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SPECTRAL PERMANENCE

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Abstract. Several kinds of generalized inverse bounce off one another in the proof of a variant of spectral permanence for C* embeddings.

This represents an expanded version of our talk to the IMS meeting of August 2012, which in turn was based on the work [3] of Dragan Djordjevic and Szezena Zivkovic of Nis, in Serbia.

1. Gelfand property

Spectral permanence, for C* algebras, says that the spectrum of an element $a \in A \subseteq B$ of a C* algebra is the same whether it is taken relative to the subalgebra A or the whole algebra B: this discussion is sparked by the effort to prove that the same is true of a variant of spectral permanence in which the two-sided inverse, whose presence or not defines "spectrum", is replaced by a *generalized inverse*. The argument involves a circuitous tour through "group inverses", "Koliha-Drazin inverses" and "Moore-Penrose inverses"; it turns out that the induced variants of spectral permanence are curiously inter-related.

Suppose $T : A \to B$ is a *semigroup homomorphism*, where we insist that a semigroup A has an identity 1 , and that a homomorphism $T : A \to B$ respects that: we might indeed talk about a *functor* between categories. It then follows, writing A^{-1} for the invertible group in A, that

$$T(A^{-1}) \subseteq B^{-1} \ , \tag{1.1}$$

or equivalently, turning it inside out,

$$A^{-1} \subseteq T^{-1}B^{-1} \ . \tag{1.2}$$

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At its most abstract then "spectral permanence" for the homomorphism T says that (1.2) holds with equality:

$$T^{-1}B^{-1} \subseteq A^{-1} \ . \tag{1.3}$$

In words, it is tempting to describe (1.3) by saying "Fredholm implies invertible". We shall also describe (1.3) as the *Gelfand property*, since it also holds, famously, when

$$T = \Gamma : A \to C(X) \subseteq \mathbf{C}^X \tag{1.4}$$

is the *Gelfand representation* of a commutative Banach algebra A; here of course $X = \sigma(A)$ is the "maximal ideal space" of the algebra A. We might notice a secondary instance of spectral permanence in the embedding

$$C(X) \subseteq \mathbf{C}^X \tag{1.5}$$

of continuous functions among arbitrary functions; similarly, for a Banach space X, the embedding

$$B(X) \subseteq L(X) \tag{1.6}$$

of bounded operators among arbitrary linear operators has spectral permanence, but only thanks to the ministrations of the *open mapping theorem*. Another elementary example is the *left regular representation*

$$L : A \to A^A \tag{1.7}$$

of the semigroup A as mappings, where, for $a \in A$,

$$L_a(x) = ax \ (x \in A) \ . \tag{1.8}$$

Less familiar is a *commutant embedding*

$$J : A = \mathrm{comm}_B(K) \to B \ , \tag{1.9}$$

where

$$\mathrm{comm}_B(K) = \{b \in B : a \in K \Longrightarrow ba = ab\} \tag{1.10}$$

and of course $J(a) = a$: here spectral permanence reflects the fact that two-sided inverses double commute:

$$a \in B^{-1} \Longrightarrow a^{-1} \in \mathrm{comm}^2_B(a) \ . \tag{1.11}$$

If in particular the semigroup A is a ring, having therefore a background "addition" and a distributive law, then we can quotient out the *Jacobson radical*

$$\mathrm{Rad}(A) = \{a \in A : 1 - Aa \subseteq A^{-1}\} \ , \tag{1.12}$$

in which every possible expression $1 - ca$ has an inverse: now it is easily checked that

$$K : a \mapsto a + \mathrm{Rad}(A) \ (A \to A/\mathrm{Rad}(A)) \tag{1.13}$$

has spectral permanence. Our final example will be the most familiar, if not by any means the most elementary: it is the *determinant*

$$\det : \mathbf{C}^{n \times n} \to \mathbf{C} \ , \tag{1.14}$$

which indeed "determines" whether or not a square matrix is invertible.

