify and provide compelling evidence for their existence. Plausible patterns, on the other hand, are not unique to a set of data.

Conjecturing consists of two interrelated actions: developing conjectures and testing conjectures. A conjecture is defined as an attempt to apply some regularity seen in a set of data to values beyond that set of data. Conjecture testing is when student tasks ask for a given conjecture to be tested through the location of one or more examples that meet the criteria of the conditions surrounding the conjecture.

A valid argument opportunity asked students to complete a problem that consisted of three components: a set of accepted statements; modes of argumentation; and modes of argument representation [29]. Accepted statements consist of theorems, definitions, axioms, etc. Modes of argumentation can be considered the glue that holds the argument together and consist of logical rules of inference (e.g., modus tollens), use of cases, indirect reasoning, etc. Modes of argument representation are the ways in which the argument can be communicated to others and can consist of written text, pictures, algebraic symbols, etc. A total of five valid argument categories appear within the framework. First, proof writing exercise opportunities contained the three characteristics of a valid argument as well as the further condition that they were not new for students. That is, students had seen similar arguments within the exposition sections of the textbook or had been asked previously to develop similar arguments.

Second, a valid argument opportunity of the argument-specific type occurred if students were asked to complete a task that contained the three components of a valid argument and the assertion that was to be proved was of a specific nature. Third, a valid argument opportunity of the argument-general type occurred if students were asked within an exercise section to create a valid proof and the assertion validated was of a general nature. Fourth, counterexamples were considered a valid argument category and were considered to exist if an exercise asked students to construct a counterexample to a given statement or to show that a given general statement was not always true. Fifth, a proof subcomponents opportunity was coded if students were asked to develop one or more statements or one or more explanations within a set of statements that contained the three components of a valid argument.

The different components of the framework described above may be interconnected with one another. That is, students' pattern identification opportunities may be connected to conjecture development and testing opportunities. If patterns led to conjecture opportunities they were considered to be conjecture precursors, otherwise they were denoted as conjecture non-precursors. In a similar vein, conjecturing opportunities may lead to students' construction of valid arguments that are of the specific or general variety as described earlier. Conjecturing opportunities that were connected to argument opportunities were considered to be argument precursors otherwise they were denoted as argument non-precursors. The process of constructing and testing a conjecture may also be bypassed and lead directly to the construction of an argument.

Technology such as Geogebra can play a role in the identification of patterns, construction of conjectures, and development of arguments. For instance, the graphical representations of a function can lead students to identify patterns about the relationship between the number of minima or maxima that cubic functions can have. The matrix capabilities of a graphing calculator could be used to locate a counterexample to show that matrix multiplication is, in general, non-commutative.

The last component of the framework consists of reasoning-and-proving objects. Reasoning-and-proving objects consist of definitions, corollaries, theorems, etc. These objects are connected to the argument construction category as students can work on developing the wording for a theorem after developing an argument to validate its existence. At the same time, students may construct definitions, hence, this category could stand apart from the development of valid arguments. Moreover, reasoning-and-proving objects have the potential to be objects within which students identify patterns, formulate and test conjectures, and construct arguments.

4. Methodology

4.1. **Choice of Topic Focus.** The topic area of focus for this study was complex number. This content area was chosen due to its appearance as an entire unit across three ordinary level and three higher level texts as well as the fact that a teaching and learning plan was created for it. The leaving certificate syllabus [20] contains the following learning outcome for students at the ordinary and higher levels: investigate the operations of addition, multiplication, subtraction and division with complex numbers $\mathbb{C}$ in rectangular form $a+ib$ (p. 25). The word investigate was linked with the potential for students to engage in the components of identifying patterns, formulating conjectures, and creating arguments. In addition, the syllabus states that students learning about the number content strand, within which complex numbers appears should frequently encounter the following actions: explore patterns and formulate conjectures, explain findings, and justify conclusions (p. 27).

4.2. **Materials.** There are three textbooks that Irish schools can choose from for ordinary level students at the foundation and ordinary level: *Texts & Tests 3* (TT) [19]; *New Concise Project Maths 3B* (NC) [12]; and *Active Maths 3: Book 1* [15]. The Active Maths textbook also has a companion text that serves as a set of activities for students to complete: *Active Maths 3 Activity Book* [14]. Analyses for both of these Active Maths texts will be combined and be denoted in the results section by AM. Individuals associated with Project Maths have created a series of teaching learning plans (TLP) designed for teachers to implement. They have created one such TLP for complex numbers [25]. This TLP encompasses 52 pages and consists of six activities for students along with information for teachers about how to implement these plans.

Three texts were examined at the higher level: *Active Maths 4: Book 1* [17], *Text & Tests 6* (TTH) [5], and *New Concise Project Maths 5* (NCH) [13]. Similar to the ordinary level the Active Maths textbook also had a companion higher level textbook: *Active Maths 4: Activity Book* [16]. The results for the Active Maths program at the higher level will include both books and will be denoted by AMH.

4.3. **Coding.** The six textbook complex number units and teaching and learning plan were examined for RP tasks by the author. A variety of words associated with the framework such as *pattern, describe, conjecture, proof, proving, prove, show, verify, explain*, *investigate* and *justify* were used to identify potential RP tasks. These potential RP elements were more carefully examined using the descriptions of the framework components to determine if they were indeed RP tasks. Tasks using the words *pattern* and *conjecture* were considered to be candidates for the identification of a pattern and development of a conjecture categories. The word *test* needed to be present for tasks to be considered conjecture test candidates. Tasks involving the words *proof, proving, prove, show, verify, explain*, and *justify* were considered to be candidates for argument-general, argument-specific, and proof-writing exercises. The work involved in solving tasks using the word *investigate* was examined in a more open-ended fashion to see which of the RP categories it fit.

4.4. **Inter-rater Reliability.** While the author coded the textbook tasks another researcher familiar with the framework was asked to use the methodology described above to locate and categorize RP tasks within the ordinary level textbook and from a higher level textbook in order to determine inter-rater reliability. Overall, the inter-rater reliability was excellent as Cohen's Kappa was 0.9248 for the ordinary level text excerpt and 0.9135 for the higher level text excerpt. This is a high level of agreement as a value of 1 would denote perfect alignment.

4.5. **Analysis.** Analysis of student tasks began with counting the number of student tasks on each page of each textbook unit and the teaching and learning plan. The number of RP tasks identified during the coding stage were also counted and the percentages of tasks that were coded as RP was calculated. The number of reasoning-and-proving objects that required an argument in order to be validated (e.g., theorem) were counted and the number that either were accompanied by a valid argument presented in the textbook or were left to students to prove were counted in order to calculate the percentage of reasoning-and-proving objects proved. A Chi Square analysis was conducted on the frequency of non-RP and RP tasks in the three ordinary texts and the teaching and learning plan. A Chi Square analysis was conducted on the frequency of non-RP and RP tasks in the three higher level texts. An $\alpha = 0.05$ was used for all statistical tests.

