Irish Mathematical Society Bulletin

Editor: Anthony G. O'Farrell

The aim of the *Bulletin* is to inform Society members, and the mathematical community at large, about the activities of the Society and about items of general mathematical interest. It appears twice each year. The *Bulletin* is supplied free of charge to members; it is sent abroad by surface mail. Libraries may subscribe to the *Bulletin* for 30 euro per annum.

The *Bulletin* seeks articles written in an expository style and likely to be of interest to the members of the Society and the wider mathematical community. Short research papers are also welcome. All areas of mathematics will be considered, pure and applied, old and new. The *Bulletin* is typeset using LaTeX. Authors must submit their articles LaTeX. See the inside back cover for instructions.

Correspondence concerning the *Bulletin* should be sent, in the first instance, by e-mail to the Editor at

ims.bulletin@gmail.com

and only if not possible in electronic form to the address

The Editor
Irish Mathematical Society Bulletin
Department of Mathematics and Statistics
NUI, Maynooth
Co. Kildare

Further information about the Irish Mathematical Society and its *Bulletin* can be obtained from the IMS webpage

http://www.maths.tcd.ie/pub/ims/

Printed in the University of Limerick

CONTENTS

EDITORIAL

This "Winter" issue appears just after the Equinox, when, as the song says, all the fields are fresh and green. The delay is regretted, but not the rising sap. One may hope that the longer days and pleasant weather will stimulate more members to complete the articles they have been planning and drafting for the Bulletin. As usual, we welcome:

- informative surveys of active research areas, written for the general mathematically-literate reader,
- biographical and historical articles related to Irish mathematics, including obituaries and interviews with senior figures.
- informative and factual articles, and letters with views, about important developments and events in Irish mathematics,
- short research articles
- classroom notes
- thesis summaries or abstracts from Irish schools and departments in the mathematical areas, and
- book reviews.

A new Bulletin LaTeXpackage may be downloaded from the Bulletin webpage. This should facilitate authors (and ease the labour of producing the Bulletin). The `bimsart` class is based on amsart. The main modification is the addition of a 14pt option. We needed this because the printers wanted to take an A4 pdf original and reduce it to A5. The `bims` style formats the print area appropriately, and imposes a uniform layout. I appreciate that the result is far from perfect, and feedback is welcome.

The Winter number includes the usual reports and thesis summaries. I am aware of numbers of theses that have not been submitted, and ask supervisors and senior staff around the country to remind students to send in their abstract as soon as the thesis has been successfully defended. There is a template on the webpage.

There have been some enhancements to the electronic version. Hypertext links within papers and across the internet are now live. With this in mind, it would seem useful to put a page in the Bulletin giving Irish links relevant to prospective mathematics research students. Ideally, each University and Institute involved in research should have at least one point of contact on the page, preferably a contact who is a mathematician, as opposed to some official in the central administration. I invite those responsible for postgraduate research student recruitment to send me a short text including a url. For instance, something like:

```
NUIM: \url{http://www.maths.nuim.ie/pghowtoapply}
```

or:

```
QUB: \url{http://www.qub.ac.uk/puremaths/Funded_PG_2012.html}
```

These will appear as:
NUIM: http://www.maths.nuim.ie/pghowtoapply
QUB: http://www.qub.ac.uk/puremaths/Funded_PG_2012.html
with live links (— so please test the links).

AOF. Department of Mathematics and Statistics, NUI, Maynooth, Co. Kildare
E-mail address: ims.bulletin@gmail.com

NOTICES FROM THE SOCIETY

Officers and Committee Members

President	Dr S. Wills	Dept. of Mathematics NUI Cork
Vice-President	Dr M. Mathieu	Dept. of Pure Mathematics QUB
Secretary	Dr R. Quinlan	Dept. of Mathematics NUI Galway
Treasurer	Dr S. Breen	Dept. of Mathematics St Patrick's College Drumcondra

Dr S. Breen, Prof S. Buckley, Dr J. Cruickshank, Dr. J. Gleeson, Dr B. Guilfoyle, Dr C. Hills, Dr N. Kopteva, Dr M. Mackey, Dr M. Mathieu, Dr R. Quinlan, Dr S. O'Rourke, Dr C. Stack, Prof A. Wickstead, Dr S. Wills

Local Representatives

Belfast	QUB	Dr M. Mathieu
Carlow	IT	Dr D. Ó Sé
Cork	IT	Dr D. Flannery
	UCC	Prof. M. Stynes
Dublin	DIAS	Prof Tony Dorlas
	DIT	Dr C. Hills
	DCU	Dr M. Clancy
	St Patrick's	Dr S. Breen
	TCD	Prof R. Timoney
	UCD	Dr R. Higgs
Dundalk	IT	Mr Seamus Bellew
Galway	UCG	Dr J. Cruickshank
Limerick	MIC	Dr G. Enright
	UL	Mr G. Lessells
Maynooth	NUI	Prof S. Buckley
Tallaght	IT	Dr C. Stack
Tralee	IT	Dr B. Guilfoyle
Waterford	IT	Dr P. Kirwan

Applying for I.M.S. Membership

(1) The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society, the Irish Mathematics Teachers Association, the New Zealand Mathematical Society and the Real Sociedad Matemática Española.

(2) The current subscription fees are given below:

Institutional member	160 euro
Ordinary member	25 euro
Student member	12.50 euro
I.M.T.A., NZMS or RSME reciprocity member	12.50 euro
AMS reciprocity member	15 US$

The subscription fees listed above should be paid in euro by means of a cheque drawn on a bank in the Irish Republic, a Eurocheque, or an international money-order.

(3) The subscription fee for ordinary membership can also be paid in a currency other than euro using a cheque drawn on a foreign bank according to the following schedule:

If paid in United States currency then the subscription fee is US$ 30.00.
If paid in sterling then the subscription is $20.00.
If paid in any other currency then the subscription fee is the amount in that currency equivalent to US$ 30.00.
The amounts given in the table above have been set for the

current year to allow for bank charges and possible changes in exchange rates.

(4) Any member with a bank account in the Irish Republic may pay his or her subscription by a bank standing order using the form supplied by the Society.

(5) Any ordinary member who has reached the age of 65 years and has been a fully paid up member for the previous five years may pay at the student membership rate of subscription.

(6) Subscriptions normally fall due on 1 February each year.

(7) Cheques should be made payable to the Irish Mathematical Society. If a Eurocheque is used then the card number should be written on the back of the cheque.

(8) Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.

(9) Please send the completed application form with one year's subscription to:

The Treasurer, I.M.S.
Department of Mathematics
St Patrick's College
Drumcondra
Dublin 9, Ireland

Deceased Members

It is with regret that we report that Jonathan Campbell died on 25th July 2010. He was a member of the Society for 4 years.

IRISH MATHEMATICAL SOCIETY

President's Report

Introduction: Since the last President's Report, the committee of the IMS has undergone numerous changes. James Cruickshank finished his term of office as president, passing care of the society to myself. I must express my gratitude for the work he has done during this time. We have also had the transition of editorship of the Bulletin from Martin Mathieu to Tony O'Farrell. They have worked hard to ensure a very smooth handover, with Tony already bringing his own ideas for future developments in the Bulletin. Finally, Richard Timoney has reached the end of a very long stint on the committee, and his input will be missed.

Meetings: The Annual Meeting of the Presidents of the National Member Societies of the European Mathematical Society took place in Bilbao, 7–9 May, organised by the Royal Spanish Mathematical Society in commemoration of the centenary of their foundation. The IMS was represented by our Vice President, Martin Mathieu, who found the meeting informative and useful. More detailed information can be found here:

`http://www.euro-math-soc.eu/node/1045`

Reciprocity Agreements: During the meeting in Bilbao, Martin began negotiations for a new reciprocity agreement with the German Mathematical Society (Deutsche Mathematiker-Vereinigung — DMV), the details of which have now been finalised. Members of the IMS are now able to become reciprocity members of the DMV at half of their current subscription rate (so €50 instead of €100).

Funding: The current world economic situation continues to impact of the availability of funding for mathematics research. At the meeting in Bilbao it was suggested that the European Science Foundation will cease to exist in the near future. Closer to home, Science Foundation Ireland are in the process of reorganising the Research Frontiers Programme and Principal Investigator Programme into a single scheme. The former was open to all branches of pure/basic science, the latter only funded projects based on ICT, biotechnology and energy engineering. The details of the new scheme are still scarce, but there is a suggestion that more applied fields may be emphasised leading to a decrease in funds for mathematics.

Fergus Gaines' Cup: This award, for the best performance in the selection test for the Irish team for the International Mathematical Olympiad, was presented to Colin Egan on 29 August at the IMS meeting in UL. Also in attendance, indeed speaking that day, were Kevin McGerty (University of Oxford) and David Conlon (University of Cambridge), both previous members of Irish IMO teams.

Conference Support: The Society supported the following conferences in 2010:

- 9th Annual Workshop on Numerical Methods for Problems with Layer Phenomena, DCU, 3–4 February.
- Women in Mathematics Day Ireland, UL, 18 April.
- 6th Annual Conference in Mathematics and Statistics Service Teaching & Learning Conference, GMIT, 5 May.
- History of Mathematics Conference, NUIM, 12 May.
- 18th International Conference of Adults Learning Mathematics (Mathematical Eyes: A Bridge Between Adults, the World and Mathematics), IT Tallaght, 26–29 June.
- 14th Galway Topology Conference, QUB, 15–17 August.
- Nonlinear Dynamics Conference in Memory of Alexei Pokrovskii, UCC, 5–9 September.
- Fourth Conference on Research in Mathematics Education (MEI 4 Mathematics Teaching Matters), SPD, 22–23 September.
- 6th Annual Workshop of the Irish Mathematics Learning Support Network, UCD, 16 December.

Conference support continues to be one of our main roles; details for applications are circulated on mathdep on a regular basis by Sinéad Breen, Treasurer of the IMS.

Future Meetings: Next year's 'September' meeting will take place in IT Tallaght (27 & 28 August), in 2013 we will meet in NUI Maynooth and in 2014 we will meet in QUB.

Minutes of the Annual General Meeting

30 August 2011

The Irish Mathematical Society held its Annual General Meeting from 11.30 to 13:00 on Tuesday 30 August 2011 in the University of Limerick. There were 15 members present at the meeting.

(1) **Minutes of AGM 3 September 2010.**

The minutes of the last AGM were approved and signed.

(2) **Matters Arising.**

10: Regarding possible membership of the ICIAM the President said that it is planned to consult with our members soon. So far no objections have been made known to the committee.

(3) **Correspondence.**

The meeting was told about an introductory one-year free membership of the EMS for PhD students. The President said that students should be encouraged to join.

(4) **Membership Applications.**

Linda Kavanagh (Graduate of DIT), Dr Gerry Keane (Statistician, CIE), David Conti (Shannon Institute).

(5) **President's Report.**

The President thanked M. Mathieu for editing the Bulletin, and his successor A. O'Farrell for taking on the job. J. Cruickshank was also thanked for his contribution to the Society as President and as long-standing committee member. B. Guilfoyle and N. Kopteva were also thanked for their service on the committee.

The Society funded (or was poised to fund) between eight and nine conferences in 2011. A new reciprocity agreement with the German Mathematical Society (Deutsche Mathematiker-Vereinigung) was in process.

(6) **Treasurer's Report.**

In the absence of S. Breen (who sent her apologies) S. Wills presented the Treasurer's report for 2010. It showed a surplus of €471.63.

It was mentioned that while the membership rate was raised four years ago, many members are still not paying the increased amount. Members were asked to apply pressure at local level to update their standing orders.

(7) **Bulletin.**

The incoming editor A. O'Farrell thanked his predecessor M. Mathieu and G. Lessells for their respective contributions to the Bulletin.

He also announced that issue 67 of the Bulletin was almost ready for circulation.

(8) **Elections to Committee.**

The following were elected unopposed to the committee:

Committee Member	Proposer	Seconder
R. Quinlan (Secretary)	S. Wills	M. Mathieu
J. Gleeson	G. Lessells	S. Wills

S. Breen agreed to serve another two-year term as Treasurer.
S. Buckley, R. Quinlan and A. Wickstead agreed to serve another two-year term on the Committee.

As editor, A. O'Farrell will be invited to committee meetings. The total number of years each existing member will have been on the committee as of 31 December 2011 will be : J. Cruickshank (9), N. Kopteva (6), B. Guilfoyle (6), S. Breen (5), S. O Rourke (5), S. Buckley (4), C. Hills (4), A. Wickstead (4), S. Wills (4), C. Stack (3), M. Mackey (2), R. Quinlan (2).

The following will then have one more year of office: S. Wills, M. Mathieu, C. Hills, M. Mackey, S. O Rourke, C. Stack.

(9) **EMS Presidents' Meeting.**

M. Mathieu represented the Society at the EMS meeting in Bilbao. He reported that a lack of research funding from the EU was a problem. The ESF (European Science Foundation) was in danger of being closed.

(10) **SFI**

SFI was restructuring its grant scheme, with the likely combining of the RFP (Research Frontiers Programme) with the PI (Principal Investigator) programme. The President intends to make a submission on behalf of the Society to minimise adverse consequences for Mathematics.

(11) **Future Meetings.**

Next year's September Meeting and AGM will take place in IT, Tallaght, most likely on 27-28 August 2012. The following two September Meetings will be in NUI-Maynooth (2013) and QUB (2014).

(12) **Any other business.**

R. Timoney reported that the SFI have no intention of supporting 'useless' subjects such as mathematics. The President said that he will do what he can to represent our views.

