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EXAMINING REASONING-AND-PROVING IN THE TREATMENT OF COMPLEX NUMBERS IN IRISH SECONDARY MATHEMATICS TEXTBOOKS

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ABSTRACT. This study examined student tasks in the area of complex number operations in six Irish secondary mathematics textbooks and a Project Maths teaching and learning plan for reasoning-and-proving (RP) opportunities. At the ordinary level, 9.1% of 1274 student tasks were coded as RP. At the higher level, 13.3% of 1373 tasks were coded as RP. The majority of argument opportunities in ordinary level materials were the lowest form of argument - proof-writing exercises. At the higher level, the majority of argument opportunities were within argument-specific or argument-general categories. Less than 2% of the tasks at the ordinary or higher level involved pattern identification or conjecture development. Students were not asked to test conjectures, construct counterexamples, develop proof subcomponents, or formulate RP objects in any of the seven sets of materials. Only one RP task appearing across all seven sets of materials involved the use of technology. The implication of these results as well as how textbook materials could be redesigned using the RP framework are discussed.

1. Introduction

The construction of proofs and its related set of actions: identification of patterns, construction of conjectures [30], and reasoning [18] are important fundamental practices that mathematicians frequently use to construct mathematical ideas. Furthermore mathematicians [26], mathematics educators [1], as well as national reform documents [4, 22, 21] have pointed out that these practices should also be important components of school mathematics classrooms.

Research in mathematics classrooms around the world has established that textbooks are an important force in shaping teachers'

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classroom lessons [2, 21, 32, 33]. Consequently, as educational systems promote mathematics education reform through national level curriculum documents, textbooks are often redesigned to operationalize that change for teachers and their students. An important section of Ireland's reform of its secondary mathematics program, Project Maths, is synthesis and problem-solving skills. This component appears within each of the five mathematics content strands spanning the leaving certificate syllabus and expects students to explore patterns, formulate conjectures, explain findings, and justify conclusions [20]. These are all components of Sylianides' reasoning-and-proving (RP) framework [30]. The purpose of the study described here is to examine the nature of reasoning-and-proving in the area of complex numbers in six different Irish secondary mathematics textbooks including a teaching and learning plan developed by individuals associated with the national Project Maths reform effort.

2. Background

2.1. Textbooks' Influence on Classroom Instruction. While there might not be an isomorphism between the mathematics content and how that content is presented in textbooks and the classroom lesson or enacted curriculum [27] the former plays a strong role in determining the latter. For example, a recent survey in the United States involving 7,752 science and mathematics teachers had the following findings: over 80% of mathematics teachers used a commercially published textbook; at least 67% of mathematics teachers covered 75-100% of the content of their textbooks; and nearly half of the mathematics teachers surveyed used their textbooks 75% of the class time [2]. This strong influence of textbooks on teachers is also prevalent in Ireland as more than 75% of the Irish secondary school teachers in one survey used one mathematics textbook daily [24].

2.2. Examining Proof-Related Constructs in Textbooks. A number of themes appear within research involving the presence of proof related processes in textbooks around the world. First, students are provided with limited opportunities to engage in the development of proof-related processes within textbook exercise sections [6, 23, 30, 31]. For instance, only 6% of over 9000 tasks appearing across twenty U.S. secondary mathematics textbooks provided students with opportunities make conjectures, develop arguments, find counterexamples, or correct mistakes in arguments [31]. Second, there is variability in students' proof opportunities across different mathematics content areas. For instance, middle school students (ages 12-14) were provided with more opportunities to engage in the development of valid arguments or proofs in number than in algebra within a reform-oriented mathematics textbook in the United States [30]. Third, proof related processes differ by textbook. For example, within a polynomial functions unit a U.S. reform-oriented secondary mathematics textbook contained five times as many instances of reasoning-and-proving as a more conventional reform-oriented secondary mathematics textbook [6].

3. Framework

A framework adapted from the work of Stylianides [30] was used in this study. It consists of five main components: pattern identification; conjecture development; argument construction; technological tools; and reasoning-and-proving objects. These components are described briefly here, but the interested reader can get more details in [6]. Pattern identification instances within student tasks of the textbook occur when data is presented or students are asked to generate a set of data in a variety of different representational forms and locate regularities within them. Those regularities can be of two different forms: plausible and definite. Definite patterns are those which an expert can identify and provide compelling evidence for their existence. Plausible patterns, on the other hand, are not unique to a set of data.