5. Results

Table 1 shows the breakdown in reasoning-and-proving for student tasks within the complex number unit across the four ordinary level sets of materials and the three sets of higher level materials. Overall, students at the higher level were provided with more opportunities to engage in RP than ordinary level students as the average for ordinary level materials was 9.1% while the average for higher level materials was 13.3%. There was less variation in the percentage of student tasks coded as RP within the higher level texts when compared with the ordinary level materials. That is, the higher level texts RP task percentage varied from 11.8% to 15.0% while ordinary level materials varied from 4.5% to 15.5%. Indeed, Chi Square analyses on the ordinary level materials revealed that there were statistically significant differences in terms of the RP tasks afforded to students, $\chi^2(3) = 31.852$, $p < 0.001$. At the higher level, Chi Square analyses illustrated that the RP task opportunities were monolithic across the different texts as differences were not statistically significant, $\chi^2(2) = 2.188$, $p = .338$.

Table 1. Total Tasks and RP Tasks in Ordinary and Higher Level Materials

Textbook	Tasks	RP Tasks	Percent RP Tasks
		Ordinary	
AM	418	19	4.5%
TT	312	20	6.4%
NC	336	52	15.5%
TLP	208	25	12.0%
Total	*1274*	*116*	*9.1%*
		Higher	
AMH	473	60	12.7%
TTH	373	44	11.8%
NCH	527	79	15.0%
Total	*1373*	*183*	*13.3%*

Table 2 shows the breakdown in different categories for RP student tasks appearing in the ordinary level and higher level materials. In this table P-WE refers to proof-writing exercises, P refers to definite patterns, C refers to conjecture development, ArgS refers to argument-specific instances, and ArgG refers to argument-general instances. Elements separated by dashes such as P-C represent tasks that provide students with pattern identification and conjecture development opportunities. The majority of RP tasks within the AM and NC complex number units were coded as proof-writing exercises, which is the lowest level of valid arguments. In the TT complex number unit, there was an even split between proof-writing exercises and argument-specific/argument-general categories. The analysis found that 26.3% ($\frac{5}{19}$) of AM unit RP tasks, 0% ($\frac{0}{20}$) of TT unit RP tasks, 3.8% ($\frac{2}{52}$) of NC unit RP tasks, and 44% ($\frac{11}{25}$) of TLP RP tasks involved the identification of patterns or construction of conjectures. While AM provided more opportunities in pattern identification and conjecture development than the TT and NC textbook units, the TLP provided students with over twice as many opportunities to engage in these components of the framework than the AM textbook unit. For instance, the exposition sections of the NC, TT, and AM Activity Book presented the fact that multiplying a complex number by i results in an anticlockwise rotation of the complex number by 90 degrees. The TLP, on the other hand, provided students with the following tasks.

1. If $z = 3+4i$, what is the value of iz, i^2z, i^3z, i^4z? Represent your results on an Argand Diagram joining each point to the origin $o = 0 + 0i$.
2. Investigate what is happening geometrically when z is multiplied by i to get iz? Use geometrical instruments and/or calculation to help you in your investigation. (p. 44).

Consequently, a fact that was presented in the other textbook units became an investigation for students within the TLP providing them with an opportunity to detect a definite pattern.

There were also differences across textbook units within the argument-specific and argument-general categories across the ordinary level units as seen in the following percentages: AM textbook 26.3% ($\frac{5}{19}$); TT textbook 50.0% ($\frac{10}{20}$); NC textbook 42.3% ($\frac{22}{52}$); and TLP 28% ($\frac{7}{25}$). The majority of these instances, however, were within the argument-specific category. The NC textbook unit did not ask students to construct proofs of the argument-general type. Consequently, within the three ordinary level texts student tasks requiring proofs of the argument-general type were rare occurrences.

The TLP contained the greatest percentage of RP tasks within the argument-general category at 20% ($\frac{5}{25}$) when compared to the three ordinary level texts. This is seen in the proof of the idea that the sum of a complex number and its conjugate is always a non-complex real number. In the TLP students were given the general form of a complex number $z = a+bi$ and asked to find its conjugate, $\bar{z} = a - bi$. Next students were asked to calculate the sum $z + \bar{z}$. In the TT and NC textbook units students were told in the exposition section that the sum of a complex number and its conjugate are always a real number without an accompanying proof. In the AM complex number unit, students were asked to verify that $z + \bar{z}$ is a real number for a specific case when $z = 13 + 2i$ and $\bar{z} = 13 - 2i$.

There were interesting differences in students' opportunities to identify patterns and formulate conjectures across the three higher level texts. The percentage of student tasks that were coded as identification of patterns or formulation of conjectures was 3.3% ($\frac{2}{60}$) in AMH and 6.3% ($\frac{5}{79}$) in NCH. On the other hand, 25% or ($\frac{11}{44}$) of the tasks in TTH involved the identification of pattern or construction of conjectures. In the AMH textbook unit, there was an even split between proof-writing opportunities and students' opportunities to develop specific or general arguments. In the TTH and NCH textbook units, argument-specific and argument-general opportunities outnumbered proof-writing exercises.

Recall that technology can be used to provide students with opportunities to engage in various components of the RP framework. Out of a total of 299 RP tasks only one of these used technology. Patterns did not always lead to conjectures or arguments. At the ordinary level, there were a total of 11 tasks that involved pattern development and three were coded as conjecture or argument precursors. There were a total of nine conjecture opportunities at the ordinary level and seven of these were tied to arguments. At the higher level, seventeen tasks involved pattern identification and a total of eleven of these tasks involved conjecture or argument development. There were twelve tasks that involved conjecture development at the higher level. Of these tasks, five were connected to argument development. There were no tasks across the seven sets of Irish secondary mathematics materials that specifically asked students to test conjectures, develop reasoning-and-proving objects, construct proof subcomponents, or create counterexamples.

Textbook	RP Tasks	P-WE	P	C	P-C	P-C-ArgS	P-C-ArgG	C-ArgS	ArgS	ArgG
						Ordinary				
AM	19	13	0	0	1	1	0	3	0	1
TT	20	10	0	0	0	0	0	0	8	2
NC	52	29	1	0	0	0	0	1	21	0
TLP	25	9	8	1	0	1	0	1	0	5
						Higher				
AMH	60	29	0	0	1	1	0	0	5	24
TTH	44	11	4	0	4	0	3	0	7	17
NCH	79	35	2	0	2	0	0	1	14	25

TABLE 2. Breaking Down Tasks into RP Categories

6. DISCUSSION

The similarity between this framework and frameworks used in for the analysis of proof related constructs in other countries enables cross-country comparisons. The next few paragraphs describe some of these comparisons. Overall, students at the higher level were provided with more opportunities to engage in RP at the higher level (13.3%) than at the ordinary level (9.1%). These percentages are lower than what was found in a set of U.S. reform-oriented mathematics textbooks designed for students ages 12-14 (40% of 4855 tasks involved RP) [30] as well as what was found in a polynomial functions unit from a reform-oriented U.S. mathematics textbook designed for students ages 14-18 (22% of 1158 tasks involved RP) [6]. Davis also found that 4% of 1129 tasks involved reasoning-and-proving in a more conventional U.S. mathematics textbook designed for students ages 14-18 [6].

Among 9742 tasks appearing in 20 U.S. textbooks in the areas of exponents, logarithms, and polynomials 3.1% involved developing and evaluating arguments [31]. A total of 3.5% of tasks appearing in the ordinary level materials and 7.1% of tasks appearing in the higher level materials involved argument development.