24TH ANNUAL MEETING OF THE IRISH MATHEMATICAL SOCIETY 29–30 AUGUST 2011

The 24th Annual Scientific Meeting of the Irish Mathematical Society was held at the University of Limerick. The meeting was organised by the Department of Mathematics and Statistics at the University of Limerick, in cooperation with the Mathematics Applications Consortium for Science and Industry (MACSI). The organisers gratefully acknowledge financial support from the National Centre for Excellence in Mathematics and Science Teaching and Learning (NCE-MSTL), and the Departments of CSIS and Physics & Energy at UL. The meeting brought together researchers from a wide range of mathematical disciplines and across Ireland and the United Kingdom. Below we give the meeting timetable and abstracts of the talks. We also refer the reader to the meeting website `http://www.maths.ul.ie/IMS_ul_2011/`.

Timetable

Monday 29 August

10:30–11:00	Registration; Tea & Coffee
	Chair : Alan Hegarty
11:00–11:10	**Kieran Hodnett**, *Dean of the Faculty of Science and Engineering, University of Limerick* Opening address
11:10–12:00	**Anne Brosnan**, *Project Maths Development Team* Project Maths Development Team: Supporting Good Practice
12:00–12:50	**Duncan Lawson**, *Coventry University* Who'd be a mathematician?
12:50–14:00	Lunch

	Chair : Bernd Kreussler
14:00–14:30	**David Conlon**, *University of Cambridge* Combinatorial theorems in sparse random sets
14:30–15:00	**Francesca Shearer**, *Queen's University Belfast* Electron detachment from negative ions in a short laser pulse
15:00–15:40	**Kevin McGerty**, *University of Oxford* Langlands duality and quantum groups at a root of unity
15:40–16:00	Tea & Coffee
	Chair : Gordon Lessells
16:00–16:50	**Philip Maini**, *University of Oxford* Modelling aspects of tumour growth
16:50–17:20	**Kevin Hutchinson**, *University College Dublin* The dilogarithm and SL_2
17:20–17:25	Presentation of the Fergus Gaines Cup
19:30	Dinner at the Castletroy Park Hotel

Tuesday 30 August

	Chair : Natalia Kopteva
9:20–10:10	**Martin Stynes**, *University College Cork* Convection-diffusion problems and their numerical solution
10:10–10:40	**Michel Destrade**, *National University of Ireland Galway* Acoustics of soft solids
10:40–11:00	Tea & Coffee
	Chair : Stephen Buckley
11:00–11:30	**Ailish Hannigan**, *University of Limerick* Modelling survival time: an application to data from Irish cancer patients
11:30–12:00	**Jean Charpin**, *University of Limerick* The MACSI summer school

12:00–13:00	IMS meeting
13:00–14:10	Lunch
	Chair : Stephen Wills
14:10–15:00	**John Appleby**, *Dublin City University* Preserving long-run properties when discretising stochastic and deterministic differential equations
15:00–15:45	**Richard Timoney**, *Trinity College Dublin* A brief survey on hypercyclicity
15:45	Tea & Coffee; Close

Abstracts

John Appleby, *Dublin City University*

Preserving long-run properties when discretising stochastic and deterministic differential equations

In this talk, we consider some highly schematised stochastic dynamical models of simulated annealing, bubble formation in financial markets and the propagation of fractures in metal. In each case we are lead to open mathematical problems in the asymptotic analysis of dynamical systems. In order that more realistic models of these phenomena can be developed, it is necessary to prove that the numerical methods applied will reproduce both qualitatively and quantitatively the appropriate asymptotic properties, namely pathwise stability, unbounded fluctuations and finite-time blow up of solutions. In order to minimise the computational cost, it should also be shown that, in some sense, no simpler numerical method will perform better than the method chosen. Each problem presents different challenges, requiring different numerical techniques, with the role of the simplified stochastic system being that of a test equation for the method.

Anne Brosnan, *Project Maths Development Team*

Project Maths Development Team: Supporting Good Practice

This presentation seeks to provide an understanding of the timeframe for Project Maths in the initial 24 schools and subsequent

National roll out for all other post primary schools. The presentation also seeks to provide an understanding of the work carried out by the Project Maths Development Team (PMDT). The influences that have informed and continue to inform this work are then reviewed. Arising from the work carried out by PMDT some key questions concerning Project Maths in general and the Continuing Professional Development being offered to teachers are then reviewed. The presentation concludes with a brief sketch of the integral teaching and learning resources developed by PMDT for teachers and a consideration of future work.

Jean Charpin, *University of Limerick*

The MACSI summer school

To encourage the study of mathematics in Ireland, the Mathematics Applications Consortium for Science and Industry (MACSI) organises a summer school once a year. The different aspects of this summer school are presented. Students are selected depending on their motivation, academic abilities, gender and geographical origins. Instruction and supervision is provided by academics, postdoctoral fellows and postgraduate students. The teaching programme evolves every year and reflects the interests of the people involved. Feedback from participants has been almost uniformly positive. Students favour interactive sessions and enjoy the residential aspect of the summer school. Food and accommodation are however the most costly aspects of this summer school. In this respect the support of Science Foundation Ireland has been invaluable.

Michel Destrade, *National University of Ireland Galway*

Acoustics of soft solids

Rubbers and biological soft tissues undergo large isochoric motions in service, and can thus be modelled as nonlinear, incompressible elastic solids. It is easy to enforce incompressibility in the finite (exact) theory of nonlinear elasticity, but not so simple in the weakly nonlinear formulation, where the stress is expanded in successive powers of the strain. In linear and second-order elasticity, incompressibility means that Poissons ratio is 1/2. Here we show how third- and fourth-order elastic constants behave in the incompressible limit. For applications, we turn to the propagation of elastic waves in soft incompressible solids, a topic of crucial importance in medical imaging (joint work with Ray Ogden, University of Aberdeen).

Ailish Hannigan, *University of Limerick*

Modelling survival time: an application to data from Irish cancer patients

Survival analysis describes the analysis of data that correspond to the time from a well-defined time origin until the occurrence of some particular event or endpoint. In medical research the time origin will often correspond to diagnosis with a particular disease and the endpoint is the death of the patient from that disease. Modelling survival data explores the relationship between the survival experience of the patient and explanatory variables such as age, sex, smoking history and treatment. This talk explores the issues involved in modelling survival data from cancer registry databases in particular the difficulty with inaccurate or missing death certificates. Relative survival models are introduced and applied to all women diagnosed with breast cancer in Ireland from 1994 to 2004 and followed up until the end of 2005.

Duncan Lawson, *Coventry University*

Who'd be a mathematician?

In England, mathematicians have something of an image problem. The general public does not have much understanding of what a mathematician is. The media tends to portray mathematicians as, at best, eccentric geniuses but often as anti-social loners. Research amongst school students has shown that their opinion of mathematicians is not very flattering either. Even amongst those who might be thought to have some commitment to mathematics, undergraduate students studying for degrees in the mathematical sciences, there are some quite negative opinions. The National Student Survey shows some quite disturbing results from mathematical sciences students, particularly in terms of their course giving them confidence and equipping them with broader skills. Furthermore, work amongst second year mathematics undergraduates at a range of universities has identified a range of perceptions of these students that may not be appreciated by academic staff including the worrying finding that over 20

Philip K. Maini, *University of Oxford*

Modelling aspects of tumour growth

The complex interaction of the multitude of physical and chemical processes that lead to tumour invasion is too difficult to understand by verbal reasoning along. As a result, the field of mathematical oncology is growing very fast. In this talk, mathematical models for tumour invasion, somatic evolution and tumour vasculature dynamics will be presented. They will consist of systems of partial differential equations, cellular automata and hybrid models. The results will be compared with experiment and therapeutic consequences will be explored.

Francesca Shearer, *Queen's University Belfast*

Electron detachment from negative ions in a short laser pulse

(joint work with M. C. Smyth and G. F. Gribakin)

We present an efficient and accurate method to study electron detachment from negative ions by a few-cycle linearly polarized laser pulse. The adiabatic saddle-point method of Gribakin and Kuchiev [Phys Rev A **55**, 3760 (1997)] is adapted to calculate the transition amplitude for a short laser pulse. Its application to a pulse with N optical cycles produces $2(N + 1)$ saddle points in complex time, which form a characteristic "smile". Numerical calculations are performed for H^- in a 5-cycle pulse with frequency 0.0043 a.u. and intensities of 10^{10}, 5×10^{10}, and 10^{11} W/cm^2, and for various carrier-envelope phases. We determine the spectrum of the photoelectrons as a function of both energy and emission angle, as well as the angle-integrated energy spectra and total detachment probabilities. Our calculations show that the dominant contribution to the transition amplitude is given by 5-6 central saddle points which correspond to the strongest part of the pulse. We examine the dependence of the photoelectron angular distributions on the carrier-envelope phase, and show that measuring such distributions can provide a way of determining this phase.

Martin Stynes, *University College Cork*

Convection-diffusion problems and their numerical solution

Convection-diffusion problems arise in the modelling of many physical processes. Their typical solutions exhibit boundary and/or interior layers. As the associated differential operators are linear, one might not expect much difficulty is solving these problems numerically, yet they pose questions to the numerical analyst that are still unanswered after more than 30 years of research in this area. This talk will discuss the nature of solutions to convection-diffusion problems in one and two dimensions, the inadequacy of classical numerical methods in this context, and some leading numerical techniques (including SDFEM/SUPG and Shishkin meshes) in current use for these problems.

Richard M. Timoney, *Trinity College Dublin*

A brief survey on hypercyclicity

While the phenomenon of hypercyclicity (of linear operators on a topological vector space) was originally observed in the context of complex analysis by G. D. Birkhoff in 1929 and much later for operators on Hilbert spaces by S. Rolewicz (in 1969), it was generally regarded as a bizarre occurrence until later. Now there is a huge literature on related topics, too extensive to explain in a talk. Starting with the basic definitions and the early history, we will describe some of the ideas involved and concentrate on surveying some results which reveal that the hypercyclicity is a very common occurrence.

REPORT ON THE 14TH GALWAY TOPOLOGY COLLOQUIUM

A.W. WICKSTEAD

Despite the name, the Galway Topology Colloquium series is held at varying locations around Ireland and the UK. The 14th session was held in Queen's University Belfast (QUB) between August 15th and 17th, 2011. It served a dual purpose by also marking the retirement from QUB of Brian McMaster who has been active in this series since its inception. The event was supported by the Irish Mathematical Society, a generous Scheme 1 grant from the London Mathematical Society and the Pure Mathematics Research Centre at QUB.

The conference was attended by a total of 25 participants, including eight postgraduate students which was very gratifying as one of the features of the Galway talks has been involving students. It was, indeed, pleasing to see the level of interaction between the invited speakers and the younger participants. In fact, one of the student participants, Shari Levine from Oxford, is currently organising the 15th colloquium in the series. There were 18 talks in total over the three days. We limit ourselves here to the describing the talks given by the invited speakers.

Alexander Arhangel'skii (Ohio and Moscow) talked on *Remainders of Various Kinds of Spaces in Compactifications*, mainly about the remainders of non-locally compact topological groups in Hausdorff compactifications.

Paul Bankston (Marquette) gave a talk titled *A Framework for Characterising Topological Spaces* which was about deciding whether or not a topological space can be characterised within a class of its "peers" by a sentence of first order logic.

Paul Gartside (Pittsburgh) spoke on *Stongly Arcwise Connected Continua* which involved spaces where for every n points there is an arc connecting them, possibly with the order being specified.

Jan van Mill (Amsterdam) gave an interesting survey on *Topological Homogeneity*, including several examples of homogeneous spaces and discussing several open problems.

Ivan Relly (Auckland) gave the centrepiece talk of the conference entitled *A Topological Antihero*. This surveyed the topic of topological anti-properties, introduced by Paul Bankston in 1979, and highlighted the contribution to the field of Brian McMaster and his students.

Dona Strauss (Leeds) talked on *The Algebra of* $\beta\mathbb{N}$ *and Polynomial Returns* which harked back to problems dating back to the 19th century and work of Hardy and Littlewood.

On the Monday evening, the conference dinner was combined with a retirement dinner for Brian and was enjoyed by all. Some photographs of both the conference and the dinner may be found at `http://www.qub.ac.uk/puremaths/Photos/Photograph_Album.html`.

Irish Math. Soc. Bulletin
Number 68, Winter 2011, 21–21
ISSN 0791-5578

ON THE QUIVER PRESENTATION OF THE DESCENT ALGEBRA OF TYPE A OR B

MARCUS BISHOP

This is an abstract of the PhD thesis *On the Quiver Presentation of the Descent Algebra of Type A or B* written by Marcus Bishop under the supervision of Götz Pfeiffer at the School of Mathematics, Statistics, and Applied Mathematics of the National University of Ireland, Galway and submitted in September 2010.

In this thesis we study the descent algebra of the Coxeter group of type A or B using a quiver presentation. The descent algebra is a quotient of the algebra of streets described in the recent paper [1]. To derive the presentation, we introduce an algebra of binary trees. The trees reflect the structure of the algebra of streets in a natural way and make it possible to construct a quiver whose path algebra is essentially the algebra of streets. Then expressing the descent algebra as a quotient of the algebra of binary trees and transferring the quotient to the path algebra provides the desired presentation. We then use the presentation to calculate the Cartan invariants and the projective indecomposable modules of the descent algebra.

References

[1] Götz Pfeiffer: *A quiver presentation for Solomon's descent algebra.* Adv. Math. 220 (5) (2009) 1428–1465. URL `http://arxiv.org/pdf/0709.3914v3`.