2. Spectral permanence

Mathematicians are thus prepared to go to a lot of trouble to establish spectral permanence. If we specialise to linear homomorphisms between (complex) linear algebras then we meet the phenomenon of *spectrum*, defining for each $a \in A$,

$$\sigma_A(a) = \{\lambda \in \mathbf{C} : a - \lambda \notin A^{-1}\} \ ; \tag{2.1}$$

the idea is to harness complex analysis to the theory of invertibility. Now we can rewrite (1.1) to say that, for arbitrary $a \in A$,

$$\sigma_B(Ta) \subseteq \sigma_A(a) \ , \tag{2.2}$$

while the Gelfand property (1.3) says that (2.2) holds with equality, giving indeed "spectral permanence".

If we specialise to isometric Banach algebra homomorphisms then there is built in a certain degree of spectral permanence, to the extent that we always get

$$\partial\sigma_A(a) \subseteq \sigma_B(Ta) \ : \tag{2.3}$$

the topological boundary of the larger spectrum is included in the smaller. Equivalently, it turns out, this means that

$$\sigma_A(a) \subseteq \eta\sigma_B(Ta) \ , \tag{2.4}$$

where the *connected hull* ηK of a compact subset $K \subseteq \mathbf{C}$ is the complement of the unbounded connected component of the complement $\mathbf{C} \setminus K$. This has spin-off: if for a particular element $a \in A$ either the larger spectrum is all boundary,

$$\sigma_A(a) \subseteq \partial\sigma_A(a) \ , \tag{2.5}$$

or the smaller spectrum fills out its connected hull,

$$\eta\sigma_B(Ta) \subseteq \sigma_B(Ta) \ , \tag{2.6}$$

then the homomorphism $T : A \to B$ has "spectral permanence at" $a \in A$, in the sense of equality in (2.2). This holds if for example the spectrum is either *real* or *finite*.

If more generally the homomorphism $T : A \to B$ is one-one there is at least inclusion

$$\text{iso}\ \sigma_A(a) \subseteq \sigma_B(Ta)\ . \tag{2.7}$$

3. Generalized permanence

If A is a semigroup we shall write

$$A^\cap = \{a \in A : a \in aAa\} \tag{3.1}$$

for the "regular" or *relatively regular* elements of A, those $a \in A$ which have a *generalized inverse* $c \in A$ for which

$$a = aca\ \ : \tag{3.2}$$

we remark that if (3.2) holds the products

$$p = ca = p^2\ ,\ q = ac = q^2 \tag{3.3}$$

are both *idempotent*. Generally if $T : A \to B$ is a homomorphism there is inclusion

$$T(A^\cap) \subseteq B^\cap \subseteq B\ , \tag{3.4}$$

and hence also

$$A^\cap \subseteq T^{-1}(B^\cap) \subseteq A\ . \tag{3.5}$$

If there is equality in (3.4) we shall say that T has *generalized permanence*. This happens for example when

$$T^{-1}(0) \subseteq A^\cap\ ,\ T(A) = B\ \ : \tag{3.6}$$

recall the implication

$$(a - aAa) \cap A^\cap \neq \emptyset \Longrightarrow a \in A^\cap\ . \tag{3.7}$$

This does not however happen when T is quotienting out the radical as in (1.10), unless the ring A is *semi simple*: for notice

$$\text{Rad}(A) \cap A^\cap = \{0\}\ . \tag{3.8}$$

It follows that spectral permanence is not in general sufficient for generalized permanence. Indeed by (3.8) spectral and generalized permanence together imply that a homomorphism $T : A \to B$ is one one; further (1.5) shows that spectral permanence and one one do not together imply generalized permanence. If A is the ring of continuous homomorphisms $a : X \to X$ on a Hausdorff topological abelian group X then it is necessary for $a \in A^\cap$ that a have *closed range*

$$a(X) = \mathrm{cl}\ a(X)\ : \tag{3.9}$$

this is because

$$a(X) = ac(X) = (1 - ac)^{-1}(0) \tag{3.10}$$

is the null space of the complementary idempotent. Thus the embedding (1.6) is another example with spectral but not generalized permanence.