The examination of a set of U.S. middle school reform-oriented texts found that 27% of RP tasks involved the identification of patterns or development of conjectures in a U.S. reform-oriented middle school mathematics textbook program [30] . A total of 15.5% of RP tasks at the ordinary level and 9.9% of RP tasks at the higher level involved the identification of patterns or development of conjectures. [31] found that 14.8% of tasks coded as reasoning involved the development of conjectures. A total of 7.8% of RP tasks at the ordinary level and 6.6% of RP tasks at the higher level involved the development of conjectures. In sum, the Irish secondary mathematics textbook complex number units examined as part of this study contained smaller percentages of RP than reform-oriented curricula in the United States. The Irish textbook units contained a higher percentage of student tasks involving RP opportunities overall and a higher percentage of student tasks involving argument opportunities than a collection of U.S. textbook units involving exponents, logarithms, and polynomials. In contrast, this collection of U.S. textbook units contained a higher percentage of conjecturing opportunities than Irish textbooks. Reform-oriented textbook materials in the U.S., designed for students ages 12-18 contained a higher percentage of student tasks coded as RP. The implications of the findings from this study for redesigning Irish secondary mathematics textbooks are discussed in the following paragraphs.

6.1. **Redesigning Textbook Activities to Make RP More Central to Learning Mathematics.** One of the most common RP categories appearing in the student tasks across the six textbook complex number units was proof-writing exercises. Below is one example of such a task.

Let $u = 2 + i$ and $w = 3i$. Show that: $u\bar{w} + \bar{u}w$ is a real number [12] (p. 6).

Such tasks provide students with opportunities to engage in a component of reasoning-and-proving, but this is presented as an end in and of itself, a destination instead of the beginning of a mathematical journey. A mathematician would want to know if this instance comprised a pattern and would proceed to determine this by testing a variety of other complex numbers. Not only does this process provide the user with more practice it also enables him or her to engage in a fundamental component of the RP process, pattern identification. Once he or she was convinced that this was more than merely a coincidence he or she would develop a conjecture that it always held and proceed to test it with examples different from the ones already encountered. Once convinced that the conjecture could be true a mathematician would attempt to construct an argument to this effect. Students representing different ability levels could engage in the necessary cognitive work to answer such a problem. For fundamental and some ordinary level students the symbolic manipulation work could be offloaded to technology such as Geogebra, an ICT tool that Project Maths has embraced as indicated by its presence on its website (`http://www.projectmaths.ie/geobra/`).

Using Geogebra to calculate $u\bar{w}+\bar{u}w$ with $u = a+bi$ and $w = c+di$ yields $2ac + 2bd$, showing that the result is indeed a real-number as it no longer involves the complex number i. This work illustrates that the result is true due to the black box nature of the Geogebra CAS [3], fulfilling a role of verification, but does not shed light on why this statement is true, the explanation purpose of proof. The CAS could be used to calculate $(a + bi)(c - di)$ and $(a - bi)(c + di)$ separately to understand why terms involving i drop out of the calculation. Higher level students could complete these calculations by hand to prove that as well as why this assertion is always true. Last, mathematicians would engage in constructing the statement of the theorem or reasoning-and-proving object that was just proved. Students of varying levels could write such a theorem such as the following: Given complex numbers u and w, $u\bar{w} + \bar{u}w$ is a real number. Providing students with opportunities to complete the different components of this investigation would require additional class time. Thus there would be less time for students to work on other problems that would presumably provide them with procedural practice, but this type of practice could take place within the context of this investigation as students develop the work from which patterns could be identified and conjectures investigated. Moreover, the inclusion of activities such as this would provide students with more authentic RP opportunities.

If textbook designers wish to design resources for ordinary level students that contain fewer argument opportunities the RP framework used in this study suggests four different options. First, textbook developers could defer proofs of mathematical ideas to a later time. However, if the developers choose to pursue this option they should note within the textbook materials that a proof for this mathematical idea exists but it is too difficult for students to understand at this point in their education. Second, textbook developers could provide students with more opportunities to engage in pattern identification and conjecture development without moving to an argument but stating that it is possible for a proof to be constructed to show this idea is always true. Third, textbook developers could provide students with opportunities to engage in the development of proof subcomponents with regard to more complex arguments. Fourth, textbook developers could provide students with opportunities to develop rationales [30] or arguments that are not complete with the caveat that these do not denote valid arguments. The low incidence or nonexistence of conjecture testing, proof subcomponents, counterexamples, technology in the development of RP components, as well as RP objects suggests that textbook materials for foundation, ordinary, and higher level students could be reimagined to include these areas. The last section examines the alignment between reasoning-and-proving as it appears within the Project Maths Leaving Certificate syllabus and the seven sets of textbook materials examined in this study.

6.2. **Alignment between Project Maths Syllabus and Textbooks: Complex Numbers.** Recall that the Project Maths leaving certificate syllabus asserts that students at all levels should be provided with opportunities for students to: explore patterns and formulate conjectures; explain findings; and justify conclusions. These actions roughly translate into the three main components of the RP framework used in this study. However, 1.4% $(\frac{18}{1274})$ of ordinary level tasks and 1.3% $(\frac{18}{1373})$ of higher level tasks involved pattern identification or conjecture development. This low percentage suggests a misalignment between the six textbook units and the Leaving Certificate syllabus. Until textbooks are redesigned the responsibility for rectifying this misalignment will fall on the shoulders of teachers who will need to think carefully about how these curricula can be supplemented with activities that provide students with opportunities in these areas in order to align students' classroom experiences with the ideal visualized within the leaving certificate syllabus.

The TLP was more aligned with the syllabus as 5.3% ($\frac{11}{208}$) of its tasks provided students with opportunities to engage in pattern identification and conjecture formulation. However, students work with patterns and conjectures often ended with these RP components instead of moving to the development of an argument or at least a statement in the materials stating that a proof of this mathematical idea exists but is beyond the ability level of the students. That is, out of a total of 11 tasks that involved pattern formulation or conjecture development, 9 or 82% did not lead to an argument opportunity. Thus these tasks may lead students to believe that a pattern is sufficient to show that a mathematical idea is always true, promoting what is referred to as an empirical proof scheme [10].

There were also differential opportunities for students of various ability levels to engage in the development of valid arguments within student task sections across the six textbooks. The Project Maths syllabus asks students at the higher level to develop more arguments [7] and explicitly states that at the higher level, particular emphasis can be placed on the development of powers of abstraction and generalisation and on the idea of rigorous proof, hence giving learners a feeling for the great mathematical concepts that span many centuries and cultures [20] (p. 13). However, students at the foundation and ordinary levels may develop a false impression of mathematics when the vast majority of mathematical ideas that they encounter in the complex number unit are not justified. These students may not feel that mathematical ideas need to be justified and, consequently, they may struggle to see the purpose behind their teachers' demands for justification. The large differences in the number of proofs between textbooks designed for ordinary level students and textbooks designed for higher level students can also influence teachers' perceptions about students' abilities. That is, when the textbooks that teachers follow on a regular basis [24] expect ordinary level students to engage in reasoning-and-proving so infrequently, teachers themselves may not believe that students are capable of work of this nature.