Ruhr-Universität Bochum, Fakultät für Mathematik, Universitätsstrasse 150, D-44780, Bochum, Germany.
E-mail address: `marcus.bishop@rub.de`

2010 *Mathematics Subject Classification.* 20F55, 16G20.
Key words and phrases. Coxeter group, quiver, presentation, path algebra.
Received on 18-10-2011; revised 20-2-2012.
Support from the Irish Research Council for Science, Engineering, and Technology through the Embark Postgraduate Scholarship Scheme is gratefully acknowledged.

Irish Math. Soc. Bulletin
Number 68, Winter 2011, 22–22
ISSN 0791-5578

COMPUTING THE TABLE OF MARKS OF A FINITE GROUP

LIAM NAUGHTON

This is an abstract of the PhD thesis *Computing the Table of Marks of a Finite Group* written by Liam Naughton under the supervision of Goetz Pfeiffer at the School of Mathematics, Statistics and Applied Mathematics, NUI, Galway and submitted in September 2010.
In this thesis we introduce a new method for constructing the table of marks of a finite group S from the table of marks of A a normal subgroup of S of index p, a prime. The first step in any such approach is to compute a list of representatives of the conjugacy classes of subgroups of S. In this spirit we describe a new algorithm which computes the conjugacy classes of subgroups of S from the conjugacy classes of subgroups of A. We then present a series of algorithms which compute the table of marks of S from the table of marks of A. Computer programs based on the theory described in this thesis have been used to compute the table of marks of S_{13} from the table of marks of A_{13}.

School of iMathematics, Statistics and Applied Mathematics, NUI, Galway
E-mail address: `liam.naughton@nuigalway.ie`

2010 *Mathematics Subject Classification.* 20B05.
Key words and phrases. Table of Marks, Finite Group.
Received on 1-11-2011.

Irish Math. Soc. Bulletin
Number 68, Winter 2011, 23–23
ISSN 0791-5578

PROPERTIES OF SUBSPACE LATTICES RELATED TO REFLEXIVITY

SAVVAS PAPAPANAYIDES

This is an abstract of the PhD thesis *Properties of Subspace Lattices related to Reflexivity* written by S. Papapanayides under the supervision of Dr I.G. Todorov at the School of Mathematics and Physics, Queen's University Belfast, Belfast and submitted in September 2011.

This PhD focuses on issues such as the reflexivity of the tensor product of two lattices, the lattice tensor product formula (LTPF) and property (p). Property (p) was firstly introduced by Shulman and Todorov in [3] and the LTPF by Hopenwasser in [2]. The main results are that the tensor product of a subspace lattice having the ultraweak rank one density property and a commutative subspace lattice has property (p) and that for an atomic boolean subspace lattice having the ultraweak rank one density property and a reflexive subspace lattice, the LTPF holds. We also investigate other properties related to subspace lattices such as semistrong closedness. Furthermore, following the work of Arveson in [1], we study and describe algebras associated to certain classes of subspace lattices.

References

[1] W. Arveson, *Operator Algebras and Invariant Subspaces*, Ann. of Math. (2) 100 (1974), 433–532.
[2] A. Hopenwasser, *Tensor products of reflexive subspace lattices*, Michigan Math. J. 31 (1984), no. 3, 359–370.
[3] V. S. Shulman and I. G. Todorov, *On subspace lattices I. Closedness type properties and tensor products*, Int. Eq. Op. Th. 52 (2005), no. 4, 561–579.

(from June 2012) School of Mathematics and Physics, Queen's University Belfast
E-mail address, S. Papapanayides: spapapanayides01@qub.ac.uk

2010 *Mathematics Subject Classification.* 47L30, 47L75.
Key words and phrases. Subspace lattices.
Received on 21-3-2012.
Support from my supervisor, Dr I.G. Todorov, is gratefully acknowledged.

Irish Math. Soc. Bulletin
Number 68, Winter 2011, 24–24
ISSN 0791-5578

AUTOMORPHISMS OF PAIRWISE COMBINATORIAL DESIGNS

PADRAIG Ó CATHÁIN

This is an abstract of the PhD thesis *Automorphisms of Pairwise Combinatorial Designs* written by Pádraig Ó Catháin under the supervision of Dr Dane Flannery at the School of Mathematics, Statistics, and Applied Mathematics, National University of Ireland, Galway and submitted in August 2011.

The thesis investigates group actions on certain families of pairwise combinatorial designs, in particular Hadamard matrices and symmetric 2-$(4t-1, 2t-1, t-1)$ designs.

A Hadamard matrix H is called cocyclic if a certain quotient of the automorphism group contains a subgroup acting regularly on the rows and columns of H. We develop an algorithm for constructing all CHMs of order $4t$ based on a known relation between CHMs and relative difference sets. This method is then used to produce a classification of all CHMs of order less than 40. This is an extension and completion of work of de Launey and Ito.

If H is a CHM developed from a difference set then the automorphism group of H is doubly transitive. We show that the only CHMs with non-affine doubly transitive automorphism group are those that arise from the Paley Hadamard matrices. As a corollary of this result, we show that twin prime power difference sets and Hall sextic residue difference sets each give rise to a unique CHM.

We classify all difference sets which give rise to Hadamard matrices with non-affine doubly transitive automorphism group. In the process, we uncover a new triply infinite family of skew-Hadamard difference sets.

School of Mathematics, Applied Mathematics and Statistics, National University of Ireland, Galway, Ireland.
E-mail address: p.ocathain@gmail.com

2010 *Mathematics Subject Classification.* 05B05, 05B10, 05B20, 20B20, 20B25.
Key words and phrases. Block designs, Difference sets, Hadamard matrices, automorphisms, permutation groups.
Received on 15-3-2012.

Patrick Francis Hodnett

October 15th, 1939 – July 5th, 2011

It was with much sadness that we learned last summer of the untimely passing of our friend and colleague Frank Hodnett, RIP. Frank is sadly missed by his wife Diane, daughter Nicola, son David, relatives, colleagues and friends. For most of his life, Frank played a major role in the Irish applied mathematical community and was the founding Head of the Department of Mathematics and Statistics (as it is now called) at the University of Limerick. What follows are memories of Frank, written by six colleagues who walked with him along parts of the path of his life.

Finbarr Holland, *Professor Emeritus* (UCC)– Frank's early years in Cork

Patrick Francis Hodnett, first Professor of Applied Mathematics at The University of Limerick, was born in the Square, Bantry, Co Cork, on October 15, 1939, the eldest in a family of three girls and three boys; he died from cancer of the oesophagus on July 5, 2011.

Frank's father was a Post Office engineer, and sometime in the late 1940s or early 1950s, his job took him to Cork City, and the family took up residence in Mercier Park just on the outskirts of the city, a stone's throw away from the Turner's Cross Soccer grounds. In many ways this was a judicious choice by the family. On the one hand, their home was very close to the bus route that connected that suburb to the north side of the city, one that passed the North Monastery, where Frank finished his primary education, and received his secondary education. On the other hand, Mercier Park was just outside the boundary dividing the city from the county, which entitled him to compete later in his career for Cork County Council Scholarships, instead of the more competitive City Council Scholarships. Being so near to the soccer grounds, also, must surely have influenced his choice to make soccer his main sporting interest in his youth. In fact, up to the age of eighteen he played with the local soccer club, Tramore Athletic, and was a member of that club's team that won the Evans Cup, a national competition for under fifteens. Later in 1958, when practically the same team reached the final of the FAI Minor Cup which it won, Frank lost out because, close to the final, having played in all the earlier matches, he came off the panel to concentrate on the Leaving Certificate Examination and the Entrance Scholarship Examination for University College, Cork.

Already, even at this early stage, we see evidence of his strength of character, and his independence of mind. Thus, from an early age he travelled across the city on his own by bus to go to school, by-passing the nearby Scoil Chríost Rí, which was just around the corner from his home. As a pupil at the North Mon—a school renowned in those days for its achievements in hurling and adherence to gaelic culture, which he attended on the advice of friends of the family who had a love of Irish—he played soccer to a high level, something that wouldn't have endeared him to the Christian Brothers, and placed him somewhat outside the pale. Remember, the infamous GAA Ban was in force at that time, and ensured that he wouldn't be eligible to line out with any of the Mon's teams. Again, in deciding to concentrate on his preparation for impending important examinations, we can but admire his decision to give up a once-in-a-life's opportunity to win a coveted FAI Minor trophy.

The latter decision paid off handsomely for him, however, because later that year he earned a prestigious Entrance Scholarship to UCC, and a Cork County Scholarship to boot. These scholarships enabled him to register in the Science Faculty at UCC in October, 1958, where, alongside many more freshmen who were inspired by Sputnik etc., to pursue a career in Science, including this writer, he settled down to do Chemistry, Experimental Physics, Mathematics and Mathematical Physics, all at honours level, and Science German. I remember he did Chemistry through Irish in his first year at UCC—no doubt a carry-over from his experience in the Mon, where the Leaving Certificate science subjects were taught through Irish; indeed, he was one of only a handful of students who did Chemistry through Irish—another instance of his single-mindedness–and so received special attention from the Lecturer, Dr. Réamonn Ó Cinnéide. On the basis of his First Year Examination results, Frank continued to hold the Cork County Scholarship in the succeeding two years during which time he studied for an honours degree in Mathematics and Mathematical Physics, and was duly conferred in 1961 with a First Class Honours BSc degree in these subjects.

Throughout his time in UCC, Frank took an active part in the running of the UCC Soccer Club, both on and off the field of play, and in many ways he was the heart-beat of the club. He played with the club in the Munster Senior League and in the Intervarsity competition for the Collingwood Cup, a trophy UCC didn't win until 1973.[1] Sometime during his early undergraduate career, he damaged a cartilage in one of his knees which necessitated him spending a bout in hospital, but as soon as he was discharged and declared fit to play, he resumed his playing activities, anxious to assist the club in whatever way he could. Also, in those early days, he decided to take up weight-lifting to strengthen his muscles, a novel thing then for students to do—many of whom would have been more accustomed

[1]Finbarr Holland is grateful to Bernard McLoughlin who supplied facts about Frank's soccer activities, and to Tony Deeney who helped refresh his memory about other details mentioned in this article.

to lifting pints—and this became something of an obsession with him. But this, too, singled him out from the rest of the student body, and evinced the single-minded attitude he brought to bear on everything he did, and the seriousness with which he approached whatever interested him at the time.

Upon receipt of his primary degree, Frank registered at UCC in October, 1961, to do his MSc, which entailed a further two years study, the customary period in those days to study for a taught master's degree, and to compete for the NUI Travelling Studentship in Mathematical Science. But at the beginning of his second year, in order to earn some money, he took up an appointment as a Teaching Assistant at Magee College, Derry, where he had the good fortune to meet his future wife, Diane—a Cornish woman who was unable to use her grant from Cornwall County Council to attend TCD, the university of her choice, because it was deemed to be in a foreign country!—before re-entering UCC in January, 1963, where he resumed his studies. Later that year, the NUI awarded him the MSc degree in Mathematical Science with First Class Honours. On the strength of that he was offered an Assistant Lectureship at Leeds University where he also commenced his PhD in Applied Mathematics, but I leave it to others to take up the story at that point.

A final remark: long after our student days together, but still a long time ago, he and I wrote a joint paper with King-Hele. My interest back then was sparked by a paper Frank delivered at an *ad-hoc* seminar he gave to the Applied Mathematicians in UCC, after which I made a small observation which he graciously acknowledged by putting my name to the paper! Alas, I've no memory of the content of that paper now, though.

Prof. Derek Ingham (Leeds)– Frank's time in Leeds

Frank came to Leeds in the mid 1960's at the start of the rapid expansion in UK universities. He was appointed as an 'Assistant Lecturer in the Department of Mathematics' but in the Applied Mathematics section. His presence was like 'a breath of fresh air' coming to a Department that had a long tradition of staff that had been there for many years. Frank held an MSc qualification but had great ambitions and desperately wanted to study for a PhD. At the same time as Frank came to Leeds, Professor Allin Goldsworthy came as a new Research Professor in Applied Mathematics and

Frank soon realised that he was the person under whom to study. At that time there were weekly research meetings where all the ‘young’ active researchers presented their work and discussed ways forward for their research work. Frank was always determined to be the first to discuss his work. He was always extremely keen and very enthusiastic. However, in addition to his great drive to succeed academically, Frank was a very outward looking person and enjoyed life. For example, he was one of the founding members of the Department of Mathematics football team where he was the centre forward. No matter where he was on the field, he wanted the ball and ‘led the line’ in the way only Frank could do. We played against such illustrious teams as the Meat Market, Yorkshire Post, the Police, etc. In addition, we will never forget his exploits with his old cars. Coming from Éire, he could drive on his Irish licence for one year but he failed his driving test in the UK a couple of times, even though he went to several different test centres! Hence, even after one year of being in the UK he continued to drive with no UK licence - his defence being that he had not actually been driving for one year in the UK although he had been living here for more than one year! Further, his car was old and required an MOT, and the hand brake did not work, therefore he charged us all to find an MOT garage that was not near any hills so that the garage would not test the efficiency of the hand brake. This we did and the car passed.

Frank was a great success in Leeds with his research work, his lecturing and his very likeable character. Yet, Frank had dreams/ideas that Leeds could not match and after a few years he was off to the US. We all knew that it would not be long before he would return to his beloved Éire to become a real national and international force in Applied Mathematics. However, ever since leaving Leeds, Frank has been in regular contact with us and we have exchanged research ideas, performed examining activities, planned how best to promote Applied Mathematics, etc. We, in Leeds, are proud to have been associated with Frank and to have started him on his path to achieving all his goals. However, all this he could not have done without Diane, his wife and his family, all of whom we dearly loved.