4. Simple permanence

If in particular there is $c \in A$ for which

$$a - aca = 0 = ac - ca\ , \tag{4.1}$$

then $a \in A$ is very special; this happens if $a \in A$ is either invertible, or idempotent, or more generally the commuting product of an invertible and an idempotent. When (4.1) holds we shall say that $a \in A$ is *simply polar*: in Banach-algebra-land $0 \in \mathbf{C}$ can be at worst a simple pole of the *resolvent mapping*

$$(z - a)^{-1} : \mathbf{C} \setminus \sigma(a) \to A\ . \tag{4.2}$$

In the group theory world the product cac is referred to as the *group inverse* for $a \in A$. We remark that it is necessary and sufficient for $a \in A$ to be simply polar that

$$a \in a^2 A \cap Aa^2\ : \tag{4.3}$$

indeed [15],[19],[20] there is implication

$$a^2 u = a = va^2 \Longrightarrow au = va\ ,\ aua = a = ava\ , \tag{4.4}$$

giving (4.1) with $c = vau$.

We shall write $\mathrm{SP}(A)$ for the simply polar elements of a semigroup A and observe, for homomorphisms $T : A \to B$, that

$$T\ \mathrm{SP}(A) \subseteq \mathrm{SP}(B) \subseteq B\ , \tag{4.5}$$

and hence

$$\mathrm{SP}(A) \subseteq T^{-1}\mathrm{SP}(B) \subseteq A\ ; \tag{4.6}$$

when there is equality in (4.5) we shall say that $T : A \to B$ has *simple permanence*. The counterimage $T^{-1}\mathrm{SP}(B) \subseteq A$ is sometimes known [2],[18],[16] as the "generalized Fredholm" elements of A.

We remark that spectral permanence does not in general, or even together with one-one-ness, imply simple permanence: return to (3.8) and (1.5).

In general

$$\mathrm{SP}(A) \subseteq A^{\cup} \equiv \{a \in A : a \in aA^{-1}a\} \ , \tag{4.7}$$

and hence

$$\mathrm{SP}(A) \cap A^{-1}_{left} = A^{-1} = \mathrm{SP}(A) \cap A^{-1}_{right} \ . \tag{4.8}$$

This will show again that spectral permanence together with one one is not sufficient for generalized permanence:

Theorem 4.1. *If $B^{-1}_{left} \neq B^{-1}_{right}$ then there exist A and $T : A \to B$ for which T is one one with spectral but not generalized permanence.*

Proof. If A is commutative then $A^{\cap} = \mathrm{SP}(A)$ and hence

$$T(A^{\cap}) \subseteq \mathrm{SP}(B) \subseteq B^{\cap} \ ,$$

and if

$$T(A^{\cap}) \cap B^{-1}_{left} \setminus B^{-1} \neq \emptyset$$

then T does not have generalized permanence. Thus find $a \in B^{-1}_{left} \setminus B^{-1}$ and, recalling (1.9), take

$$T = J : \mathrm{comm}^2_B(a) \subseteq B$$

□

The familiar example is to take $B = L(X)$ to be the linear mappings on the space $X = \mathbf{C}^{\mathbf{N}}$ of all complex sequences and $a \in B$ to be the *forward shift*. Conversely however simple permanence together with one-one-ness does imply spectral permanence:

Theorem 4.2. *For semigroup homomorphisms*

$$\textit{one one and simple permanence implies spectral permanence} \ , \tag{4.9}$$

while conversely

$$\textit{simple and spectral permanence implies one one} \ . \tag{4.10}$$

Proof. The last implication is (3.8); conversely observe

$$\mathrm{SP}(A) \cap T^{-1}B^{-1}_{left} \subseteq A^{\cup} \cap T^{-1}B^{-1}_{left} \subseteq A^{-1} + T^{-1}(0) \tag{4.11}$$

□

When we specialise to rings of mappings then simple polarity is characterized by "ascent" and "descent":