7. Conclusion

This study examined the prevalence of reasoning-and-proving in complex number operations within seven different Irish textbook materials. Despite the fact that reasoning-and-proving is a central act of practicing mathematics [26] and the Project Maths syllabus has advocated the importance of this concerted set of actions in learning mathematics [20], the analyses here suggest that this reality is not reflected in Irish mathematics text materials for ordinary and higher level students. Indeed, the current set of materials underemphasizes the importance of pattern identification, conjecture development and testing, argument construction, the use of technology in RP, and the construction of RP objects. The framework used in this study as well as the concrete suggestions provided in this paper can be used by textbook developers to redesign Irish textbook materials so that RP occupies a much more central and prominent location in learning school mathematics.

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TO WHAT DOES THE HARMONIC SERIES CONVERGE?

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ABSTRACT. We consider what value the harmonic series will converge to if evaluated in the obvious way using some standard computational arithmetic types.

1. Introduction

In Niall Madden's article about John Todd [5], he mentions a topic we might bring up in an introductory course on analysis: the question of convergence of the harmonic series and what would happen if it was naïvely evaluated on a computer using finite precision. In particular, the obvious way to evaluate

$$s_N = \sum_{n=1}^{N} \frac{1}{n}$$

is to use an algorithm similar to Algorithm 1. If we run this algorithm using integer arithmetic where division commonly rounds down, then the sum converges to 1, as after the first term the reciprocal becomes zero. If we use a different rounding, where $1/2$ rounds to 1, then the sum will converge to 2.

Algorithm 1 Simple algorithm for calculating the harmonic series.

$\mathtt{s} \leftarrow 0$, $\mathtt{N} \leftarrow 1$
for ever **do**
  $\mathtt{s} \leftarrow \mathtt{s} + \frac{1}{\mathtt{N}}$, $\mathtt{N} \leftarrow \mathtt{N} + 1$
end for

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2. Floating Point

Of course, no one is (deliberately) likely to try and evaluate this sum using integer arithmetic. However, using floating point arithmetic seems more promising. Students, when learning numerical programming, sometimes assume that the answer given by the computer is exact, and the harmonic series can be a good example to help them understand what can go wrong.

We usually ask them to reason as follows. We know that if `s` and `N` were double precision floating point numbers, the sum will gradually grow while what we add gradually decreases. Also, because the sum and most reciprocals cannot be represented exactly as floating point numbers, the reciprocal and the sum will be rounded. Thus, if we round to the nearest representable number, we see that the sum must eventually become constant. We reach a similar conclusion if we round toward zero or toward $-\infty$.

This is an existence proof for the limit, but it might be of interest to know what value the sum converges to, or at what term the sequence becomes constant. Floating point behaviour is well defined [2, 3], so can we use this to make an estimate?

Let's consider the commonly encountered format of *binary64* double precision floating point numbers. In this format, one bit is used to store the sign of the number, 11 bits are used to store a binary exponent and the remaining 52 bits are used to store bits of the binary expansion of the number. The value of a particular pattern of bits is, usually, interpreted as:

$$(-1)^S \left(1 + \sum_{i=1}^{52} \frac{B_i}{2^i}\right) 2^{E-1023},$$

where S is the sign bit, $B_1, \ldots B_{52}$ are the bits of the binary expansion of the number and E is the value of the exponent bits interpreted as positive binary integer. Note, the leading bit is not stored, because if the number is non-zero then it must always be one. Some numbers, including zero, are stored as special cases, which are indicated by making all the exponent bits zero or one. Other special types of number stored in this way include plus/minus infinity, NaN values (to indicate results of invalid operations) and special small *denormalised* numbers.

For floating point numbers of the format described above, we can consider the change in the value resulting from flipping the least significant bit, B_{52}. This change is a change in the *unit of least precision*, and a change of this size is often referred to as one ULP. Where the difference between the floating point value and the exact value is important, we will use the typewriter font for $\mathtt{N}$ and $\mathtt{s}$.

3. Estimates

As a crude estimate, since we know a double precision floating point has a significand of 52 bits, and the partial sums $\mathtt{s}_N$ are always bigger than 1, it must converge before term 2^{52+1}, as at this point adding each $1/\mathtt{N}$ will give a result that is small enough so it won't be rounded up to the next number.

We can improve on this estimate, as we know that the sum will be somewhat bigger than one. Denoting by $\lfloor x \rfloor_2$, the largest power of two less than x, the sequence will stop increasing when

$$\frac{1}{\mathtt{N}} < 2^{-52-1} \lfloor \mathtt{s}_N \rfloor_2$$

By using Euler's estimate for the harmonic sum [1], we know $\mathtt{s}_N \approx s_N \approx \log(N) + \gamma$ where

$$\gamma = 0.57721566490153286060651209008240243104215933593992\ldots$$

We can convert this to

$$2^{53} < N(\lfloor \log(N) + \gamma \rfloor_2),$$

which happens when $N > 2^{48}$. We expect a value of $\mathtt{s}_N$ close to

$$\log(2^{48}) + \gamma = 33.8482803317789077\ldots,$$

though there may have been significant accumulation of rounding errors in the sum. We can, however, get a bound on this error; in round-to-nearest mode the error in each operation should be bounded by 0.5 ULP. Concentrating on the errors associated with the sum[1], for each term when the sum is between 1 and 2, we will have an error of at most 2^{-53}. Similarly, for each term when the sum is between 2 and 4, the error will will be at most 2^{-52}, and so on. Thus, if T_r is

$$T_r = \min \{ n \in \mathbb{N} : \mathtt{s}_n > r \},$$

[1] We could also calculate the error associated with evaluating the reciprocals, however this will be considerably less than 2^{-53} for most terms. Consequently, the error terms from the sum dominate.

then we can accumulate these error bounds and bound the error between $\mathtt{s}_n$ and the exact sum by

$$\sum_{n=1}^{\infty} (\min(N, T_{2^n}) - \min(N, T_{2^{n-1}}))\, 0.5 \times 2^{-52+n-1}. \tag{1}$$

Here, the first expression in parentheses counts the number of terms before N with the given error. Again, if we are willing to make the approximation $\mathtt{s}_n \approx \log(n) + \gamma$ then we find $T_r \approx e^{r-\gamma}$, and we can easily evaluate this expression numerically and find that we expect the error at term 2^{48} to be less than $0.921256\ldots$.

One might also attempt to consider the rounding errors to be independent random variables. Using $\mathrm{ULP}^2/4$ as a bound on their variance, one might hope to use the central limit theorem to predict a distribution for the error. However, as n becomes large, the bits of $1/n$ and $1/(n+1)$ that are being rounded are increasingly similar, and so independence appears a poor assumption.

4. Numerical Results

Evaluating a sum with 2^{48} terms is feasible on modern computer hardware. Naturally, if parallelisation of the calculation was possible, the sum could be evaluated on a cluster of computers or CPUs. However, the intermediate values are important in establishing what rounding occurs and the final value of the sum.

We evaluated the sum using an AMD Athlon 64 CPU, clocked at 2.6GHz. To avoid the use of extended (80-bit) precision, which is common on platforms based on the i387 floating point processor, the operating system was in 64-bit mode, where i387 instructions are not used. A C program, which explicitly selected the to-nearest rounding mode, was used to find when the sum converged and to what value. As predicted, the sum became constant when $N > 2^{48}$. In this environment, the calculation took a little more than 24 days.