Prof. James Flavin, *Professor Emeritus* (NUIG)– Frank's academic career

Career (Postdoctoral)

Following his years at Leeds, Frank's career may be summarised as follows:

1968:	Research Associate, Aerospace Research Lab., Dayton, Ohio, U.S.A.
1969-73:	Assistant/Associate Professor, Aerospace Engineering, University of Connecticut, U.S.A.
1974-83:	Lecturer/Senior Lecturer, Applied Mathematics, N.I.H.E., Limerick.
1984-97:	Head, Department of Mathematics and Statistics, University of Limerick (N.I.H.E. Limerick until 1989).
1986-2005:	Associate Professor/Professor, Applied Mathematics, University of Limerick.

Frank joined N.I.H.E., Limerick (as it then was) as the first member of staff in applied mathematics. According to the folklore, this arose as follows: Ed Walsh (Director of N.I.H.E, Limerick), who was looking for a mathematician for his faculty, rang Paddy Quinlan – Frank's former professor in Cork – while Frank happened to be visiting Paddy's office, and the rest is history! Applied Mathematics was first part of the Department of Electronics until it became, under Frank's guidance, a fully-fledged department - one of the largest and most vibrant centres of applied mathematics in Ireland. While spectacularly developing applied mathematics at Limerick and pursuing an extensive research programme, he progressed through the academic *cursus honorum* - in the face of competition - as summarised above.

Research

Frank's research, while largely centred on various aspects of fluid dynamics, extended over a very broad area.
His research publications may roughly be classified as follows:

(1) Classical fluid dynamics including slow flow of compressible fluids.
(2) Ocean dynamics.
(3) Rossby waves.
(4) Solutions of the Korteweg- de Vries equation.

(5) Electricity load forecasting.
(6) Miscellaneous-including industrial applications of mathematics.

A brief description of his work in some of these areas now follows:

Classical fluid dynamics: He considers flow of compressible fluids, with thermal effects, in regions bounded internally by a circular cylinder, and bounded internally and externally by concentric circular cylinders. He identifies a small, dimensionless parameter for each physical problem at hand, and develops solutions in terms of the small parameter, using the method of matched asymptotic expansions where relevant (*e.g.* for slow flow, the Reynolds number is used as the small parameter). Examples of this work are reported in the following papers, all in his own name except the last which is in his name and that of a doctoral student D. Rose:

- The slow compressible flow past a circular cylinder, *Physics of Fluids*, 11, 1636-1647, 1968.
- Low Reynolds number flow of a variable property gas past a heated circular cylinder, *J. Fluid Mech.*, 39, 465-476, 1969.
- Low Reynolds number flow past a heated circular cylinder, *Physics of Fluids*, 13, 2429-2431, 1970.
- Natural convection between horizontal heated, concentric circular cylinders, *ZAMP*, 24,507-516, 1973.
- (with D. Rose) Unsteady heat transfer from a circular cylinder in a low Reynolds number flow. *ZAMP*, 25, 179-188, 1974.

Ocean dynamics: The basic model employed is that of a perfect fluid (but in the context of Boussinesq's approximation) in steady motion, with Coriolis effects due to the earth's rotation, and incorporating temperature variations. The independent variables are the latitudinal and longitudinal angular coordinates together with the depth, while the dependent variables are the velocity components in the coordinate directions together with the pressure and temperature. Various approximations to the relevant partial differential equations are considered which are deemed appropriate to different oceanic zones, *e.g.* the thermocline, the abyssal region. Solutions of the resulting equations are considered by analytic and numerical means. Solutions of relevant boundary value problems are considered by both these means and the results are usually compared with observations. Frank's published work on this subject may conveniently be grouped into five areas:

(1) Papers on the advective model of the thermocline, two in his own name and one with F. Holland and J.A. King-Hele : In the first of these *(J. Marine Research*, 36,1,185-198,1978), by employing the temperature (T) as an *independent* variable and a depth variable (z) as a *dependent* variable, Frank elegantly reduced the five p.d.e.s to a single, fully nonlinear, third order equation in just one variable ($G = p - Tz$, p being the pressure). This would enable him to generate a large class of solutions to the original five equations. The second of the papers, with Holland and King-Hele, (*Proc. R. Ir. Acad.* , 80A, 1,57-62,1980) showed how to obtain a general first integral for the aforementioned equation in G. The third paper (*Proc. R. Ir. Acad.*, 82A, 2,187-199, 1982) considered boundary value problems, by analytic means, in the zonally uniform case (*i.e.* independent of longitude).

(2) In a paper (*Geophys. Astrophys. Fluid Dynamics*, 78, 73-93, 1994) with A.R. Ansari, the interaction of a simple ocean model with eastern boundary shapes is considered.

(3) In two papers with Y. Yuan, one (*Proc. R. Ir. Acad.*, 97A, 2,193-207, 1997) considers the effect of small thermal diffusion in a zonally uniform ocean model, while another (*Proc. R. Ir. Acad.,* 100A, 2,115-138,2000) considers a model of the northern ocean with eastern boundary currents.

(4) In three papers with R. McNamara, one (*Proc. R. Ir. Acad.,* 100A, 1 85-104, 2000) treats a modified Stommel–Arons model of the abyssal ocean circulation, the second deals with zonal influences in a similar context (*Proc. R. Ir. Acad.,* 102A,1,1-27) while another (*Proc. R. Ir. Acad.,* 103A, 2, 217-230,2003) discusses the variation of the vertical thermal diffusion coefficient in a simple ocean model.

(5) Work related to the above general themes is discussed in two other papers *(Bull. I.M.A.,*16,2-3,68-72,1980); *J.Math.Ed.Sci. Tech.*,38,1029-1049,2007).

Rossby Waves: Frank co-authored with a doctoral student, W.M. O'Brien, three papers on the instability of Rossby waves and allied waves. These papers were published in *Beitr. Phys. Atmosph.* as follows: 62,90-98,1989; 64,49-54,1991; 64,261-271,1992. The flavour of this work is summarised as follows: Rossby waves represent slow, large scale motions of the atmosphere similar to what is observed in some weather systems; thus the stability or otherwise of these waves is of interest. The instability of Rossby waves had been investigated by Lorentz (1972) by linear analysis and by others (Loesch, 1978; Deininger and Loesch, 1982) using a weakly nonlinear analysis. Hodnett and O'Brien extended these studies to the fully nonlinear regime by solving numerically the non-divergent barotropic vorticity equation and tracing the evolution over a long time period (800 hours). They found (a) that Lorentz's analysis correctly predicts only the initial growth rate of the solution (when unstable), (b) that the nonlinear solution eventually settles into a bounded oscillating state for all values of the wave amplitude A (similar to what Loesch and Deininger and Loesch found for values of A slightly in excess of A_c, the critical amplitude of instability) and (c) that A_c (determined by linear theory) is not an accurate indicator of stability/instability for the nonlinear solution.

Solutions of the Korteweg -de Vries equation: the Korteweg -de Vries equation, which arises in the theory of water waves, for example, is

$$u_t + 6uu_x + u_{xxx} = 0.$$

The solitary wave, or soliton solution, is

$$u = 2a^2\text{sech}^2\theta$$

where $\theta = ax - 4a^3t + \delta$, where a and δ are constants. Another solution, called the "two soliton solution" was derived by Hirota. Frank together with a doctoral student, T.P. Moloney, (*J. Phys. A.*: *Math.Gen.*,19, 18, L1129-L1135, 1986) showed that this latter solution may be written as a linear superposition of two functions each of which has a form similar to the soliton (or solitary wave) solution. This decomposition facilitates the elucidation of the interaction of two solitons. Extensions and generalisations of these interesting ideas are discussed in a series of papers by Moloney and Hodnett (*SIAM J. Appl. Math.*, 49, 4, 1174- 1187, 1989; *Proc. R. Ir. Acad.*, 89A, 2, 205-217, 1989; *SIAM J. Appl.Math.*, 51, 4, 940-947, 1991).

Electricity load forecasting: This was work done with a doctoral student, O. Hyde, in response to an ESB requirement for forecasts of systems demand or electrical load. Typically, a forecasting model was developed which identified a "normal" or weather-insensitive load component and a weather-sensitive load component. Linear regression analysis of past load and weather data was used to identify the normal load model. The weather sensitive component was estimated by using the parameters of the regression analysis. This work resulted in some four publications.

Miscellaneous: Frank's publications in this category extend over a very broad spectrum: they range from magneto-fluid dynamics through the aerodynamics of foil chaff, design of an air journal bearing, to wave induced washout of submerged vegetation in shallow lakes.

Five people completed their Ph.D.s under Frank's supervision: D. Rose (1973), Univ. of Connecticut; T.P. Moloney (1989), W.M. O'Brien(1990), O. Hyde (1994), R. McNamara (2000), Univ. of Limerick.

Frank's research at UL was supported by numerous grants:

(1) Two NBST Research Grants (1982-84, 1986-87).
(2) Forbairt Basic Research Grant (1995-97).
(3) Eolas Higher Education-Industry Cooperation Grants with the ESB (1989-91,1996-97).
(4) European Commission Grants under the Marine Science and Technology Programme (1993-96, 1991-93).

Other Publications

In addition to research reports to funding agencies, Frank published articles on the role of mathematics in engineering education, on workshops on interactions with industry held at Limerick. He also published an interesting article on "Osborn Reynolds, 1842-1912" in *Creators of Mathematics, The Irish Connection*, University College Dublin Press, 71-77, 2000.

External Interaction

Frank made over forty presentations of his research at scientific conferences, and gave over twenty invited lectures at conferences and other universities.

Frank was active in societies promoting mechanics and applied mathematics. He was Chairman of the Irish Mechanics Group (subsequently Society) from 1977-80. He served as Chairman, from 1996-2000, of the National Committee for Theoretical and Applied Mechanics of the Royal Irish Academy, having served as its Secretary from 1992-96. Indeed, this body was one of the sponsors of the highly prestigious IUTAM (International Union of Theoretical and Applied Mechanics) symposium "Advances in Mathematical Modelling of Atmospheric and Ocean Dynamics" which was hosted by Frank at U.L. in 2000; specifically, he was Chairman of both the Organising and Scientific Committees of the symposium. He also edited the proceedings:

Hodnett , P. F. (Ed.), *Proceedings of the IUTAM Symposium on "Advances in Mathematical Modelling of Atmospheric and Ocean Dynamics*", Kluwer, Dordrecht, pp.1-298, 2001).

He was also active in ECMI —Sean McKee writes about this.

Frank served as External Examiner for the NCEA, at the University of Strathclyde, and at Queen's University, Belfast. He was also a member of the Board of Associate Referees, *Journal of Engineering Mathematics.*

Conclusion and general

Frank had many sporting interests: he was a keen soccer player in his youth, and in later years developed a great interest in horse racing while he was also an enthusiastic fan of Munster rugby.

He was a highly regarded member of the Irish mathematical community whose passing leaves a void in the lives of all who knew him. But above all, he will be missed by his wife Diane, his children David and Nicola and his grandchildren.

Leaba i measc na naomh go raibh aige.

Gordon Lessells (University of Limerick)– Frank's involvement in ILIAM

For 10 years, Frank was heavily involved with ILIAM, the Information Linkage between Industry and Applied Mathematics. Originally the brainchild of Dr. John Carroll (NIHE, Dublin), this was an attempt to bring together industrialists and applied mathematicians to promote cooperation and information flow between industry and academia. *ILIAM 1* was held in June 1984. At *ILIAM 2,* a joint coordinating committee was set up consisting of Frank Hodnett (NIHE Limerick), John Carroll (NIHE Dublin) and Tony

Donegan (University of Ulster) with a view to rotating future ILIAM seminars between the three institutions. These meetings continued for at least ten years. During those years, a huge variety of talks were given from Aircraft Design in 1986 to Statistical Analysis of the Foreign Exchange Market in 1993. *ILIAM 10* was held at the University of Limerick in May 1993. The success of these meetings was a spur for Frank to get involved with a larger project with the same philosophy *viz.* ECMI; Sean McKee takes up this story.

Prof. Sean McKee (University of Strathclyde)– Frank's involvement in ECMI

Professor Frank Hodnett, representing Ireland, was one of eleven Europeans to found the European Consortium for Mathematics in Industry, ECMI. This took place in Neustadt-Mussbach, a pleasant wine growing region of Germany on the 14th of April 1986. In addition to Frank Hodnett, the other ten men were A. Bensoussan (INRIA, Paris), A. Fasano (University of Florence), M. Hazewinkel (CWI, Amsterdam), M. Heilio (Lappeenranta University, Finland), H. Martens (Norwegian Institute of Technology, Trondheim), S. McKee, (University of Strathclyde, Scotland), H. Neunzert (University of Kaiserslautern, Germany), D. Sundström (The Swedish Institute of Applied Mathematics, Stockholm), A. Tayler (University of Oxford, England) and H. Wacker (University of Linz, Austria). For 10 years, Frank remained on the Executive Board travelling, twice a year, to various parts of Europe, providing good humoured common sense to some of our more excitable European partners. In 1991, he organised, funded and ran the sixth ECMI Conference in Limerick. Influenced by Hansjorg Wacker's work on optimising hydro-electric systems, Frank Hodnett and I used to meet periodically in Dublin to put together a European proposal. Although never successful, this led to both parties obtaining funding elsewhere: in particular, Frank built up a good relationship with the Irish Electricity Supply Board (ESB) and this resulted in a number of publications mainly concerned with the prediction of electricity demand.