Conjecturing consists of two interrelated actions: developing conjectures and testing conjectures. A conjecture is defined as an attempt to apply some regularity seen in a set of data to values beyond that set of data. Conjecture testing is when student tasks ask for a given conjecture to be tested through the location of one or more examples that meet the criteria of the conditions surrounding the conjecture.

A valid argument opportunity asked students to complete a problem that consisted of three components: a set of accepted statements; modes of argumentation; and modes of argument representation [29]. Accepted statements consist of theorems, definitions, axioms, etc. Modes of argumentation can be considered the glue that holds the argument together and consist of logical rules of inference (e.g., modus tollens), use of cases, indirect reasoning, etc. Modes of argument representation are the ways in which the argument can be communicated to others and can consist of written text, pictures, algebraic symbols, etc. A total of five valid argument categories appear within the framework. First, proof writing exercise opportunities contained the three characteristics of a valid argument as well as the further condition that they were not new for students. That is, students had seen similar arguments within the exposition sections of the textbook or had been asked previously to develop similar arguments.

Second, a valid argument opportunity of the argument-specific type occurred if students were asked to complete a task that contained the three components of a valid argument and the assertion that was to be proved was of a specific nature. Third, a valid argument opportunity of the argument-general type occurred if students were asked within an exercise section to create a valid proof and the assertion validated was of a general nature. Fourth, counterexamples were considered a valid argument category and were considered to exist if an exercise asked students to construct a counterexample to a given statement or to show that a given general statement was not always true. Fifth, a proof subcomponents opportunity was coded if students were asked to develop one or more statements or one or more explanations within a set of statements that contained the three components of a valid argument.

The different components of the framework described above may be interconnected with one another. That is, students' pattern identification opportunities may be connected to conjecture development and testing opportunities. If patterns led to conjecture opportunities they were considered to be conjecture precursors, otherwise they were denoted as conjecture non-precursors. In a similar vein, conjecturing opportunities may lead to students' construction of valid arguments that are of the specific or general variety as described earlier. Conjecturing opportunities that were connected to argument opportunities were considered to be argument precursors otherwise they were denoted as argument non-precursors. The process of constructing and testing a conjecture may also be bypassed and lead directly to the construction of an argument.

Technology such as Geogebra can play a role in the identification of patterns, construction of conjectures, and development of arguments. For instance, the graphical representations of a function can lead students to identify patterns about the relationship between the number of minima or maxima that cubic functions can have. The matrix capabilities of a graphing calculator could be used to locate a counterexample to show that matrix multiplication is, in general, non-commutative.

The last component of the framework consists of reasoning-and-proving objects. Reasoning-and-proving objects consist of definitions, corollaries, theorems, etc. These objects are connected to the argument construction category as students can work on developing the wording for a theorem after developing an argument to validate its existence. At the same time, students may construct definitions, hence, this category could stand apart from the development of valid arguments. Moreover, reasoning-and-proving objects have the potential to be objects within which students identify patterns, formulate and test conjectures, and construct arguments.

4. Methodology

4.1. **Choice of Topic Focus.** The topic area of focus for this study was complex number. This content area was chosen due to its appearance as an entire unit across three ordinary level and three higher level texts as well as the fact that a teaching and learning plan was created for it. The leaving certificate syllabus [20] contains the following learning outcome for students at the ordinary and higher levels: investigate the operations of addition, multiplication, subtraction and division with complex numbers $\mathbb{C}$ in rectangular form $a+ib$ (p. 25). The word investigate was linked with the potential for students to engage in the components of identifying patterns, formulating conjectures, and creating arguments. In addition, the syllabus states that students learning about the number content strand, within which complex numbers appears should frequently encounter the following actions: explore patterns and formulate conjectures, explain findings, and justify conclusions (p. 27).

4.2. **Materials.** There are three textbooks that Irish schools can choose from for ordinary level students at the foundation and ordinary level: *Texts & Tests 3* (TT) [19]; *New Concise Project Maths 3B* (NC) [12]; and *Active Maths 3: Book 1* [15]. The Active Maths textbook also has a companion text that serves as a set of activities for students to complete: *Active Maths 3 Activity Book* [14]. Analyses for both of these Active Maths texts will be combined and be denoted in the results section by AM. Individuals associated with Project Maths have created a series of teaching learning plans (TLP) designed for teachers to implement. They have created one such TLP for complex numbers [25]. This TLP encompasses 52 pages and consists of six activities for students along with information for teachers about how to implement these plans.