The value of the sum was

$$\mathtt{s}_{2^{48}} = 34.1220356680478715816207113675773143768310546875,$$

or, the bits of the floating point number were `0x40410f9edd619b06`. The difference between this and Euler's approximation is $0.2737\ldots$, within our estimated error bound of 0.921256.

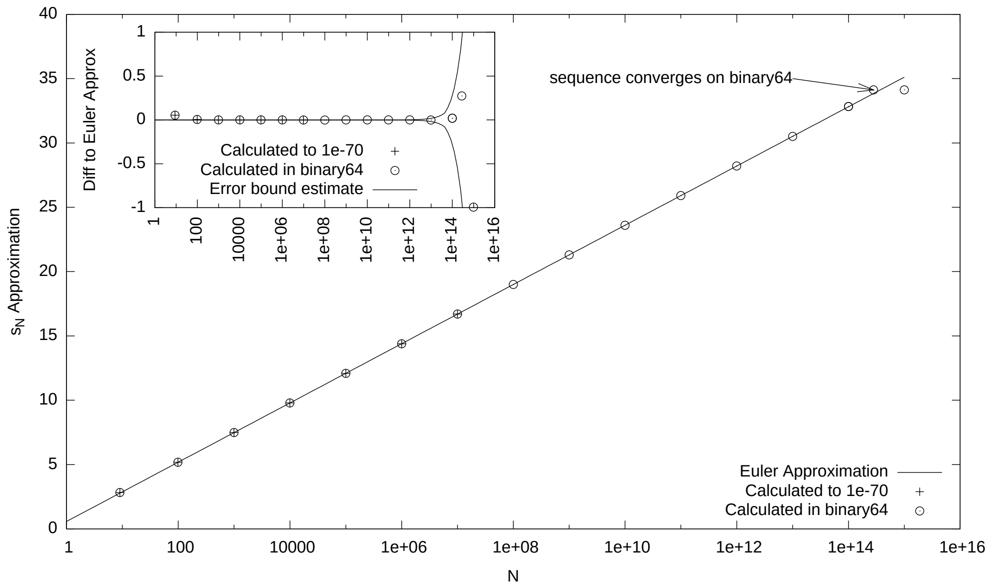

Figure 1. Approximations to the partial harmonic sum. The inset graph shows the distance between Euler's estimate and the numerical estimates.

Figure 1 shows Euler's approximation of the partial harmonic sums, $\log(N) + \gamma$, along with the numerical values of $\mathtt{s}_N$. For comparison, we also used an arbitrary precision calculator to estimate the value of s_N when the rounding error at each step is bounded by 10^{-70} for smaller values of N. We see that the values agree well, until we come close to the point where the numerical $\mathtt{s}_N$ converges.

Also shown in Figure 1 is the difference between Euler's approximation and the numerical values. For comparison, the error bound from (1) is plotted. The differences all fall within the error bound estimates. We also note that until $N > 10^{13}$ the accumulated errors are unlikely to be a serious concern. Finally, observe that both numerical values diverge from Euler's approximation when N is small. This is because the error in Euler's formula is known to be $O(1/N)$. If the approximation is improved by including the next term, $\gamma + \log(N) + 1/2N$, this discrepancy is corrected.

5. Rounding Up

We can also ask what happens in other rounding modes. If we round down to the nearest representable floating point number, then the story will be similar to the previous case. In this case, since the sum is always rounded down, the sum will become constant when

$$\frac{1}{\mathtt{N}} < 2^{-52}\lfloor \mathtt{s}_N \rfloor_2$$

which we can estimate happens when $N > 2^{47}$. Again, we could make an estimate of the range of possible numerical values. Using Euler's formula directly we can get the upper end of the possible range. For the lower end, calculating the error due to rounding, we can use an expression similar to eq. 1, but where the error per term can be 1ULP, rather than 0.5ULP. Checking numerically[2], we find the sum converges to

$$32.8137301342568008521993760950863361358642578125$$

with bit pattern `0x404068284f1d338a` at term 2^{47}. Figure 2 shows this to be within the range we expect.

If we round up, the situation is a little more interesting. In this case, when

$$\frac{1}{\mathtt{N}} < 2^{-52}\lfloor \mathtt{s}_N \rfloor_2$$

we get an increase of one ULP each time the reciprocal is added. Once this condition is met, at term 2^{47}, the sum begins to act like a counter, beginning at $\mathtt{s}_{2^{47}}$ and incrementing by one ULP for each value of N, beginning at 2^{47}. However, when N reaches 2^{53}, not all integers can be exactly represented as floating point numbers. So, because we are in round-up mode, each increment of `N` actually results in `N` being increased by *two* (which is one ULP for `N`). Thus, after a further 2^{52} steps, we reach 2^{54} when the increments in `N` result in `N` being increased by 4, and so on.

Thus, after 2^{52} loop iterations, `N` and `s` are both incremented by one ULP for each iteration. At iteration 2^{52}, `N` has value 2^{52} and `s` has value of $\mathtt{s}_{2^{47}}$ increased by $2^{52} - 2^{47}$ ULP increments. As $\mathtt{s}_{2^{47}} \ll 2^{47}$, we see that `N` begins at a larger value, and so will remain ahead in this curious race. The largest non-special case value of the exponent is 1023, and will increase by one every 2^{52} iterations, so following another $(1024-52)\times 2^{52}-1$ iterations `N` will reach the largest finite double value of $2^{1024} - 2^{1024-53}$.

On the next iteration `N` will become `inf`, and then `1/inf` is zero. At this point, the sum will no longer increase! We conclude that

[2]Some additional care is required here, because if the compiler does constant folding, it will probably be done in the default mode (round to nearest). To avoid such issues, we disabled optimisation when compiling.

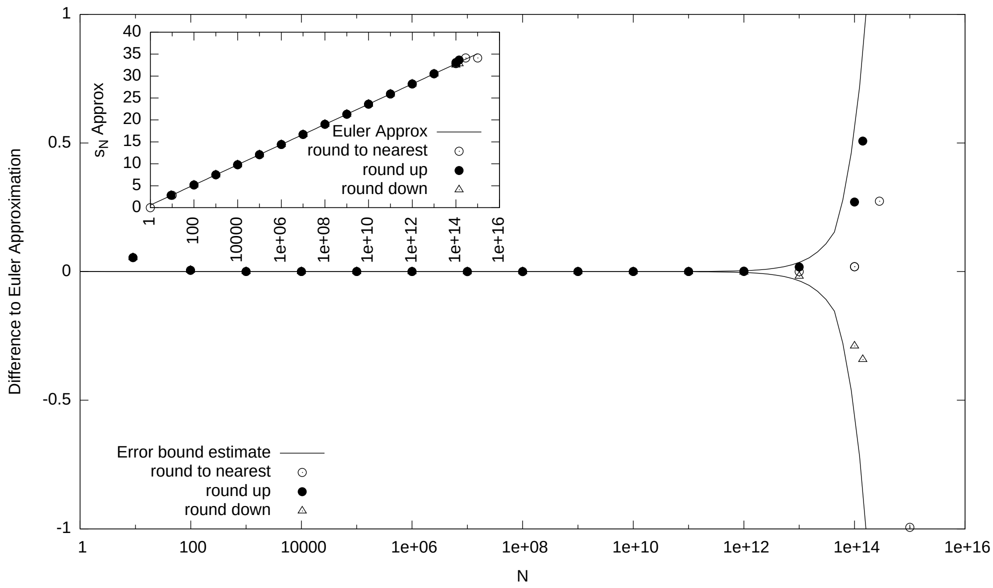

FIGURE 2. Differences between floating point estimates of the partial harmonic sum and Euler's approximation in round-to-nearest, round-up and round-down mode. The inset graph shows the actual values of the estimates.