Michael Wallace, *Professor Emeritus* (UL)– Frank's career at NIHE/UL

Dr. Frank Hodnett arrived in the National Institute for Higher Education, Limerick in the spring of 1974 as the first lecturer in Applied Mathematics. For the next 31 years, he worked tirelessly for the cause of Mathematics and its applications at initially NIHE, later to become the University of Limerick (in 1989). His objectives and achievements progressed in parallel. Initially he set about establishing suitable streams of Mathematics for existing programmes *i.e.* Electronic and Materials Engineering, Business Studies and Administrative Systems. Frank strongly believed that all Mathematics should be taught by Mathematics faculty and it is to his credit that this maxim is still in place at UL. As student enrolment increased (240 in 1974), further appointments were made in the areas of Statistics, Operations Research, Control Theory, Algebra etc. Although initially members of the Electronics Dept., Frank insisted that we met as a separate group with the intense belief that the Mathematics faculty should have its own department, later to become a realisation in 1984 with Frank as its first head, a position held until 1997. He also believed that the Mathematics faculty should have its own programme and was very proud in 1979 to welcome the first cohort of students on the BSc degree in 'Industrial and Management Mathematics', a title that caused much debate and explanation. All these achievements in his first ten years at NIHE were not without internal opposition and it was his tenacity allied to his strongly held views that won the day. An example of this was the decision taken at an Academic Council meeting in June 1980 to discontinue the Mathematics programme, despite having two cohorts of students enrolled. However, at its meeting the following month, the Governing Authority overturned this decision!

Despite all the time given to these developments and when, in the early years, it was not a main priority, Frank managed to progress his research activities both in Ireland and Europe. A testimony to his standing in this regard was his hosting of several major and international conferences *e.g.* ECMI in 1991 and IUTAM in 2000. He published regularly, spoke at major conferences and supervised many PhD students. (Jim Flavin deals with this area in more detail). In recognition of all these achievements, Frank was deservedly appointed as the first Professor of Applied Mathematics at the University of Limerick in 1994.

Away from his academic responsibilities Frank was very sociable and great company. His interest in sport knew no bounds and his company at coffee on Mondays was sought for his insightful assessments of the week-end games from Cork hurling to Munster rugby and everything in between. He loved a good party and whether it was the staff at Christmas or the graduating students in summer, he will be long remembered for his rousing rendition of ‘The Bould Thady Quill’:

For ramblin’, for rovin’, for football or courtin’
For drinkin’ black porter as fast as you’d fill....

Ar dheis Dé go raibh a anam dílis.

Collected and compiled by Eugene Gath, University of Limerick.

Irish Math. Soc. Bulletin
Number 68, Winter 2011, 39–54
ISSN 0791-5578

QUADRATIC MONOMIAL ALGEBRAS AND THEIR COHOMOLOGY

DAVID O'KEEFFE

Abstract. The aim of this note is to discuss and highlight the use of projective modules and projective resolutions in homological algebra. Using a minimal projective resolution by Sköldberg in [8] , we describe the calculation of the cohomology groups for the class of quadratic monomial algebras. The cohomology groups of an associative algebra are invariants of an algebra and provide a fundamental description of the structure of the algebra.

1. Introduction

In a forthcoming paper with Emil Sköldberg, the cohomology groups and cohomology ring structure are explicitly described for a particular class of associative algebras: the class of *quadratic monomial algebras.* Until recently, little was known about the multiplicative structure of the Hochschild cohomology ring for most classes of associative algebras. During the last decade or so, more light has been shed on this topic, with several papers published regarding the structure of the Hochschild cohomology rings for various classes of associative algebras, see for instance [1] and [2] . In [2], Claude Cibils computes the cohomology groups and ring structure for the class of radical square zero algebras. The aforementioned algebras are a subclass of algebras considered in this paper.

2. QUIVERS AND QUADRATIC MONOMIAL ALGEBRAS

In this section we begin by recapitulating the following definition of a quiver and from there we will present the main objects of interest in this article, namely the class of quadratic monomial algebras.

2010 *Mathematics Subject Classification.* 16E40.
Key words and phrases. Quadratic Monomial Algebras, Hochschild cohomology.
Received 20-9-2011; revised 1-3-2012.

Definition 1. A *quiver* $\Delta = (\Delta_0, \Delta_1)$, is an oriented graph, where Δ_0 denotes the set of vertices, and Δ_1 the set of arrows between the vertices. The origin and terminus of an arrow $a \in \Delta_1$, is denoted by $o(a)$ and $t(a)$ respectively.

We shall deal with finite connected quivers, that is the sets Δ_0 and Δ_1 are finite and the undirected graph will be connected. A *path* α in Δ, is an ordered sequence of arrows, $\alpha = a_1 \cdots a_n$, $a_i \in \Delta_1$ with $t(a_i) = o(a_{i+1})$ for $i = 1, \cdots, n-1$. We shall write $o(\alpha) = o(a_1)$ and $t(\alpha) = t(a_n)$ for the initial and terminal vertices of α respectively. An *oriented* cycle in Δ, is a path α, where $o(\alpha) = t(\alpha)$. The *length* or *degree* of α, denoted $|\alpha|$ is equal to the number of arrows in α and the set of all paths of length n, is denoted Δ_n. A vertex $e \in \Delta_0$ is considered to be a path of length zero with $o(e) = t(e) = e$. We shall allow Δ to have oriented cycles and multiple arrows between vertices.

Now we would like to make a semigroup out of the paths, and we will do this by first defining the set $\hat{\Delta}$ by

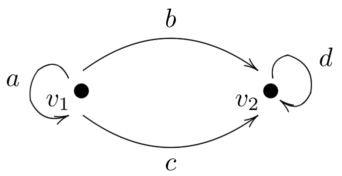
$$\hat{\Delta} = \{\perp\} \cup \bigcup_{i=0}^{\infty} \Delta_i$$

The multiplication in $\hat{\Delta}$ is, for $\gamma \in \Delta_i$, $\delta \in \Delta_j$, defined by $\gamma \cdot \delta = \gamma\delta$, if $t(\gamma) = o(\delta)$ and $\gamma \cdot \delta = \perp$ otherwise. For all $\alpha \in \hat{\Delta}$ we have $\alpha \cdot \perp = \perp \cdot \alpha = \perp$. For k a commutative ring, we may now form the semigroup-algebra $k\hat{\Delta}$, and then we may view $k\Delta$, the *path* or *quiver* algebra on Δ, as the quotient algebra $k\hat{\Delta}/(\perp)$. The benefit of this definition is that $k\Delta$ becomes a $\hat{\Delta}-$graded algebra. There is also an $\mathbb{N}-$grading on $k\Delta$, where $\deg_{\mathbb{N}} \alpha = n$, if $\alpha \in \Delta_n$. The paths of length 0, Δ_0 generate a subalgebra $k\Delta_0$ of $k\Delta$; hence $k\Delta$ is a $k\Delta_0-$bimodule. The identity element in $k\Delta$ is given by the sum of vertices. The class of algebras studied here are quotients of path algebras.

Definition 2. A *quadratic monomial algebra* A is a quotient of a quiver algebra $k\Delta$, $A = k\,\Delta/I$, where $I = (\alpha_1, \ldots, \alpha_n)$ is a two sided homogeneous ideal generated by a set of paths of length two in Δ.

Since I is a two sided homogeneous ideal with respect to the standard grading, A is a graded algebra and we may describe a canonical $k-$basis for A and denote it $B(A)$. Such a basis consists of all paths that do not contain a path from I. A typical basis element in A may be written as $a_1 \ldots a_n$, where $a_i a_{i+1} \notin I$ for $1 \leq i \leq n-1$.

Example 1. Let k be any field and Δ the following quiver:



We may construct a quadratic monomial algebra which we shall denote by A by choosing an ideal I, generated by paths of length two and then forming the quotient $A = k\,\Delta/I$. For instance if we let $I = (aa, dd)$ then the following is an example of multiplication in A:

$$v_1 \cdot v_1 = v_1,\ \ v_1 \cdot v_2 = 0,\ \ v_1 \cdot b = b,\ \ a \cdot b = ab,\ \ b \cdot a = 0, \quad \text{etc.}$$

The products

$$a \cdot a = aa, \quad d \cdot d = dd$$

are equal to 0 in A, since both of these paths are contained in I.

3. PROJECTIVE MODULES AND PROJECTIVE RESOLUTIONS

A central theme in the world of homological algebra is the exploration of the structure of rings and modules. Projective modules are a basic tool which are used extensively in this examination, since any module may be viewed as an epimorphic image of a projective module - just choose a set of generators $\{g_i\}$ for M and map a projective module on a corresponding set of generators $\{e_i\}$ to M by sending e_i to g_i. In this way it is easy to compare any module to a projective module: if $d : P \longrightarrow M$ is an epimorphism, then we may say that P differs from M by the kernel of d. For more on this, see for example [3]. We will explore this idea in the present section. We will assume that we are working with left modules unless stated otherwise. There are several equivalent definitions of a projective module but the one that will be of most interest to us at present will be the following:

Definition 3. Let A be a ring. An A module P is *projective* if it is isomorphic to a direct summand of a free A module.

We will now introduce the following notation with the quadratic monomial algebra A of example 1 in mind.

Let

$$P_{v_1} := \langle \text{ All paths in } A \text{ starting at } v_1, \text{ including the zero path } v_1 \rangle$$
$$P_{v_2} := \langle \text{ All paths in } A \text{ starting at } v_2, \text{ including the zero path } v_2 \rangle$$
$$P_{v_1} \cdot J := \langle \text{ All paths in } A \text{ starting at } v_1 \text{ of length} \geq 1 \rangle.$$

Since a path may only begin at either vertex v_1 or v_2, we have

$$P_{v_1} + P_{v_2} = A \quad \text{and} \quad P_{v_1} \bigcap P_{v_2} = 0$$

and so

$$A = P_{v_1} \bigoplus P_{v_2}$$

P_{v_1} and P_{v_2} are direct summands of the left module A (which is free over itself) and so we may regard them as (indecomposable) projective left A modules. In the following we write $\operatorname{im} d$ to denote the image of the homomorphism d and $\ker d$ to denote the kernel of d.

Definition 4. Let M be an A-module. The sequence P_* of A-modules and A-module homomorphisms

$$P_* : \quad \cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

is called a *complex* if $\operatorname{im} d_{i+1} \subseteq \ker d_i$ for each i. P_* is said to be *exact* if the $\operatorname{im} d_{i+1} = \ker d_i$ for each i. P_* is called a *projective resolution* of M over A, if it is exact for all $i \geq 0$ and each P_i is a projective A-module.

For now, P_* will denote a projective resolution of M and when we write *deleted* projective resolution, we shall mean P_* with the M term removed. We will be mainly interested in the case when $M = A$. Projective resolutions are utilised in homological algebra as a way of approximating a module by using "well behaved" projective modules. It is well known that any A-module admits a projective resolution, see for instance [3] and we illustrate this below for the

$A-$module $P_{v_1}/P_{v_1}\cdot J$ taking the quadratic monomial algebra A in example 1 on page 41, as a demonstration.

We begin by considering the epimorphism ε in the following exact sequence:

$$P_{v_1} \xrightarrow{\varepsilon} P_{v_1}/P_{v_1}\cdot J \longrightarrow 0$$

P_{v_1} differs from $P_{v_1}/P_{v_1}\cdot J$ and this difference is recorded in $\ker\varepsilon$.

We may now form the exact sequence:

$$\ker\varepsilon \hookrightarrow P_{v_1} \xrightarrow{\varepsilon} P_{v_1}/P_{v_1}\cdot J \longrightarrow 0 \tag{1}$$

but in the above, $\ker\varepsilon = P_{v_1}\cdot J$ is not projective as an $A-$module, since it is not a direct summand of the free module A. We may correct this blemish however by finding a projective resolution of $P_{v_1}\cdot J$. Consider the following sequence

$$\ker d_1 \hookrightarrow P \xrightarrow{d_1} P_{v_1}\cdot J \longrightarrow 0$$

We would now like to construct a projective module P and epimorphism d_1, making the aforementioned sequence exact. Note any path in $P_{v_1}\cdot J$ can be expressed using the generating set $\{v_1a\alpha,\, v_1b\beta,\, v_1c\gamma\}$, where $\alpha\in P_{v_1}$, $\beta\in P_{v_2}$, and $\gamma\in P_{v_2}$ (see quiver on page 41). If we replace P with the direct sum of projective modules $P_{v_1}\oplus P_{v_2}\oplus P_{v_2}$ (which again results in a projective module), then a typical element in $P_{v_1}\oplus P_{v_2}\oplus P_{v_2}$ is of the form

$$v_1\alpha + v_2\beta + v_2\gamma$$

and we may define a surjective homomorphism d_1 on a generating element as follows:

$$\begin{aligned} d_1(v_1\alpha + v_2\beta + v_2\gamma) &= d_1(v_1\alpha) + d_1(v_2\beta) + d_1(v_2\gamma) \\ &= v_1a\alpha + v_1b\beta + v_1c\gamma \end{aligned}$$

It is easy to see that d_1 is surjective and $\operatorname{im} d_1 = \ker\varepsilon = P_{v_1}\cdot J$ and so replacing $\ker\varepsilon$, with $P_{v_1}\oplus P_{v_2}\oplus P_{v_2}$ in (1), the projective resolution of $P_{v_1}/P_{v_1}\cdot J$ now becomes:

$$\ker d_1 \hookrightarrow P_{v_1}\oplus P_{v_2}\oplus P_{v_2} \xrightarrow{d_1} P_{v_1} \xrightarrow{\varepsilon} P_{v_1}/P_{v_1}\cdot J \longrightarrow 0$$

Again as before, the $\ker d_1$ is not projective as an $A-$module. We may again tackle this situation using the same approach as previously: forming a projective resolution of $\ker d_1$. The kernel of d_1

is generated by $v_1 \cdot a$. We need to define a homomorphism and projective module that maps onto $\ker d_1$:

$$P_{v_1} \xrightarrow{d_2} \ker d_1 \longrightarrow 0 \quad \text{with} \quad v_1 \xrightarrow{d_2} v_1 a$$

and now the projective resolution of $P_{v_1}/P_{v_1} \cdot J$ takes the form:

$$\cdots \xrightarrow{d_3} P_{v_1} \xrightarrow{d_2} P_{v_1} \oplus P_{v_2} \oplus P_{v_2} \xrightarrow{d_1} P_{v_1} \xrightarrow{\varepsilon} P_{v_1}/P_{v_1} \cdot J \longrightarrow 0$$

If we delete $P_{v_1}/P_{v_1} \cdot J$ from the above exact complex, we get a projective resolution $P_{v_1}/P_{v_1} \cdot J$. We may view this deleted resolution as an approximation of the simple module $P_{v_1}/P_{v_1} \cdot J$.