Theorem 4.3. *If $A = L(X)$ is the additive, or linear, operators on an abelian group, or vector space, X then necessary and sufficient for $a \in A$ to be simply polar is that it has ascent ≤ 1,*

$$a^{-2}(0) \subseteq a^{-1}(0) \ ; \ \textit{equivalently}\ a^{-1}(0) \cap a(X) = O \equiv \{0\} \ , \quad (4.12)$$

and also descent ≤ 1,

$$a(X) \subseteq a^2(X) \ ; \ \textit{equivalently}\ a^{-1}(0) + a(X) = X \ . \quad (4.13)$$

The same characterization is valid when $A = B(X)$ for a Banach space X.

Proof. The complementary subspaces $a^{-1}(0)$ and $a(X)$ determine the idempotent $p : X \to X$, defined by setting

$$p(\xi) \in a(X) \ ; \ \xi - p(\xi) \in a^{-1}(0)$$

for each $\xi \in X$, whose boundedness, together with the closedness of the range $a(X)$, follows ([7] Theorem 4.8.2) from the open mapping theorem; and finally, if $\xi \in X$,

$$c(\xi) = cp(\xi) \ ; \ ca(\xi) = p(\xi)$$

□

We remark that, on incomplete spaces, the conditions (4.5) and (4.6) are not sufficient for simple polarity: indeed it is possible for $a \in B(X)$ to be one one and onto but not in $B(X)^{\cap}$: the obvious example is the "standard weight" $a = w$ on $X = c_{00} \subseteq c_0$ defined by setting

$$w(\xi)_n = (1/n)\xi_n \ .$$

Even together with the assumption $a \in A^{\cap}$, however, the conditions (4.5) and (4.6) are ([7] (7.3.6.8)) not sufficient for simple polarity (4.1) when $A = B(X)$ for an incomplete normed space X.

5. Drazin permanence

More generally if there is $n \in \mathbf{N}$ for which a^n is simply polar we shall also say that $a \in A$ is "polar", or *Drazin invertible*. If $a \in A$ is polar then there is $c \in A$ for which $ac = ca$ and $a - aca$ is *nilpotent*. If we further relax this to "quasinilpotent" we reach the condition that $a \in A$ "quasipolar". Specifically if we write

$$\mathrm{QN}(A) = \{a \in A : 1 - \mathbf{C}a \subseteq A^{-1}\} \quad (5.1)$$

for the *quasinilpotents* of a Banach algebra A then $a \in \mathrm{QN}(A)$ if and only if

$$\sigma_A(a) \subseteq \{0\} \ ,$$

while with some complex analysis we can prove that if $a \in \mathrm{QN}(A)$ then

$$\|a^n\|^{1/n} \to 0 \ (n \to \infty) \ . \tag{5.2}$$

In the ultimate generalization of "group invertibility", we shall write $\mathrm{QP}(A)$ for the *quasipolar* elements $a \in A$, those which have a *spectral projection* $q \in A$ for which (cf [8])

$$q = q^2 \ ; \ aq = qa \ ; \ a + q \in A^{-1} \ ; \ aq \in \mathrm{QN}(A) \ . \tag{5.3}$$

Now [17] the spectral projection and the *Koliha-Drazin inverse*

$$a^\bullet = q \ , \ a^\times = (a+q)^{-1}(1-q) \tag{5.4}$$

are uniquely determined and lie in the double commutant of $a \in A$. It is easy to see that if (5.3) is satisfied then

$$0 \notin \mathrm{acc}\ \sigma_A(a) \ : \tag{5.5}$$

the origin cannot be an accumulation point of the spectrum; conversely if (5.5) holds then we can display the spectral projection as a sort of "vector-valued winding number"

$$a^\bullet = \frac{1}{2\pi i}\oint_0 (z-a)^{-1} dz \ , \tag{5.6}$$

where we integrate counter clockwise round a small circle γ centre the origin whose connected hull $\eta\gamma$ is a disc whose intersection with the spectrum is at most the point $\{0\}$. Now generally for a homomorphism $T : A \to B$ there is inclusion