$\mathtt{s}_N$ converges to a fixed value at this point. This happens after $2^{52} + (1024 - 52) \times 2^{52}$ loop iterations, when $\mathtt{s}_\mathtt{N}$ will have increased by $2^{52} - 2^{47} + (1024 - 52) \times 2^{52} = 973 \times 2^{52} - 2^{47}$ ULPs above its value at $\mathtt{s}_{2^{47}}$.

While evaluating 937×2^{52} loop iterations is not practical, we can evaluate $\mathtt{s}_{2^{47}}$ numerically in round-up mode, and we get

$$33.66247161023633083232198259793221950531005859375,$$

with bit pattern `0x4040d4cbdea63f2b`. If we treat this pattern as a hexadecimal integer, then each ULP increment will increase it by one, giving
`0x4040d4cbdea63f2b` $+ 937 \times 2^{52} - 2^{47} =$ `0x7d1054cbdea63f2b`. Interpreting this as a floating point number, we get

$$4596834217901867 \times 2^{926}.$$

Figure 2 shows the values of the sum in round-up, round-down and round-to-nearest mode compared to the value of Euler's approximation, up to the point where they stop increasing or increase by one ULP. The upper error estimate is for the sum in round-up mode and the lower error estimate is the corresponding value for round-down mode. Unsurprisingly, at a particular N values the rounded-up $\mathtt{s}_N$

values are larger than the round-to-nearest values, which are larger than the round-down values.

6. Conclusion

We have looked at the behaviour of the floating point versions of the harmonic series, and shown that we can actually give bounds on how they should behave. Modern PCs are capable of evaluating enough terms of the sequence to establish the values that the numeric sequence converges to in each rounding mode. We leave open the question of what happens if we use Algorithms other than Algorithm 1, for example if we sum the terms in the order suggested by Huffman Coding [4].

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Jeremy Gray: Henri Poincaré: A Scientific Biography, Princeton University Press, 2013, ISBN:978-0-691-15271-4

REVIEWED BY PETER LYNCH

Henri Poincaré was a mathematician of the highest calibre. He was also a leading physicist, adumbrating Einstein on a number of key points in relativity, a prominent philosopher of science, and an excellent expository writer. Gray's book is a comprehensive scientific biography of Poincaré. It embraces the broad scope of Poincaré's work, from his philosophical speculations to his popular writing, and gives a thorough overview of his extensive mathematical researches.

The book is very 'scholarly', by which I mean very heavy going in places. The opening Chapter, at 126 pages, is the longest in the book. From the title 'The Essayist', I was lulled into hoping for a gentle and readable account of Poincaré's popular writing, much of which is highly accessible, entertaining and thought-provoking. In fact, the chapter is laced with heavy, stolid philosophical discussion. It was a great struggle to wade through this chapter. I am of the view that it detracts from the book: it is misplaced, over-long and unlikely to appeal to most mathematicians. Fortunately, Chapters 2 to 18 are very different in style. Only in the final chapter does the author falter again, becoming immersed once more in philosophical cogitations.

Poincaré published about 500 papers and more than 30 books covering mathematics, theoretical physics and astronomy. In his mid-twenties, he introduced ideas that led to the transformation of several areas of mathematics: complex function theory, differential equations in the complex domain and non-Euclidian geometry. He did not stay working in a single field, deepening his knowledge of it. Nor did he flit from one field to another. Rather, he took up new interests while maintaining contact with earlier ones, so that his command of mathematics ultimately became vast in scope. As is well known, E. T. Bell described him as 'the last universalist'.

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In 1880, Poincaré submitted a 46 page essay for a prize competition, the goal of which was to advance the theory of ordinary differential equations. He amended and extended Fuchs' work, discovering in the process a connection between singularities in the complex plane and non-Euclidian geometry. The functions he considered now come under the rubric of automorphic functions, generalisations of elliptic functions that are invariant under groups of transformations, and that are of topical interest.

Poincaré had a life-long involvement with the dynamics of planetary motion. He made definitive contributions to the three-body problem, showing that what is now called chaos can make long-range prediction impossible. In 1885 King Oscar II of Sweden sponsored a competition calling for original mathematical contributions in one of four areas. First on the list was the problem of the long-term stability of the solar system.

Poincaré's submission was judged the winning entry. Weierstrass's conclusion that its publication 'will inaugurate a new era in the history of celestial mechanics' turned out to be prescient. Poincaré's competition entry was a profound work, and it led on to his three-volume *Les Méthodes Nouvelles de la Méchanique Céleste*, which remains of great influence even today.

Poincaré's study of the three-body problem led him to the question of small divisors. In the course of this, he systematised the rigorous treatment of divergent series. Asymptotic analysis has proved of inestimable value ever since.

Poincaré discovered many ideas that we now automatically associate with Einstein: how observers can compare measurements by exchanging light signals; how the speed of light is an impassable limit; the Lorentz group of transformations between frames; and the invariance of Maxwell's Equations under this group. All these discoveries were made by Poincaré independently of Einstein and mostly before him. Gray discusses the complex conjunction of circumstances that prevented Poincaré from taking more decisive steps to develop special relativity. He was nominated in 1910 for the Nobel Prize in Physics. But, although he won a 'handsome number of votes', he did not get the prize.

One of Poincaré's most profound contributions was to topology. He introduced the algebraic objects associated with a manifold, the homotopy and homology groups. His monumental *Analysis Situs* of

1895, together with five supplements published over the following decade, were of enormous influence. He used algebraic structures to distinguish non-homeomorphic spaces, inventing the subject of algebraic topology. The Poincaré Conjecture emerged from this work. In fact, Poincaré raised it as a question rather than a conjecture: Is a closed 3-manifold with trivial fundamental group necessarily homeomorphic to the 3-sphere? The conjecture was recently proved true by Grigori Perelman.

In relation to his mathematical inventiveness, Poincaré wrote several popular accounts of how he benefited from 'flashes of inspiration'. These came with 'brevity, suddenness and immediate certainty', but always following a period of intense dedication to a problem. In a similar vein, Felix Klein is quoted as describing how, sitting up late one night afflicted by asthma, the boundary circle theorem (Grenzkreis Theorem) appeared suddenly before him. Within a week, Klein had written it up and sent it off for publication.

Poincaré pioneered qualitative analysis of differential equations. One of his abiding principles was that a qualitative analysis of a problem must precede a quantitative treatment, 'for if it is by logic that one proves, it is by intuition that one invents'. He was keenly aware that understanding a proof is not the same thing as checking its logical consistency. But he was no slouch when it came to the hard graft of putting theory on sound foundations: 'In mathematics, rigour is not everything, but without it there is nothing'.