Three texts were examined at the higher level: *Active Maths 4: Book 1* [17], *Text & Tests 6* (TTH) [5], and *New Concise Project Maths 5* (NCH) [13]. Similar to the ordinary level the Active Maths textbook also had a companion higher level textbook: *Active Maths 4: Activity Book* [16]. The results for the Active Maths program at the higher level will include both books and will be denoted by AMH.

4.3. **Coding.** The six textbook complex number units and teaching and learning plan were examined for RP tasks by the author. A variety of words associated with the framework such as *pattern, describe, conjecture, proof, proving, prove, show, verify, explain*, *investigate* and *justify* were used to identify potential RP tasks. These potential RP elements were more carefully examined using the descriptions of the framework components to determine if they were indeed RP tasks. Tasks using the words *pattern* and *conjecture* were considered to be candidates for the identification of a pattern and development of a conjecture categories. The word *test* needed to be present for tasks to be considered conjecture test candidates. Tasks involving the words *proof, proving, prove, show, verify, explain*, and *justify* were considered to be candidates for argument-general, argument-specific, and proof-writing exercises. The work involved in solving tasks using the word *investigate* was examined in a more open-ended fashion to see which of the RP categories it fit.

4.4. **Inter-rater Reliability.** While the author coded the textbook tasks another researcher familiar with the framework was asked to use the methodology described above to locate and categorize RP tasks within the ordinary level textbook and from a higher level textbook in order to determine inter-rater reliability. Overall, the inter-rater reliability was excellent as Cohen's Kappa was 0.9248 for the ordinary level text excerpt and 0.9135 for the higher level text excerpt. This is a high level of agreement as a value of 1 would denote perfect alignment.

4.5. **Analysis.** Analysis of student tasks began with counting the number of student tasks on each page of each textbook unit and the teaching and learning plan. The number of RP tasks identified during the coding stage were also counted and the percentages of tasks that were coded as RP was calculated. The number of reasoning-and-proving objects that required an argument in order to be validated (e.g., theorem) were counted and the number that either were accompanied by a valid argument presented in the textbook or were left to students to prove were counted in order to calculate the percentage of reasoning-and-proving objects proved. A Chi Square analysis was conducted on the frequency of non-RP and RP tasks in the three ordinary texts and the teaching and learning plan. A Chi Square analysis was conducted on the frequency of non-RP and RP tasks in the three higher level texts. An $\alpha = 0.05$ was used for all statistical tests.

5. Results

Table 1 shows the breakdown in reasoning-and-proving for student tasks within the complex number unit across the four ordinary level sets of materials and the three sets of higher level materials. Overall, students at the higher level were provided with more opportunities to engage in RP than ordinary level students as the average for ordinary level materials was 9.1% while the average for higher level materials was 13.3%. There was less variation in the percentage of student tasks coded as RP within the higher level texts when compared with the ordinary level materials. That is, the higher level texts RP task percentage varied from 11.8% to 15.0% while ordinary level materials varied from 4.5% to 15.5%. Indeed, Chi Square analyses on the ordinary level materials revealed that there were statistically significant differences in terms of the RP tasks afforded to students, $\chi^2(3) = 31.852$, $p < 0.001$. At the higher level, Chi Square analyses illustrated that the RP task opportunities were monolithic across the different texts as differences were not statistically significant, $\chi^2(2) = 2.188$, $p = .338$.

Table 1. Total Tasks and RP Tasks in Ordinary and Higher Level Materials

Textbook	Tasks	RP Tasks	Percent RP Tasks
		Ordinary	
AM	418	19	4.5%
TT	312	20	6.4%
NC	336	52	15.5%
TLP	208	25	12.0%
Total	*1274*	*116*	*9.1%*
		Higher	
AMH	473	60	12.7%
TTH	373	44	11.8%
NCH	527	79	15.0%
Total	*1373*	*183*	*13.3%*

Table 2 shows the breakdown in different categories for RP student tasks appearing in the ordinary level and higher level materials. In this table P-WE refers to proof-writing exercises, P refers to definite patterns, C refers to conjecture development, ArgS refers to argument-specific instances, and ArgG refers to argument-general instances. Elements separated by dashes such as P-C represent tasks that provide students with pattern identification and conjecture development opportunities. The majority of RP tasks within the AM and NC complex number units were coded as proof-writing exercises, which is the lowest level of valid arguments. In the TT complex number unit, there was an even split between proof-writing exercises and argument-specific/argument-general categories. The analysis found that 26.3% ($\frac{5}{19}$) of AM unit RP tasks, 0% ($\frac{0}{20}$) of TT unit RP tasks, 3.8% ($\frac{2}{52}$) of NC unit RP tasks, and 44% ($\frac{11}{25}$) of TLP RP tasks involved the identification of patterns or construction of conjectures. While AM provided more opportunities in pattern identification and conjecture development than the TT and NC textbook units, the TLP provided students with over twice as many opportunities to engage in these components of the framework than the AM textbook unit. For instance, the exposition sections of the NC, TT, and AM Activity Book presented the fact that multiplying a complex number by i results in an anticlockwise rotation of the complex number by 90 degrees. The TLP, on the other hand, provided students with the following tasks.