$$T\ \mathrm{QP}(A) \subseteq \mathrm{QP}(B) \ , \tag{5.7}$$

while if $T : A \to B$ has spectral permanence in the sense (1.3) then it is clear from (5.5) that there is also "Drazin permanence" in the sense that

$$\mathrm{QP}(A) = T^{-1}\mathrm{QP}(B) \subseteq A \ : \tag{5.8}$$

Theorem 5.1. *For Banach algebra homomorphisms $T : A \to B$ there is implication*

$$\textit{spectral permanence} \Longrightarrow \textit{Drazin permanence} \ .$$

Proof. Equality in (2.2), together with (5.5) □

The example of Theorem 4.1 also shows that the left regular representation $L : A \to B(A)$, with $A = B(X)$ for a normed space X, does not always have generalized permanence; however we do have a sort of "closed range permanence": there is implication

$$L_a A = \text{cl } L_a A \Longrightarrow a(X) = \text{cl } a(X) \ : \tag{5.9}$$

indeed if $a\xi_n \to \eta$ and $\varphi \in X^*$ and $\varphi(\xi) = 1$ then, with $\varphi \odot \eta : \zeta \mapsto \varphi(\zeta)\eta$,

$$L_a(\varphi \odot \eta) = L_a(b) \Longrightarrow \eta = a(b\xi) \ . \tag{5.10}$$

Generally

Theorem 5.2. *If $T : A \to B$ is arbitrary then*

$$\text{QP}(A) \cap T^{-1}(B^{-1}) \subseteq A^{-1} + T^{-1}(0) \tag{5.11}$$

and if $T : A \to B$ is one one then

$$\text{QP}(A) \cap T^{-1}\text{SP}(B) = \text{SP}(A) \ . \tag{5.12}$$

Hence if $a \in B$ and $T = J : A = \text{comm}_B^2(a) \subseteq B$ then

$$A^{\cap} = T^{-1}\text{SP}(B) \ . \tag{5.13}$$

It follows that if $T^{-1}(0) = O$ then

Drazin $\Longrightarrow$ simple $\Longrightarrow$ spectral permanence .

Proof. Uniqueness guarantees that the spectral projection $T(a)^\bullet$ of $Ta \in \text{SP}(B) \subseteq \text{QP}(B)$ commutes with $T(a) \in B$, and one-one-ness guarantees the same for $a \in A$ □

For Banach algebra homomorphisms therefore there is an improved version of Theorem 4.2: of the three conditions

spectral permanence ; simple permanence ; one one ,

any two imply the third.

If we rework Theorem 4.1 with $B = B(\ell_2)$ then it is clear that isometric homomorphisms with spectral permanence need not have generalized permanence: indeed the forward shift $a = u \in B^{\cap} \setminus \text{QP}(A)$ is not even quasipolar: we recall that the spectrum of u is the closed unit disc, violating (5.5).

Theorem 4.1 was obtained in this way ([3] Theorem 3.2) in [3]. Of course (cf [9],[17]) "quasinilpotents" and"quasipolars" are only available in Banach algebras; Theorem 4.1 above, using "simply polar" elements, is conceptually much simpler.

6. Moore-Penrose permanence

We recall that a "C* algebra" is a Banach algebra which also has an *involution* $a \mapsto a^*$ which is conjugate linear, reverses multiplication, respects the identity and satisfies the "B* condition"

$$\|a^*a\| = \|a\|^2 \ (a \in A) \ . \tag{6.1}$$

Historically the term "C* algebra" was reserved for closed *-subalgebras of the algebras $B(X)$ for Hilbert spaces X; however the *Gelfand-Naimark-Segal* (GNS) representation