Gray's book is certainly a valuable addition to scholarship on the scientific work of Poincaré and will be of interest to many readers of this Bulletin. Although it is a monumental tome, there are some notable omissions. In particular, it is surprising that chaos theory is hardly discussed by Gray. This idea pervades modern science, and Poincaré's contributions were singularly original and profound. Yet the term 'chaos' does not even appear in the index. On that point, a subject index of four pages is quite inadequate for a book of this scope.

For decades, much of Poincaré's work was eclipsed, but more recent developments have changed that. There has been a revival of three-dimensional geometry, most notably with Thurston's pioneering work in the field of low-dimensional topology. Dynamical systems theory is now a topic of intense research. And chaos theory has had a profound affect on our world-view. As Gray writes, 'in each of the fields [in which] Poincaré worked, his achievements in mathematics, physics and philosophy are still alive and important today'.

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David Wells: Games and Mathematics: Subtle Connections, Cambridge University Press, 2012, ISBN:978-1-107-02460-1

REVIEWED BY MICHAEL BRENNAN

David Wells sets out to highlight the analogies between playing games and doing mathematics. In the Introduction he refers to three ways in which the mathematician does his/her work: "The mathematician as game-player observes and makes conjectures; the mathematician as scientist makes moves and spots possibilities; the mathematician as observer studies objects like the pieces in an abstract game of chess". The reader will already detect the redundancy.

The book starts by describing in everyday language features of well-known games such as chess, Hex and Go (at which the author is a master), then features of well-known maths-puzzles — the Edouard Lucas puzzles are prominent — and some results from prime numbers, Euclidean geometry, sequences and series. The narrative points to an argument that since there are parallels between the types of thinking, the obstacle-surmounting and the strategies that the mathematician and the game-player use, then. . . what? The conclusion is unclear. Wells never gets to the point of saying that in playing games one is doing mathematics, and/or vice versa. Even if such an identification is true there is no way of knowing so from this book, because crucially the author does not define what he means by a game. The reader will deduce that there are objectives and strategies involved, but these are involved in crossing a street, or boiling an egg.

Ironically, modern game theory would be happy to include street-crossing and egg-boiling in its compass. David Wells however makes no mention of game theory. Its omission robs him of such concepts as cooperative and non-cooperative games, notions that might usefully be applied to professional mathematicians, their methods, rivalries, piggy-backing, and mutual inspiration. Wells's book is reductionist

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in the way it attempts to squeeze mathematics into some notion of a "game": "To play around with a problem, to 'suck it and see',... is the very essence of mathematics. It is no coincidence of course that *playing* [his italics] is just what game players do" (P. 74). There is much innuendo by word-association: "Euler combines his dazzling ability for pattern-spotting with his equal ability to play the game of mathematics brilliantly" (P. 84); "We could say that MacLane thinks of mathematics from the start as game-like." (P.120); and there are chapters optimistically entitled "Game-like mathematics", "Euclid and the rules of his geometrical game" and "Mathematics becomes games-like", but the association is everywhere strained and comes to no meaningful insight.

Yet if all of this can be left to one side, there are some real merits to the book. Firstly however it should be said that the editing was too light-handed. Chapter 8 ("New concepts and new objects") opens, incredibly, with a list of over 150 nouns and adjectives from a mathematical dictionary: all the way from "Abelian group" through "knot" to "Zorn's lemma", just to show that mathematicians use new names for new things. The author adds the comment that the list could be longer, since some of the nouns are associated with verbs and adjectives "as in any colloquial language". Elsewhere, there are misnomers in the proof of Ceva's theorem, a referenced arrow missing on Fig. 7.8, a misplaced P′ in Figs. 14.5 and 14.5b; and Napolean's theorem is referred to, forty-five pages before we are told what it is. The lack of tightness gives an impression that the author was left somewhat to his devices. This may be because David Wells is, and has been for decades, an established scholar and author. He references twenty-three of his own publications in the book's extensive and useful bibliography.

And there's the rub. The book contains a wealth of material of mathematical interest. It is something of an Aladdin's cave of mathematical topics, theorems and investigations from a wide range of subject areas and historical periods, and from a wide canon of mathematicians. It would thus be a good reference book on any college library shelf, where undergraduates could find springboards of interest for projects or small dissertations. Some Irish student might for example, investigate if the 7×7 Hex board mentioned by Wells could be related to the 7×7 gaming board of possible Viking origin found in a crannóg at Ballinderry, Co. Westmeath in 1932 and on

daily display in the National Museum.
In the meantime students will have to gloss over the author's persistence with an ill-formed plea that mathematics and games are of the same DNA. It may well be that they are, but this is not the book to secure a conviction.

(Note: The ISBN given is for the hardback. A paperback edition is available: ISBN: 978-1-107-69091-2.)

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Alan Baker: A Comprehensive Course in Number Theory, Cambridge University Press, 2012, ISBN:978-1107603790

REVIEWED BY TOM CARROLL

Baker's superb book is a much-expanded, substantial reworking of his *A Concise Introduction to the Theory of Numbers* [1], retaining the style and character of the original 'concise' edition in this 'comprehensive' account. The choice of topics follows lines which have been well-established by many excellent texts over the years. Indeed, the first four of seventeen chapters are entitled *divisibility*, *arithmetical functions*, *congruences*, *quadratic residues*. The feel is that of classical analysis where the goal is to solve problems and establish general results rather than to formulate general theories. This is not an 'elementary' book on number theory including, as it does, aspects of both algebraic and analytic number theory as required. At the same time, Baker does not skimp in his treatment of the more popular aspects of the subject such as perfect numbers, Mersenne and Fermat primes, Carmichael numbers, and has a nice, short chapter on *factorisation and primality testing*, which includes the RSA public key cryptosystem.

Each chapter is divided into short readable sections, the last two in each chapter being, without exception, *further reading* and *exercises*. Theorems are not often formally stated. Rather than being 'theorem-proof', the expository style is primarily discursive with many paragraphs beginning with the phrase 'for the proof'. The glue that holds the exposition together is the authority of the author which is evident throughout. Since Baker is clearly as much at ease with the law of quadratic reciprocity, with the prime number theorem, and with primes in arithmetical progressions as he is with the Euclidean algorithm and the Euler ϕ-function, the exposition is never strained or forced. That said, despite the single conspicuous variation in its title compared to its predecessor, the mathematical

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narrative continues to be quite concise. This allows Baker to advance rapidly at the sole expense of requiring readers to pause often for thought and to convince themselves of the validity of his arguments. By Page 42, Baker has already established both Gauss's law of quadratic reciprocity and that every natural number can be written as the sum of four squares, so that these forty-odd pages alone would make a fine 24-lecture first undergraduate course in number theory. In fact, since the chapters are short and broken into bite-sized sections which are seldom longer than two or three pages (and are often much shorter), most chapters can reasonably be read in a single sitting. The book has a substantial feel to it so that, having worked through a chapter, one is rewarded with the sense that one now knows not only the key concepts but something significant concerning that chapter's topic. The final chapters, on *analytic number theory*, *on the zeros of the zeta-function*, *on the distribution of the primes*, *the sieve and circle methods*, *elliptic curves*, occupy the last one hundred pages, treating topics which are more advanced than those in the previous twelve chapters, always in the same assured manner.