4. Hochschild cohomology

The appropriate cohomology theory for the class of associative $k-$algebras was first described by Gerhard Hochschild in [6]. Before continuing any further, we shall introduce the following notation: Given an arbitrary associative $k-$algebra A with unit, we shall write $A^e = A \otimes_k A^{op}$ to denote the enveloping algebra of A. Here we write A^{op} to denote the *opposite* algebra of A; as vector spaces A and A^{op} are isomorphic but A^{op} is endowed with the opposite multiplication of A:

$$a^{op} b^{op} = (ba)^{op}, \text{ where } a,\, b \in A, \quad a^{op},\, b^{op} \in A^{op}$$

By $\mathrm{Hom}_{A^e}(M, N)$ we shall mean the set of all A^e-homomorphisms from M to N. This set may be endowed with the structure of an abelian group, where for $f,\, g \in \mathrm{Hom}_{A^e}(M, N)$ it may be shown that $f + g$ defined by $(f+g)(m) = f(m)+g(m)$ is an A^e-homomorphism for all $m \in M$.

$\mathrm{Ext}_{A^e}(A, M)$ may be defined as the Hochschild cohomology group $H^*(A, M)$ of A with coefficients in the $A-$bimodule M. When $M = A$, it may be shown that $\mathrm{Ext}_{A^e}(A, A)$ also possesses a rich multiplicative structure:

$$\mathrm{Ext}^m_{A^e}(A, A) \otimes_k \mathrm{Ext}^n_{A^e}(A, A) \longrightarrow \mathrm{Ext}^{m+n}_{A^e}(A, A)$$

turning $\mathrm{Ext}_{A^e}(A, A)$ into a *graded commutative* algebra. An algebra A is graded commutative (or *supercommutative*) with homogeneous elements a and b of degree m and n in A respectively if

$$ab = (-1)^{mn}\, ba.$$

To compute the aforementioned cohomology groups of A, we begin by first applying the left exact functor $\mathrm{Hom}_{A^e}(\,\cdot\,, M\,)$ to the deleted *standard* projective resolution of A, where the nth projective module has the form $P_n = A \otimes_k A^{\otimes n} \otimes_k A$ and so the resolution may be written:

$$\cdots \longrightarrow A \otimes A^{\otimes n} \otimes A \xrightarrow{d_n} \cdots \xrightarrow{d_2} A \otimes A \otimes A \xrightarrow{d_1} A \otimes A \longrightarrow 0$$

The differential d_n is given by

$$\begin{aligned} d_n(a_0 \otimes a_1 \cdots a_n \otimes a_{n+1}) &= a_0 a_1 \otimes a_2 \cdots a_n \otimes a_{n+1} \\ &+ \sum_{i=1}^{n-1} (-1)^i a_0 \otimes a_1 \cdots (a_i a_{i+1}) \cdots a_n \otimes a_{n+1} \\ &+ (-1)^n\, a_0 \otimes a_1 \cdots a_{n-1} \otimes a_n a_{n+1} \end{aligned}$$

Now using the isomorphism $\mathrm{Hom}_{A^e}(A \otimes A^{\otimes n} \otimes A, M) \cong \mathrm{Hom}_k(A^{\otimes n}, M)$, we get the so called *Hochschild complex*:

$$0 \longrightarrow A \xrightarrow{\delta^0} \mathrm{Hom}_k(A, M) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta} \mathrm{Hom}_k(A^{\otimes n}, M) \xrightarrow{\delta} \cdots$$

The nth Hochschild cohomology module of A with coefficients in M is given by

$$H^n(A, M) \cong \ker \delta^n / \mathrm{im}\, \delta^{n-1} \cong \mathrm{Ext}_{A^e}(A, M).$$

In particular, we shall be interested in the case when $M = A$ and we shall write $HH^*(A)$ instead of $H^*(A, A)$. In this instance the differential δ has the following form:

$$\delta^0 : A \longrightarrow \mathrm{Hom}_k(A, A) \quad \text{with} \quad (\delta^0 b)(a) = ab - ba \quad \text{for } a,\, b \in A.$$

and for $n \geq 1$, $\quad \delta^n : \mathrm{Hom}_k(A^{\otimes n}, A) \longrightarrow \mathrm{Hom}_k(A^{\otimes (n+1)}, A)$;

$$\begin{aligned} (\delta^n f)(a_1 \otimes \cdots a_{n+1}) &= a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) \\ &+ \sum_{1 \leq j \leq n} (-1)^j f(a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{n+1}) \\ &+ (-1)^{n+1} f(a_1 \otimes \cdots \otimes a_n) a_{n+1} \end{aligned}$$

4.1. **Interpreting the 0th and 1st cohomology groups.** In this subsection we highlight some aspects of the cohomology groups in dimensions ≤ 2. In particular the Hochschild cohomology groups may be interpreted as providing an insight into the structure of an algebra. We begin by considering $HH^0(A)$. This group is isomorphic to the kernel of δ^0 and hence consists of all those elements in A that commute with all elements in A, that is $HH^0(A)$ is the centre of A:

$$HH^0(A) \cong \{b \in A|\, ab = ba \text{ for all } a \in A\}.$$

Definition 5. A *derivation* of A to A is an $k-$module homomorphism $f : A \longrightarrow A$ that satisfies Leibnitz's rule:

$$f(ab) = af(b) + f(a)b \quad \text{ for all } a,\, b \in A$$

A derivation f of A to A is an *inner derivation* if there exists $b \in A$ such that:

$$f(a) = ab - ba \quad \text{ for all } a \in A$$

Now returning to the coboundary δ and setting

$$\delta^1(f)(a \otimes b) = af(b) - f(ab) + f(a)b$$

equal to zero, we observe that a 1-cocycle (an element in the kernel of δ^1) is also a linear map $f : A \longrightarrow A$ which satisfies Leibnitz's condition. Similarly for each $b \in A$,

$$\delta^0(b)(a) = ab - ba \quad \text{for all } a \in A$$

and so there is a one-to-one correspondence between the coboundaries lying in $\operatorname{im}\delta^0$ and the inner derivations of A. The $k-$module $HH^1(A)$ may be interpreted as the space of all bimodule derivations of A modulo the inner derivations of A.

There are also connections to algebraic geometry. In [4], Murray Gerstenhaber introduced a deformation theory for rings and algebras based on formal power series. A formal deformation of an associative algebra $(A,\, \mu)$ is an associative algebra $A[[t]]$ with a multiplication μ_t defined by

$$\mu_t(p,\, q) = \mu(p,\, q) + t\mu_1(p,\, q) + t^2\mu_2(p,\, q) + \cdots$$

where $p,\, q \in A$. The algebra is said to be *rigid* if every formal deformation is isomorphic to a trivial deformation. One may show that separable semi-simple algebras are rigid. In the same paper

Gerstenhaber observed that algebras A which satisfy $HH^2(A) = 0$ are *rigid.*

4.2. **A Projective Resolution of a Quadratic Monomial Algebra.** In general a module may have several projective resolutions. When one wishes to compute (co)homology, a *minimal* projective resolution is best. In the following we shall state a minimal projective A^e- resolution for a quadratic monomial algebra. This minimal projective resolution was constructed by Emil Sköldberg in[8]. In order to write down this resolution the following definition will be required:

Definition 6. Let $A = k\Delta/I$ be an algebra such that I is an ideal generated by quadratic monomials. Define the ideal J to be generated by all quadratic monomials that do not lie in I; then the *Koszul dual* of A, denoted by $A^!$, is defined by $A^! = k\Delta/J$.

Example 2. For the given quadratic monomial algebra A in example 1 on page 41, we have $A^! = k\Delta/J$, with $J = (ab, ac, cd, bd)$. $A^!$ has the same generators of A, with the following as an example of multiplication in $A^!$:

$$v_1 \cdot v_1 = v_1,\ v_1 \cdot v_2 = 0,\ v_1 \cdot b = b,\ a \cdot a = aa,\ d \cdot d = dd, \quad \text{etc.}$$

The products

$$a \cdot b = ab, \quad d \cdot d = dd \quad a \cdot c = ac, \quad c \cdot d = cd$$

are equal to zero in $A^!$, since all products are contained in J.

The projective modules in the minimal projective graded A^e-resolution of A have the following description. From here on, when we write P_* we shall be referring to the following minimal A^e-resolution.

Lemma 1. *If $A = k\Delta/I$ is a quadratic monomial algebra, then a minimal projective resolution of A given as a left A^e-module is*

$$P_i = A \otimes_{k\Delta_0} A_i^! \otimes_{k\Delta_0} A$$

and the A^e-linear differential is defined on the basis elements by

$$d_i(1 \otimes\ a_1 \cdots a_i \otimes\ 1) = a_1 \otimes\ a_2 \cdots a_i \otimes\ 1 + (-1)^i\, 1 \otimes\ a_1 \cdots a_{i-1} \otimes a_i$$

Proof. See [8].

□

Example 3. Consider $A = k\Delta/I$, where

$$\Delta = \begin{tikzcd} \underset{v}{\bullet} \arrow[r, bend left, "a"] & \underset{w}{\bullet} \arrow[l, bend left, "b"] \end{tikzcd}$$

$I = (ab, ba)$ and $J = (\emptyset)$. The projective modules P_i in P_*, have the form, where we write $\otimes = \otimes_{k\Delta_0}$:

$$\cdots \xrightarrow{d_4} \begin{matrix} A\otimes aba \otimes A \\ \oplus \\ A \otimes bab \otimes A \end{matrix} \xrightarrow{d_3} \begin{matrix} A\otimes ab \otimes A \\ \oplus \\ A \otimes ba \otimes A \end{matrix} \xrightarrow{d_2} \begin{matrix} A\otimes a \otimes A \\ \oplus \\ A \otimes b \otimes A \end{matrix} \xrightarrow{d_1} \begin{matrix} A\otimes v \otimes A \\ \oplus \\ A \otimes w \otimes A \end{matrix} \xrightarrow{\varepsilon} A \to 0$$

Notice that the module P_0 records the vertices of the quiver, P_1 records the arrows that generate A, P_2 records the relations of A, P_3 records the relations $(ab)a = a(ba)$ among relations of A, etc.

5. Cohomology of a Quadratic Monomial Algebra

In a forthcoming paper with Emil Sköldberg, the cohomology groups for a quadratic monomial algebra are explicitly described. In this section we illustrate this theory by calculating the Hochschild cohomology groups of the given quadratic monomial algebra A in example 3 on page 48. We begin with the following complex:

$$0 \longrightarrow \mathrm{Hom}_{A^e}(P_0, A) \xrightarrow{\delta^0} \cdots \xrightarrow{\delta^{n-1}} \mathrm{Hom}_{A^e}(P_n, A) \xrightarrow{\delta^n} \cdots \qquad (2)$$

The coboundary δ is induced by the differential d on P_*:

$$\begin{tikzcd} P_{i+1} \arrow[rr, "d_{i+1}"] \arrow[dr, dotted, "\delta^{i+1}(f)"'] & & P_i \arrow[dl, "f"] \\ & A & \end{tikzcd}$$

$$\begin{aligned} \delta^{i+1} f(1 \otimes a_1 \cdots a_{i+1} \otimes 1) &= f(d_{i+1}(1 \otimes a_1 \cdots a_{i+1} \otimes 1)) \\ &= f(a_1 \otimes a_2 \cdots a_{i+1} \otimes 1) \\ &+ (-1)^{i+1} f(1 \otimes a_1 \cdots a_i \otimes a_{i+1}) \end{aligned}$$

where $f \in \mathrm{Hom}_{A^e}(P_i, A)$. As we have seen earlier, the nth Hochschild cohomology module of A with coefficents in A, may be found by computing

$$HH^n(A) \cong \ker \delta^n / \mathrm{im}\, \delta^{n-1} \cong \mathrm{Ext}^n_{A^e}(A, A)$$

We will use the following lemma to simplify our calculation of the cohomology groups.

Lemma 2. *The map* $\phi : \mathrm{Hom}_{A^e}(P_i, A) \longrightarrow \mathrm{Hom}_{k\Delta_0^e}(\, A^!_i, A\,)$ *defined by*

$$\phi(f)(a_1 \cdots a_i) := f(1 \otimes_{k\Delta_0} a_1 \cdots a_i \otimes_{k\Delta_0} 1),$$

$f \in \mathrm{Hom}_{A^e}(P_i, A)$ *is a chain map and vector space isomorphism for each* i.

Proof. This result is proved in the authors thesis.