$$\Gamma : A \to B(\Xi_A) \tag{6.2}$$

takes an arbitrary "B* algebra" A isometrically into the algebra of operators on a rather large Hilbert space Ξ_A built from its "states": a defect of (6.2) would be that if already $A = B(X)$ we do not get back $\Xi_A = X$. In the opinion of this writer these terms "B* algebra" and "C* algebra" could easily ([7] Chapter 8) have been *Hilbert algebra*. When in particular $A = B(X)$ for a Hilbert space X then the closed range condition (3.9) is sufficient for relative regularity $a \in A^\cap$: indeed we can satisfy (2.2) by setting

$$c(\xi) = c(q\xi) \ ; \ c(a\xi) = p(\xi) \ (\xi \in X) \ , \tag{6.3}$$

where $q^* = q = q^2$ and $p^* = p = p^2$ are the orthogonal projections on the range $a(X)$ and the orthogonal complement $a^{-1}(0)^\perp$ of the null space. The element $c \in A$ given by (6.3) satisfies four conditions:

$$a = aca \ ; \ c = cac \ ; \ (ca)^* = ca \ ; \ (ac)^* = ac \ , \tag{6.4}$$

and is known as the *Moore-Penrose* inverse of $a \in B(X)$: more generally in a C* algebra A the conditions (6.4) uniquely determine at most one element

$$c = a^\dagger \in A \ , \tag{6.5}$$

lying ([11] Theorem 5) in the double commutant of $\{a, a^*\}$, and still known as a "Moore-Penrose inverse" for $a \in A$. Now it is a result of Harte and Mbekhta ([11] Theorem 6) that generally there is equality

$$A^\cap = A^\dagger \ : \tag{6.6}$$

in an arbitrary C* algebra, every relatively regular element has a Moore Penrose inverse. The argument, and a slight generalization, proceeds with the aid of the Drazin inverse.

More generally, on a semigroup A, an involution $a \mapsto a^*$ satisfies

$$(a^*)^* = a \ ; \ (ca)^* = a^*c^* \ ; \ 1^* = 1 \ . \tag{6.7}$$

In rings and algebras we also ask that the involution be additive, or conjugate linear. The B* condition (6.7) implies that, for arbitrary $a, x \in A$,

$$\|ax\|^2 \leq \|x^*\| \ \|a^*ax\| \ , \tag{6.8}$$

which in turn gives *cancellation*

$$L_{a^*a}^{-1}(0) \subseteq L_a^{-1}(0) \ . \tag{6.9}$$

Generally the *hermitian* or "real" elements of A are given by

$$\mathrm{Re}(A) = \{a \in A : a^* = a\} \ . \tag{6.10}$$

The Moore-Penrose inverse $a^\dagger$ of (6.4), if it exists, is unique and double commutes with a and a^*. We pause to notice the *star polar* elements of a semigroup A:

$$\mathrm{SP}^*(A) = \{a \in A : a^*a \in A^\cap\} \ ; \tag{6.11}$$

now we claim

Theorem 6.1. *If the involution * on the semigroup A is cancellable then*

$$A^\dagger \subseteq \mathrm{SP}^*(A) \subseteq A^\cap \ . \tag{6.12}$$

Proof. With cancellation there is implication

$$a \in \mathrm{SP}^*(A) \Longrightarrow a \in aAa^*a \subseteq Aa^*a \cap aAa \ ,$$

and equality

$$\mathrm{Re}(A) \cap \mathrm{SP}^*(A) = \mathrm{Re}(A) \cap \mathrm{SP}(A) \ ,$$

If $a = aca$ with $c = a^\dagger$ then

$$a^*a = a^*(ac)(ac)^*a = a^*acc^*a^*a \in a^*aAa^*a \ :$$

conversely, by cancellation,

$$a^*a = ada^*a \Longrightarrow a = ada^*a \ :$$

hence also

$$a \in Aa^*a \ ; \ \Longleftrightarrow a^* \in a^*aA \ .$$

Hence if $a^* = a$ then (4.2) follows □

It is now clear that an isometric C* homomorphism has "Moore-Penrose permanence":