1. If $z = 3+4i$, what is the value of iz, i^2z, i^3z, i^4z? Represent your results on an Argand Diagram joining each point to the origin $o = 0 + 0i$.
2. Investigate what is happening geometrically when z is multiplied by i to get iz? Use geometrical instruments and/or calculation to help you in your investigation. (p. 44).

Consequently, a fact that was presented in the other textbook units became an investigation for students within the TLP providing them with an opportunity to detect a definite pattern.

There were also differences across textbook units within the argument-specific and argument-general categories across the ordinary level units as seen in the following percentages: AM textbook 26.3% ($\frac{5}{19}$); TT textbook 50.0% ($\frac{10}{20}$); NC textbook 42.3% ($\frac{22}{52}$); and TLP 28% ($\frac{7}{25}$). The majority of these instances, however, were within the argument-specific category. The NC textbook unit did not ask students to construct proofs of the argument-general type. Consequently, within the three ordinary level texts student tasks requiring proofs of the argument-general type were rare occurrences.

The TLP contained the greatest percentage of RP tasks within the argument-general category at 20% ($\frac{5}{25}$) when compared to the three ordinary level texts. This is seen in the proof of the idea that the sum of a complex number and its conjugate is always a non-complex real number. In the TLP students were given the general form of a complex number $z = a+bi$ and asked to find its conjugate, $\bar{z} = a - bi$. Next students were asked to calculate the sum $z + \bar{z}$. In the TT and NC textbook units students were told in the exposition section that the sum of a complex number and its conjugate are always a real number without an accompanying proof. In the AM complex number unit, students were asked to verify that $z + \bar{z}$ is a real number for a specific case when $z = 13 + 2i$ and $\bar{z} = 13 - 2i$.

There were interesting differences in students' opportunities to identify patterns and formulate conjectures across the three higher level texts. The percentage of student tasks that were coded as identification of patterns or formulation of conjectures was 3.3% ($\frac{2}{60}$) in AMH and 6.3% ($\frac{5}{79}$) in NCH. On the other hand, 25% or ($\frac{11}{44}$) of the tasks in TTH involved the identification of pattern or construction of conjectures. In the AMH textbook unit, there was an even split between proof-writing opportunities and students' opportunities to develop specific or general arguments. In the TTH and NCH textbook units, argument-specific and argument-general opportunities outnumbered proof-writing exercises.

Recall that technology can be used to provide students with opportunities to engage in various components of the RP framework. Out of a total of 299 RP tasks only one of these used technology. Patterns did not always lead to conjectures or arguments. At the ordinary level, there were a total of 11 tasks that involved pattern development and three were coded as conjecture or argument precursors. There were a total of nine conjecture opportunities at the ordinary level and seven of these were tied to arguments. At the higher level, seventeen tasks involved pattern identification and a total of eleven of these tasks involved conjecture or argument development. There were twelve tasks that involved conjecture development at the higher level. Of these tasks, five were connected to argument development. There were no tasks across the seven sets of Irish secondary mathematics materials that specifically asked students to test conjectures, develop reasoning-and-proving objects, construct proof subcomponents, or create counterexamples.

Textbook	RP Tasks	P-WE	P	C	P-C	P-C-ArgS	P-C-ArgG	C-ArgS	ArgS	ArgG
						Ordinary				
AM	19	13	0	0	1	1	0	3	0	1
TT	20	10	0	0	0	0	0	0	8	2
NC	52	29	1	0	0	0	0	1	21	0
TLP	25	9	8	1	0	1	0	1	0	5
						Higher				
AMH	60	29	0	0	1	1	0	0	5	24
TTH	44	11	4	0	4	0	3	0	7	17
NCH	79	35	2	0	2	0	0	1	14	25