Baker's rewarding book is a pleasure to read including, as it does, so much beautiful mathematics, beautifully expounded.

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PROBLEMS

IAN SHORT

Problems

The first pair of problems were proposed by Professor Tom Moore of Bridgewater State University, USA.

Problem 71.1. For $n = 0, 1, 2, \ldots$, the triangular numbers T_n and Jacobsthal numbers J_n are given by the formulas

$$T_n = \frac{n(n+1)}{2} \quad \text{and} \quad J_n = \frac{2^n - (-1)^n}{3}.$$

(a) Prove that for each integer $n \geqslant 3$ there exist positive integers a, b, and c such that $T_n = T_a + T_b T_c$.
(b) Prove that infinitely many square numbers can be expressed in the form $J_a J_b + J_c J_d$ for positive integers a, b, c, and d.

The next problem was contributed by Finbarr Holland.

Problem 71.2. Prove that for each integer $n \geqslant 3$,

$$\int_0^\infty \frac{x-1}{x^n - 1}\, dx = \frac{\pi}{n \sin(2\pi/n)}.$$

The final problem, proposed by Anthony O'Farrell, is based on an assertion made by the late E. P. Dolženko in a Russian manuscript published in 1963 (see lines 7–8 on page 34 of the English translation in *American Mathematical Society Translations. Series 2. Vol. 97*, Amer. Math. Soc., Providence, RI, 1971, 33–41). The solution is not known to the proposer or to the editor of these problems.

Problem 71.3. Suppose that you remove from a circular disc its intersection with any number of larger circular discs. Is the perimeter of the resulting set necessarily less than or equal to the circumference of the original circular disc?

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Solutions

Here are the solutions to the problems from *Bulletin* Number 69. All three solutions were contributed by the North Kildare Mathematics Problem Club (and each was also solved by the proposer, although the methods of solution differ).

Problem 69.1. Suppose that the matrices A, b, and c are of sizes $n \times n$, $n \times 1$, and $1 \times n$, respectively. Prove that, for all complex numbers z,

$$\det(A - zbc) = \det A - zcA^*b = \det A + z\det\begin{pmatrix} 0 & c \\ b & A \end{pmatrix},$$

where A^* is the adjoint of A (that is, the transpose of the matrix of cofactors of A).

Solution 69.1. The rank of the $n \times n$ matrix bc is less than or equal to 1, which implies that the nullity (the geometric multiplicity of the eigenvalue 0) is greater than or equal to $n-1$. It follows that the algebraic multiplicity of the eigenvalue 0 is also greater than or equal to $n-1$. Since the sum of the eigenvalues of bc is equal to its trace cb, we see that the only eigenvalues of bc are cb and 0 (possibly $cb = 0$), and the characteristic polynomial is given by

$$\det(\lambda I - bc) = (\lambda - cb)\lambda^{n-1},$$

where I is the $n \times n$ identity matrix. Evaluating this equation at $\lambda = 1$ gives

$$\det(I - bc) = 1 - cb.$$

Next, replace b by $A^{-1}b$, for some $n \times n$ invertible matrix A, to give

$$\det(I - A^{-1}bc) = 1 - cA^{-1}b.$$

Then multiplying throughout by $\det A$:

$$\det(A - bc) = \det A - cA^*b.$$

Since the collection of invertible $n \times n$ matrices is dense in the space of all $n \times n$ matrices, and both sides of the above equation are continuous in A, we see that the equation is valid for any $n \times n$ matrix A. Finally, we obtain the first of the given identities on replacing b by zb, for a complex number z.

The second identity is obvious, on expanding

$$\det\begin{pmatrix} 0 & c \\ b & A \end{pmatrix}$$

along the first row. □

Problem 69.2. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \sum_{k=1}^{n} \frac{1}{k} = \zeta(3),$$

where ζ is the Riemann zeta function.

Solution 69.2. Let

$$u_n = \frac{1}{(n+1)^2} \sum_{k=1}^{n} \frac{1}{k}.$$

For $|z| < 1$, we have

$$\begin{aligned}
\frac{-\log(1-z)}{1-z} &= \left(\sum_{n=1}^{\infty} \frac{z^n}{n}\right) \cdot \left(\sum_{n=0}^{\infty} z^n\right) \\
&= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{1}{k}\right) z^n \\
&= \sum_{n=1}^{\infty} (n+1)^2 u_n z^n.
\end{aligned}$$

Thus, given $0 < x, y < 1$,

$$\sum_{n=1}^{\infty} (n+1)^2 u_n (xy)^n = \frac{-\log(1-xy)}{1-xy}.$$

Integrating once with respect to x and once with respect to y gives

$$\sum_{n=1}^{\infty} u_n (xy)^{n+1} = -\int_0^x \int_0^y \frac{\log(1-st)}{1-st}\, ds\, dt.$$

Since $\sum_{n=1}^{\infty} u_n$ converges, it follows by the Abel–Dirichlet theorem that

$$\sum_{n=1}^{\infty} u_n = -\int_0^1 \int_0^1 \frac{\log(1-st)}{1-st}\, ds\, dt.$$

Substituting $u = st$ and $v = t$, this becomes

$$\begin{aligned} -\int_0^1 \frac{1}{v}\int_0^v \frac{\log(1-u)}{1-u}\,dudv &= \frac{1}{2}\int_0^1 \frac{(\log(1-v))^2}{v}\,dv \\ &= \frac{1}{2}\int_0^1 \frac{(\log(x))^2}{1-x}\,dx \\ &= \frac{1}{2}\sum_{m=0}^{\infty}\int_0^1 x^m(\log(x))^2\,dx. \end{aligned}$$

Integrating by parts,

$$\frac{1}{2}\int_0^1 x^m(\log(x))^2\,dx = -\int_0^1 \frac{x^m\log(x)}{m+1}\,dx = \frac{1}{(m+1)^3}.$$

Therefore

$$\sum_{n=1}^{\infty} u_n = \sum_{m=0}^{\infty}\frac{1}{(m+1)^3} = \zeta(3),$$

as required. □

Problem 69.3. A rectangle is partitioned into finitely many smaller rectangles. Each of these smaller rectangles has a side of integral length. Prove that the larger rectangle also has a side of integral length.

Solution 69.3. We may assume that the larger rectangle R lies in the Cartesian plane and each of its sides is parallel to one of the axes of the plane. Any two rectangles in the partition that meet at more than one point must meet in an interval. It follows that each side of each rectangle in the partition is also parallel to one of the axes of the plane.

Consider now the double integral

$$\int_{y_1}^{y_2}\int_{x_1}^{x_2} e^{2\pi i(x+y)}\,dxdy.$$

This integral vanishes if and only if the rectangle with vertices (x_1, y_1), (x_1, y_2), (x_2, y_1), and (x_2, y_2) has a side of integral length. Therefore the integral vanishes when evaluated on each of the rectangles in the partition. It follows that the integral also vanishes on R, so R has a side of integral length. □

We invite readers to submit problems and solutions. Please email submissions to `imsproblems@gmail.com`.

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`mailto://ims.bulletin@gmail.com`

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