□

We have established through lemma 2 that calculating the cohomology of the cochain complex at (2), will yield the same results as computing the cohomology of the following cochain complex:

$$0 \longrightarrow \mathrm{Hom}_{k\Delta_0^e}(A^!_0, A) \xrightarrow{\bar{\delta}^0} \cdots \xrightarrow{\bar{\delta}^{n-1}} \mathrm{Hom}_{k\Delta_0^e}(A^!_n, A) \xrightarrow{\bar{\delta}^n} \cdots \tag{3}$$

We shall now take a moment to describe the coboundary homomorphism $\bar{\delta}$ and the $k-$module $\mathrm{Hom}_{k\Delta_0^e}(A^!, A)$ in a little more detail. An element $f \in \mathrm{Hom}_{k\Delta_0^e}(A^!, A)$ is a $k\Delta_0^e-$ linear homomorphism from $A^!$ to A. For $\alpha \in B(A^!)$ and $\beta \in B(A)$, we shall use the notation (α, β) for the morphism $\alpha \overset{f}{\longmapsto} \beta$, and $\gamma \mapsto 0$ for all other basis elements $\gamma \in B(A^!)$. We shall write $o(\alpha) = v$ and $t(\alpha) = w$. Since $f \in \mathrm{Hom}_{k\Delta_0^e}(A^!_i, A)$ is linear over the vertices, we have

$$f(\alpha) = f(v \cdot \alpha) = v \cdot f(\alpha) = v \cdot \beta$$

and

$$f(\alpha) = f(\alpha \cdot w) = f(\alpha) \cdot w = \beta \cdot w$$

and so for an $f \in \mathrm{Hom}_{k\Delta_0^e}(A^!, A)$, we shall write (α, β) such that $o(\alpha) = o(\beta)$ and $t(\alpha) = t(\beta)$. It is shown in [7] that the coboundary operator $\bar{\delta}$ on these basis elements is given by

$$\bar{\delta}(\alpha, \beta) = \sum_{a \in \Delta_1} (a\alpha, a\beta) + (-1)^{|\alpha|+1} \sum_{b \in \Delta_1} (\alpha b, \beta b)$$

From here on we shall write δ in of place of $\bar{\delta}$. We are now in a position to calculate the cohomology groups of the quadratic monomial algebra given in example 2, page 48.

Example 4. We begin with the following deleted projective resolution of A:

$$\cdots \xrightarrow{d_3} \begin{matrix} A\otimes ab\otimes A \\ \oplus \\ A\otimes ba\otimes A \end{matrix} \xrightarrow{d_2} \begin{matrix} A\otimes a\otimes A \\ \oplus \\ A\otimes b\otimes A \end{matrix} \xrightarrow{d_1} \begin{matrix} A\otimes v\otimes A \\ \oplus \\ A\otimes w\otimes A \end{matrix} \longrightarrow 0$$

Applying $\mathrm{Hom}_{A^e}(\cdot,\ A\,)$ and then using the isomorphism $\mathrm{Hom}_{A^e}(P_i, A) \cong \mathrm{Hom}_{k\Delta_0^e}(A^!_i, A)$,we get the resulting cochain complex :

$$0 \longrightarrow \begin{matrix} \mathrm{Hom}_{k\Delta_0^e}(v,\ A\,) \\ \oplus \\ \mathrm{Hom}_{k\Delta_0^e}(w,\ A\,) \end{matrix} \xrightarrow{\delta^1} \begin{matrix} \mathrm{Hom}_{k\Delta_0^e}(a,\ A\,) \\ \oplus \\ \mathrm{Hom}_{k\Delta_0^e}(\,b,\ A\,) \end{matrix} \xrightarrow{\delta^2} \begin{matrix} \mathrm{Hom}_{k\Delta_0^e}(\,ab,\ A\,) \\ \oplus \\ \mathrm{Hom}_{k\Delta_0^e}(\,ba,\ A\,) \end{matrix} \xrightarrow{d^3} \cdots$$

We would now like to calculate the cohomology in each degree of this complex but before doing this we note a k-basis for $\mathrm{Hom}_{k\Delta_0^e}(A^!, A)$ is given by all (α, β) with $o(\alpha) = o(\beta)$ and $t(\alpha) = t(\beta)$. Hence we may simplify the notation in the aforementioned complex by rewriting it as:

$$0 \longrightarrow \begin{matrix} k\cdot(v,\ v) \\ \oplus \\ k\cdot(w,\ w) \end{matrix} \xrightarrow{\delta^0} \begin{matrix} k\cdot(a,\ a) \\ \oplus \\ k\cdot(b,\ b) \end{matrix} \xrightarrow{\delta^1} \begin{matrix} k\cdot(ab,\ v) \\ \oplus \\ k\cdot(ba,\ w) \end{matrix} \xrightarrow{\delta^2} \cdots$$

We may now calculate the cohomology groups in each degree. We begin by computing $HH^0(A) \cong \ker\delta^0$:

$$\begin{aligned}\delta^0(\lambda(v, v) + \mu(w, w)) = \lambda((b,\ b) - (a,\ a)) \\ + \mu((a,\ a) - (b,\ b))\end{aligned}$$

which is equal to zero $\Leftrightarrow \lambda = \mu$ for $\lambda,\ \mu \in k$.

Hence the kernel of δ^0 is one dimensional and is generated by $(u,\ u) + (v,\ v)$ and so $HH^0(A) \cong k$.

Next we compute $HH^1(A)$. Since $\dim(\ker\delta^0) \cong k$ and δ^0 is a map from a two dimensional vector space, we have $\dim(\operatorname{im}\delta^0) \cong k$. Now

$$\delta^1(\lambda(a, a)) = \lambda(ba,\ ba) + \lambda(ab,\ ab) = 0$$

since $ba \in J$ or $ba \in I$ and $ab \in J$ or $ab \in I$. For the same reason we also have

$$\delta^1((\mu(b, b))) = \mu(ab, ab) + \mu(ba, ba) = 0$$

Hence $\ker\delta^1 \cong k \oplus k$ and so we have $HH^1(A) \cong k$.

Finally we compute $HH^2(A)$. Since the kernel of δ^1 is two dimensional, this means the dimension of the image $\operatorname{im}\delta^1$ is trivial. Computing as before, it is easy to show that the kernel of δ^2 is generated by $(ab, v) + (ba, w)$ and so $\dim(\ker\delta^2) \cong k$. Hence $HH^2(A) \cong k$.

□

6. Interpreting the 0th & 1st cohomology groups of A

As we have already seen, the zeroth cohomology group $HH^0(A)$ coincides with the centre of A. We shall illustrate this with an example.

Example 5. Let $A = k\Delta/I$ be the quadratic monomial algebra obtained from the quiver

$$\underset{u}{\bullet} \xrightarrow{a} \underset{v}{\bullet}$$

where in this instance $I = J = \{\emptyset\}$. Note the centre of an algebra A consists of all those x of A such that $xa = ax$ for all a in A.

We shall first compute the centre of the given quadratic monomial algebra:

$$(\lambda u + \varphi v + \psi a) \cdot u = u \cdot (\lambda u + \varphi v + \psi a)$$
$$\Leftrightarrow \lambda u = \lambda u + \psi a \quad \Rightarrow \psi = 0$$
$$(\lambda u + \varphi v) \cdot a = a \cdot (\lambda u + \varphi v)$$
$$\Leftrightarrow \lambda a = \varphi a \quad \Rightarrow \lambda = \varphi$$

and so the centre of A is a one dimensional vector space spanned by $\langle u + v \rangle$. Computing $HH^0(A)$ from the following complex:

$$0 \longrightarrow k \cdot (u, u) \oplus k \cdot (v, v) \longrightarrow k \cdot (a, a) \longrightarrow 0$$

it is easy to show that it is a one dimensional vector space generated by $(u, u) + (v, v)$. Hence the centre of A and $HH^0(A)$ are isomorphic as vector spaces.

□

Example 6. Let $A = k\Delta/I$ be obtained from the following quiver

$$\Delta := \quad \underset{u}{\bullet} \overset{a}{\underset{b}{\rightrightarrows}} \underset{v}{\bullet}$$

We write $f \in \mathrm{Hom}_k(A, A)$ as

$$\begin{aligned} f(u) &= \lambda_1 u + \lambda_2 v + \lambda_3 a + \lambda_4 b \\ f(v) &= \tau_1 u + \tau_2 v + \tau_3 a + \tau_4 b \\ f(a) &= \varphi_1 u + \varphi_2 v + \varphi_3 a + \varphi_4 b \\ f(b) &= \psi_1 u + \psi_2 v + \psi_3 a + \psi_4 b \end{aligned}$$

for all λ_i, τ_i, φ_i, $\psi_i \in k$. The derivations of A are defined as those $k-$module homomorphisms $f : A \longrightarrow A$ that satisfy

$$f(ab) = af(b) + f(a)\,b\ , \quad \text{for all} \ \ a, b \in A.$$

We shall denote this group of derivations by $\mathrm{Der}(A, A)$ and we begin now by computing all the derivations of the given algebra A:

$$\begin{aligned} \lambda_1 u + \lambda_2 v + \lambda_3 a + \lambda_4 b = f(u) = f(u \cdot u) &= u \cdot f(u) + f(u) \cdot u \\ &= 2\lambda_1 u + \lambda_3 a + \lambda_4 b \end{aligned}$$

This implies $\lambda_1 u - \lambda_2 v = 0$ or $\lambda_1 = \lambda_2 = 0$.

$$0 = f(u \cdot v) = u \cdot f(v) + f(u) \cdot v = \tau_1 u + \tau_3 a + \tau_4 b + \lambda_2 v + \lambda_3 a + \lambda_4 b$$

This means $\tau_3 = -\lambda_3$, $\tau_4 = -\lambda_4$, $\tau_1 = 0$, $\lambda_2 = 0$. Continuing in the same way and computing the remaining 14 derivations:

$$f(u \cdot a),\ f(u \cdot b),\ f(v \cdot u),\ f(v \cdot v),\ f(v \cdot a)\ f(v \cdot b),\ f(a \cdot a),$$

$$f(a \cdot v),\ f(a \cdot b),\ f(a \cdot u),\ f(b \cdot u),\ f(b \cdot a),\ f(b \cdot b),\ f(b \cdot v)$$

we also have $\lambda_1 = \tau_2 = \psi_1 = \psi_2 = \varphi_1 = \varphi_2 = 0$ Substituting $\tau_3 = -\lambda_3$, $\tau_4 = -\lambda_4$, the k-module of all derivations from A to A is spanned by:

$$\begin{aligned} f(u) &= \lambda_3 a + \lambda_4 b, \\ f(v) &= -\lambda_3 a - \lambda_4 b = -f(u), \\ f(a) &= \varphi_3 a + \varphi_4 b \\ f(b) &= \psi_3 a + \psi_4 b \end{aligned}$$

for λ_3, λ_4, φ_3, φ_4, ψ_3, $\psi_4 \in k$. Hence $\mathrm{Der}(A, A)$ is a 6-dimensional vector space, with basis , given by the set of all derivations
$f_i : A \longrightarrow A$, $1 \le i \le 6$;

$$f_1(x) = \begin{cases} a, & \textit{if} \quad x = u, \\ -a, & \textit{if} \quad x = v, \\ 0 & \textit{otherwise.} \end{cases} \qquad f_2(x) = \begin{cases} b, & \textit{if} \quad x = u, \\ -b, & \textit{if} \quad x = v, \\ 0 & \textit{otherwise.} \end{cases}$$

$$f_3(x) = \begin{cases} a, & if \quad x = a, \\ 0 & otherwise. \end{cases} \qquad f_4(x) = \begin{cases} b, & if \quad x = a, \\ 0 & otherwise. \end{cases}$$

$$f_5(x) = \begin{cases} a, & if \quad x = b, \\ 0 & otherwise. \end{cases} \qquad f_6(x) = \begin{cases} b, & if \quad x = b, \\ 0 & otherwise. \end{cases}$$

where x a basis element in A. The inner derivations of A are computed next.

$$\begin{array}{ll} u \longrightarrow u \cdot u - u \cdot u = 0 & u \longrightarrow u \cdot v - v \cdot u = 0 \\ v \longrightarrow v \cdot u - u \cdot v = 0 & v \longrightarrow v \cdot v - v \cdot v = 0 \\ a \longrightarrow a \cdot u - u \cdot a = -a & a \longrightarrow a \cdot v - v \cdot a = a \\ b \longrightarrow b \cdot u - u \cdot b = -b & b \longrightarrow b \cdot v - v \cdot b = b \\ = -f_3 - f_6 & = f_3 + f_6 = -(-f_3 - f_6) \end{array}$$

$$\begin{array}{ll} u \longrightarrow u \cdot a - a \cdot u = a & u \longrightarrow u \cdot b - b \cdot u = b \\ v \longrightarrow v \cdot a - a \cdot v = -a & v \longrightarrow v \cdot b - b \cdot v = -b \\ a \longrightarrow a \cdot a - a \cdot a = 0 & a \longrightarrow a \cdot b - b \cdot a = 0 \\ b \longrightarrow b \cdot a - a \cdot b = 0 & b \longrightarrow b \cdot b - b \cdot b = 0 \\ = f_1 & = f_2 \end{array}$$

The inner derivations form a 3-dimensional subspace of Der(A,A), and so the quotient space of derivations modulo inner derivations is:

$$\frac{k \oplus k \oplus k \oplus k \oplus k \oplus k}{k \oplus k \oplus k} \cong k \oplus k \oplus k$$

On the other hand, suppose we calculate the Hochschild cohomology of the complex associated with the given algebra:

$$0 \longrightarrow k(u,u) \oplus k(v,v) \xrightarrow{\delta^0} k(a,a) \oplus k(b,b) \oplus k(b,a) \oplus k(a,b) \xrightarrow{\delta^1} 0$$

we then have

$$HH^1(A) \cong \frac{k \oplus k \oplus k \oplus k}{k} \cong k \oplus k \oplus k$$

Hence as vector spaces, the space of outer derivations modulo the space of inner derivations and $HH^1(A)$ are isomorphic (as expected from section 4.1).