TABLE 2. Breaking Down Tasks into RP Categories

6. Discussion

The similarity between this framework and frameworks used in for the analysis of proof related constructs in other countries enables cross-country comparisons. The next few paragraphs describe some of these comparisons. Overall, students at the higher level were provided with more opportunities to engage in RP at the higher level (13.3%) than at the ordinary level (9.1%). These percentages are lower than what was found in a set of U.S. reform-oriented mathematics textbooks designed for students ages 12-14 (40% of 4855 tasks involved RP) [30] as well as what was found in a polynomial functions unit from a reform-oriented U.S. mathematics textbook designed for students ages 14-18 (22% of 1158 tasks involved RP) [6]. Davis also found that 4% of 1129 tasks involved reasoning-and-proving in a more conventional U.S. mathematics textbook designed for students ages 14-18 [6].

Among 9742 tasks appearing in 20 U.S. textbooks in the areas of exponents, logarithms, and polynomials 3.1% involved developing and evaluating arguments [31]. A total of 3.5% of tasks appearing in the ordinary level materials and 7.1% of tasks appearing in the higher level materials involved argument development.

The examination of a set of U.S. middle school reform-oriented texts found that 27% of RP tasks involved the identification of patterns or development of conjectures in a U.S. reform-oriented middle school mathematics textbook program [30] . A total of 15.5% of RP tasks at the ordinary level and 9.9% of RP tasks at the higher level involved the identification of patterns or development of conjectures. [31] found that 14.8% of tasks coded as reasoning involved the development of conjectures. A total of 7.8% of RP tasks at the ordinary level and 6.6% of RP tasks at the higher level involved the development of conjectures. In sum, the Irish secondary mathematics textbook complex number units examined as part of this study contained smaller percentages of RP than reform-oriented curricula in the United States. The Irish textbook units contained a higher percentage of student tasks involving RP opportunities overall and a higher percentage of student tasks involving argument opportunities than a collection of U.S. textbook units involving exponents, logarithms, and polynomials. In contrast, this collection of U.S. textbook units contained a higher percentage of conjecturing opportunities than Irish textbooks. Reform-oriented textbook materials in the U.S., designed for students ages 12-18 contained a higher percentage of student tasks coded as RP. The implications of the findings from this study for redesigning Irish secondary mathematics textbooks are discussed in the following paragraphs.

6.1. **Redesigning Textbook Activities to Make RP More Central to Learning Mathematics.** One of the most common RP categories appearing in the student tasks across the six textbook complex number units was proof-writing exercises. Below is one example of such a task.

Let $u = 2 + i$ and $w = 3i$. Show that: $u\bar{w} + \bar{u}w$ is a real number [12] (p. 6).

Such tasks provide students with opportunities to engage in a component of reasoning-and-proving, but this is presented as an end in and of itself, a destination instead of the beginning of a mathematical journey. A mathematician would want to know if this instance comprised a pattern and would proceed to determine this by testing a variety of other complex numbers. Not only does this process provide the user with more practice it also enables him or her to engage in a fundamental component of the RP process, pattern identification. Once he or she was convinced that this was more than merely a coincidence he or she would develop a conjecture that it always held and proceed to test it with examples different from the ones already encountered. Once convinced that the conjecture could be true a mathematician would attempt to construct an argument to this effect. Students representing different ability levels could engage in the necessary cognitive work to answer such a problem. For fundamental and some ordinary level students the symbolic manipulation work could be offloaded to technology such as Geogebra, an ICT tool that Project Maths has embraced as indicated by its presence on its website (`http://www.projectmaths.ie/geobra/`).

Using Geogebra to calculate $u\bar{w}+\bar{u}w$ with $u = a+bi$ and $w = c+di$ yields $2ac + 2bd$, showing that the result is indeed a real-number as it no longer involves the complex number i. This work illustrates that the result is true due to the black box nature of the Geogebra CAS [3], fulfilling a role of verification, but does not shed light on why this statement is true, the explanation purpose of proof. The CAS could be used to calculate $(a + bi)(c - di)$ and $(a - bi)(c + di)$ separately to understand why terms involving i drop out of the calculation. Higher level students could complete these calculations by hand to prove that as well as why this assertion is always true. Last, mathematicians would engage in constructing the statement of the theorem or reasoning-and-proving object that was just proved. Students of varying levels could write such a theorem such as the following: Given complex numbers u and w, $u\bar{w} + \bar{u}w$ is a real number. Providing students with opportunities to complete the different components of this investigation would require additional class time. Thus there would be less time for students to work on other problems that would presumably provide them with procedural practice, but this type of practice could take place within the context of this investigation as students develop the work from which patterns could be identified and conjectures investigated. Moreover, the inclusion of activities such as this would provide students with more authentic RP opportunities.