Theorem 6.2. *If* $T : A \to B$ *has simple permanence then*

$$T^{-1}B^\dagger \subseteq A^\dagger \ . \tag{6.13}$$

Proof. We claim

$$A^\dagger = \{a \in A : a^*a \in \mathrm{SP}(A)\}\ , \tag{6.14}$$

with implication

$$a^*a \in \mathrm{SP}(A) \Longrightarrow a^\dagger = (a^*a)^\times a^*\ .$$

If $a \in A^\dagger$ with $a = aca$ and $(ca)^* = ca$ then, with $d = cc^*$, we have

$$a^*ad = a^*acc^* = a^*c^* = a^*c^*a^*c^* = ca$$

and

$$da^*a = cc^*a^*a = ca\ .$$

Conversely if $a^*a = a^*ada^*a$ with $a^*ad = da^*a$ with (wlog) $d = d^*$ then, using cancellation, with $c = da^*$,

$$aca = ada^*a = a \text{ and } ca = da^*a = a^*ad = a^*c^*\ .$$

Now if $a \in A$ there is implication

$$Ta \in B^\dagger \Longrightarrow T(a^*a) \in \mathrm{SP}(B) \Longrightarrow a^*a \in \mathrm{SP}(A) \Longrightarrow a \in A^\dagger$$

□

Our main result is a slight generalization, and a new proof, of the Harte/Mbekhta result (6.6), and at the same time "generalized permanence", equality in (3.4), for isometric C* homomorphisms. One way to go, thanks to the Gelfand/Naimark/Segal representation, is to look first in the very special algebra $D = B(X)$ of bounded Hilbert space operators:

Theorem 6.3. *If $d \in D = B(X)$ for a Hilbert space X then*

$$(d^*d)^{-1}(0) \subseteq d^{-1}(0) \tag{6.15}$$

and

$$\mathrm{cl}\ d(X) + d^{*-1}(0) = X\ ; \tag{6.16}$$

hence if $\mathrm{cl}\ d(X) = d(X)$ *then*

$$d^*(X) = d^*d(X)\ , \text{and} \mathrm{cl}\ d^*d(X) = d^*d(X)\ . \tag{6.17}$$

There is inclusion

$$\mathrm{Re}(D) \cap D^\cap \subseteq \mathrm{SP}(D)\ ; \tag{6.18}$$

hence

$$d \in D^\cap \Longrightarrow d \in \mathrm{SP}^*(D) \Longrightarrow d^*d \in \mathrm{SP}(D) \Longrightarrow d \in D^\dagger\ . \tag{6.19}$$

Proof. For arbitrary $\xi \in X$ there is [3] inequality

$$\|d\xi\|^2 \leq \|\xi\| \ \|d^* d\xi\| \ ,$$

and also

$$\text{cl } d(X) = d^{*-1}(0)^\perp$$

□

Both of the Harte/Mbekhta observations now follow:

Theorem 6.4. *If $T : A \to B$ is isometric then*

$$T^{-1}(B^\cap) \subseteq A^\dagger \ . \tag{6.20}$$

Proof. With $S : B \to D = B(X)$ a GNS mapping we argue, using again Theorem 4.2, together with "spectral permanence at" $a^* a$ (which has of course real spectrum),

$$Ta \in B^\cap \Longrightarrow ST(a^* a) \in \text{SP}(D) \Longrightarrow a^* a \in \text{SP}(A) \Longrightarrow a \in A^\dagger$$

□

In the situation of (6.14),

$$a = a^* \in A^\cap \Longrightarrow a^\dagger = a^\times \ ; \ 1 - a^\dagger a = a^\bullet \ . \tag{6.21}$$

Theorem 6.4 has an obvious extension to homomorphisms with closed range:

Theorem 6.5. *If $T : A \to B$ has closed range then there is implication, for arbitrary $a \in A$,*

$$T(a) \in B^\cap \Longrightarrow a + T^{-1}(0) \in (A/T^{-1}(0))^\cap \ . \tag{6.22}$$

Proof. Apply Theorem 6.4 to the bounded below $T^\wedge : A/T^{-1}(0) \to B$ □

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