If textbook designers wish to design resources for ordinary level students that contain fewer argument opportunities the RP framework used in this study suggests four different options. First, textbook developers could defer proofs of mathematical ideas to a later time. However, if the developers choose to pursue this option they should note within the textbook materials that a proof for this mathematical idea exists but it is too difficult for students to understand at this point in their education. Second, textbook developers could provide students with more opportunities to engage in pattern identification and conjecture development without moving to an argument but stating that it is possible for a proof to be constructed to show this idea is always true. Third, textbook developers could provide students with opportunities to engage in the development of proof subcomponents with regard to more complex arguments. Fourth, textbook developers could provide students with opportunities to develop rationales [30] or arguments that are not complete with the caveat that these do not denote valid arguments. The low incidence or nonexistence of conjecture testing, proof subcomponents, counterexamples, technology in the development of RP components, as well as RP objects suggests that textbook materials for foundation, ordinary, and higher level students could be reimagined to include these areas. The last section examines the alignment between reasoning-and-proving as it appears within the Project Maths Leaving Certificate syllabus and the seven sets of textbook materials examined in this study.

6.2. **Alignment between Project Maths Syllabus and Textbooks: Complex Numbers.** Recall that the Project Maths leaving certificate syllabus asserts that students at all levels should be provided with opportunities for students to: explore patterns and formulate conjectures; explain findings; and justify conclusions. These actions roughly translate into the three main components of the RP framework used in this study. However, 1.4% ($\frac{18}{1274}$) of ordinary level tasks and 1.3% ($\frac{18}{1373}$) of higher level tasks involved pattern identification or conjecture development. This low percentage suggests a misalignment between the six textbook units and the Leaving Certificate syllabus. Until textbooks are redesigned the responsibility for rectifying this misalignment will fall on the shoulders of teachers who will need to think carefully about how these curricula can be supplemented with activities that provide students with opportunities in these areas in order to align students' classroom experiences with the ideal visualized within the leaving certificate syllabus.

The TLP was more aligned with the syllabus as 5.3% ($\frac{11}{208}$) of its tasks provided students with opportunities to engage in pattern identification and conjecture formulation. However, students work with patterns and conjectures often ended with these RP components instead of moving to the development of an argument or at least a statement in the materials stating that a proof of this mathematical idea exists but is beyond the ability level of the students. That is, out of a total of 11 tasks that involved pattern formulation or conjecture development, 9 or 82% did not lead to an argument opportunity. Thus these tasks may lead students to believe that a pattern is sufficient to show that a mathematical idea is always true, promoting what is referred to as an empirical proof scheme [10].

There were also differential opportunities for students of various ability levels to engage in the development of valid arguments within student task sections across the six textbooks. The Project Maths syllabus asks students at the higher level to develop more arguments [7] and explicitly states that at the higher level, particular emphasis can be placed on the development of powers of abstraction and generalisation and on the idea of rigorous proof, hence giving learners a feeling for the great mathematical concepts that span many centuries and cultures [20] (p. 13). However, students at the foundation and ordinary levels may develop a false impression of mathematics when the vast majority of mathematical ideas that they encounter in the complex number unit are not justified. These students may not feel that mathematical ideas need to be justified and, consequently, they may struggle to see the purpose behind their teachers' demands for justification. The large differences in the number of proofs between textbooks designed for ordinary level students and textbooks designed for higher level students can also influence teachers' perceptions about students' abilities. That is, when the textbooks that teachers follow on a regular basis [24] expect ordinary level students to engage in reasoning-and-proving so infrequently, teachers themselves may not believe that students are capable of work of this nature.

7. Conclusion

This study examined the prevalence of reasoning-and-proving in complex number operations within seven different Irish textbook materials. Despite the fact that reasoning-and-proving is a central act of practicing mathematics [26] and the Project Maths syllabus has advocated the importance of this concerted set of actions in learning mathematics [20], the analyses here suggest that this reality is not reflected in Irish mathematics text materials for ordinary and higher level students. Indeed, the current set of materials underemphasizes the importance of pattern identification, conjecture development and testing, argument construction, the use of technology in RP, and the construction of RP objects. The framework used in this study as well as the concrete suggestions provided in this paper can be used by textbook developers to redesign Irish textbook materials so that RP occupies a much more central and prominent location in learning school mathematics.

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