$\square$

References

[1] Ames, Guillermo, Cagliero Leandro, Tirao Paulo, The cohomology ring of truncated quiver algebras, *arXiv:math.KT/0603056v1*

[2] Cibils, Claude, Hochschild cohomology algebra of radical square zero algebras, *Algebras and modules, II (Geiranger, 1996), CMS Conf. Proc., vol.24, Amer.Math.Soc., Providence, RI, 1998, pp.93-101.*

[3] Eisenbud David, Commutative Algebra with a View Toward Algebraic Geometry, *Springer* (1996).

[4] Gerstenhaber, M., On the deformations of rings and algebras, Ann. of Math. 79 (1964), 59-103.

[5] Happel, Dieter, Hochschild cohomology of finite dimensional algebras, *Book Series lecture Notes in Mathematics publisher Springer Berlin* (1989).

[6] Hochschild Gerhard, On the cohomology groups of an associative algebra, *Ann. of Math. (2) Vol.46, No.1, (1945) 58-67.*

[7] O'Keeffe David, The Hochschild Cohomology Ring of a Quadratic Monomial Algebra, Ph.D Thesis, NUI Galway (2009).

[8] Sköldberg Emil, *The Hochschild homology of truncated and quadratic monomial algebras,* J.London Math.Soc., 59 (1999), 76-86.

David O'Keeffe received his primary degree at National University of Ireland, Maynooth, in 1998, and his Ph. D. at National University of Ireland, Galway in 2009.

School of Mathematics, Statistics, and Applied Mathematics, National University of Ireland Galway

E-mail address: `david.okeeffe@nuigalway.ie`

Irish Math. Soc. Bulletin
Number 68, Winter 2011, 55–63
ISSN 0791-5578

A SOLUTION OF $\Delta u + f(u) = 0$ ON A TRIANGLE

JOSEPH A. CIMA AND WILLIAM DERRICK

Abstract. We use moving planes and thin domain maximum principles to prove the maximum value of a positive solution to the equation $\Delta u + f(u) = 0$ on a symmetric-convex domain Ω , with $u = 0$ on the boundary of Ω, lies on the line of symmetry of the domain. If the domain has two or more lines of symmetry the maximum is at their intersection.

Introduction

In the last several years two novel tools used in the theory of Partial Differential Equations (PDE) have yielded interesting information: the use of "moving planes" and the concept of "maximal principles for thin domains". These ideas have been incorporated in the works of the authors Berestycki, Nirenberg, Varadhan, Ni, Gidas, Fraenkel and Du (see the papers [1]–[3] and [5]). We refer the reader to those papers for more detail on what we will address in this paper, including a very detailed account of some of their work in the book of Fraenkel [4]. Many of these results are based on solutions of an elliptic PDE on a certain special class of bounded domains in $\mathbb{R}^n$. We will work only in $\mathbb{R}^2$ and we will define the special properties of these domains in the following section. We will call these domains "symmetric-convex" and use the notation S-C for any domain that satisfies the appropriate conditions.

One of the key results of the above papers and books is to prove that the directional derivatives of the solutions in the convex directions are negative when one leaves the line of symmetry and approaches the boundary in the convex direction. The purpose of this paper is to study the location of the critical points where the solution takes its maximum. In general, if the domain in question has

2010 *Mathematics Subject Classification.* 35?02, 35B50.

Key words and phrases. Maximum Principle, Moving Planes, Symmetric-Convex Domains.

Received on 18-2-2011; revised 9-12-2011.

two possible directions of symmetry then there is a definitive answer to the question. If not, then one knows only that the maximum is taken on the intersection of the line (plane) of symmetry interior to the domain. We will show in a basic case (an isosceles triangle) that the maximum must be achieved on a fixed subset of this line of symmetry. Further, we will include computational work that indicates the solution must be unique and in a significant way is independent of the given function $f(u)$ for the problem.

1. Definitions and the methods

First we define a symmetric-convex (S-C) domain:

Definition 1.1. Let Ω be a bounded, simply-connected domain in the plane, and assume m and k are unit vectors in the plane which are orthogonal. For each point P on the boundary of Ω let L_P be the subset of Ω consisting of the points that lie on the line $\{P+rm : r \in \mathbb{R}\}$, i.e.

$$L_P = \Omega \cap (P + \mathbb{R}m).$$

Similarly, for Q on $\partial\Omega$, let

$$L'_Q = \Omega \cap (Q + \mathbb{R}k).$$

The domain Ω is said to be S-C (with respect to (m, k)) if each nonempty L'_Q (for $Q \in \partial\Omega$) is a segment and there is a point $P_0 \in \partial\Omega$ for which L_{P_0} is a segment and Ω is symmetric under orthogonal reflection in this segment, i.e. the segment L_{P_0} bisects every nonempty L'_Q.

For example, a disc is S-C for any two orthogonal directions while a Star of David with the standard orientation (i.e., with center of the star at the origin and the y axis joining two opposite points of the star) has three S-C orientations. One is the m vector with $\theta = 90^\circ$ and the other two are for the angles $\theta = \pm 30^\circ$.

Let Ω be a connected bounded S-C domain where, for simplicity, we will assume U is the vector (1,0) and V is (0,1).

We sketch the way the "moving planes" and "thin domain" tools are used in the proofs of the above mentioned authors. For simplicity, consider the strictly elliptic, second order operator on the domain Ω:

$$L = \Delta + c(x, y). \tag{1}$$

The term $c(\cdot)$ is assumed uniformly bounded by a number, say c_0.

Definition 1.2. A *maximum principle* holds for L in Ω if

$$Lw \geq 0 \tag{2}$$

in Ω, with $w \in C^2(\Omega) \cap C(\bar{\Omega})$, and

$$w(x, y) \leq 0$$

on $\partial\Omega$, implies that $w(x, y) \leq 0$, for all $(x, y) \in \Omega$.

Varadhan observes the following:

Proposition 1.3. (Thin Domain Principle.) *Assume* $\operatorname{diam}(\Omega) \leq d$. *There exists* $\delta > 0$ *depending only on* d, c_0, *such that the maximum principle holds for* L *in* Ω *provided meas*$(\Omega) = |\Omega| < \delta$.

Let $u : \bar{\Omega} \to \mathbb{R}$ be a positive solution of

$$\Delta u + f(u) = 0, \quad u = 0 \text{ on } \partial\Omega, \tag{3}$$

with $u \in C(\bar{\Omega}) \cap C^2(\Omega)$, where $f : [0, \infty] \to R$ is C^1 (thus also Lipschitz) and monotone increasing. We will give a brief overview of how the maximum principles above and the moving plane methods fit together. Assume $0 \in \Omega$ without loss of generality. Then a line $L_a = [(x, y) : x = a]$ meets the domain Ω if a is small in modulus. In particular, L_0 is the y axis. Since the domain Ω is assumed to be bounded there is a number a such that the line L_a meets Ω and cuts a small "cap" (an open subset of Ω), say $\Sigma(a) = \{(x, y) \in \Omega | x < a\}$ from Ω.

We examine only one component of this open subset of Ω and assume $a < 0$. Reflecting the domain $\Sigma(a)$ about L_a, we see that the reflected domain $\Sigma(a)^{\perp} \subset \Omega$, each point $P \in \Sigma(a)$ having a reflected point $P^{\perp} \in \Sigma(a)^{\perp}$. Define

$$w(P; a) \equiv u(P) - u(P^{\perp}) \tag{4}$$

for $P \in \overline{\Sigma(a)}$. Since f is Lipschitz w satisfies $\Delta w + \gamma(P; a)w = 0$, on $\Sigma(a)$, with

$$\gamma(P, a) = \begin{cases} \frac{f(u(P)) - f(u(P^{\perp}))}{u(P) - u(P^{\perp})}, & P \in \Sigma(a), \\ 0, & P \in L_a. \end{cases} \tag{5}$$

By the Maximum principle (or thin domain principle), it follows that $w \leq 0$ in $\overline{\Sigma(a)}$. Next, using a strong maximum principle (see [4, Theorem 2.13]), we extend this to $w < 0$ on $\Sigma(a)$.

This leads to the key inequality $u(P) < u(P^{\perp})$ for all P in $\Sigma(a)$. Finally, this shows that $u_x(P) > 0$ in $\Sigma(a)$. Consider the largest (non-positive) value of this parameter a for which this is true. If this value is 0, then one can use symmetry to show, by replacing the argument in the "opposite" direction, that in fact the solution $u(p)$ is strictly decreasing as the point p begins on the y axis and tends to the boundary in either horizontal direction.

But if it is assumed the values of the parameter a stop at, say, $a_0 < 0$, one must work to get a contradiction. The idea is that, although the cap $\Sigma(a_0)$ itself may not have measure smaller than the required value in Proposition 1.3, it is possible by a clever argument to excise a compact subdomain K of $\Sigma(a_0)$ and apply the "thin maximal principle" to the domain $\Sigma(a_0) \setminus K$. This will yield a contradiction to the assumption about a_0 and so we obtain the result concerning strictly decreasing in the symmetric direction.

We use this method to get the following result.

Proposition 1.4. *Assume Ω is an S-C domain in $\mathbb{R}^2$ and that it has at least two S-C orientations, say in the m and M directions, with orthogonal convex directions k and K respectively (i.e., the lines of symmetry have directions m and M). Then the solution of equation* (3) *has a unique point in the domain where it achieves its maximum.*

Proof. Let the notation $D_V(u)(P)$ denote the directional derivative of the function u in the V direction and evaluated at a point $P \in \Omega$. The results above indicate that if P is not on the line generated by m then it is strictly decreasing in the convex direction k. That is,

$$D_k(u)(P) = \nabla u(P) \cdot k < 0,$$

so the gradient is nonzero. Similarly for P on M. Hence only points on m and M can be maximums, and their intersection P satisfies $\nabla u(P) = 0$. Note that the maximum point P is not changed if we change the function f; it is a property of the domain and not of the forcing function. □

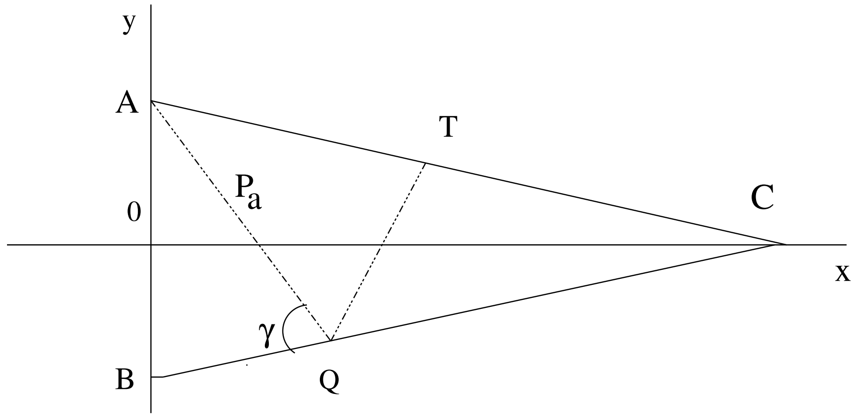
2. Domain with one S-C orientation

One can easily construct domains in $\mathbb{R}^2$ which have only one S-C orientation. Our example is an isosceles triangle Λ (which is not equilateral – in this case one has three S-C orientations) with base on the y axis and points $A = (0, a)$, $B = (0, -a)$ and $C = (c, 0)$, $c > 0$. This is S-C with the x axis as the line of symmetry and the y axis as its associated convex direction. From the above theory we know that the solution to (3) on Λ must have any maximum points on the real axis segment $0 < x < c$. We also know, by symmetry and the moving plane method that

$$u_y > 0 \quad \text{for} \quad y < 0$$
$$u_y < 0 \quad \text{for} \quad y > 0$$

and that $u_x > 0$ near the y axis and $u_x < 0$ near c.

Now writing the angle at the vertex A as 2α (and also at B), we construct a four-sided figure (the sides are all line segments) as follows. Draw the line from A by bisecting the angle at A. The line meets the side BC at a point, say Q. The segment AQ together with the segments BQ and AB form a triangle Γ. Reflect Γ along the line AQ, so that the line QB reflects along QT The four-sided figure has vertices at A, B, Q, T, and lies in Λ. We replace the original domain Λ by this domain $\Omega = [A, B, Q, T]$, a subdomain.



In our case we are considering the domain Ω and lines orthogonal to the segment AQ which is the line of symmetry. The convex direction is the direction orthogonal to the segment AQ. Using the technique above we have the L_0 line given by AQ and beginning with the point B we select lines L_a parallel to L_0. For each admissible choice of the parameter a, we construct a cap $\Sigma(a)$ and define for each P in the cap, with one vertex at B, determined by the line L_a, the function

$$w\big((x, y) \,:\, a\big) = u(P) - u(P^{\perp}), \tag{6}$$

where $P^{\perp}$ is the reflection of P in L_a. Of course, the triangle determined by this value of a has its reflection in the line L_a inside $\Omega \subseteq \Lambda$. Thus two of the sides of $\Sigma(a)$ are parts of the boundary of

Λ and the remaining two sides lie in Ω. Thus $w(\cdot\,:\,a)$ is well defined and satisfies $\Delta w + \gamma w = 0$ with γ defined by equation (5). In checking the required inequality on the boundary of $\overline{\Sigma(a)}$, we have $w(\cdot\,:\,a) = 0$ on the part of the line L_a that lies within $\overline{\Sigma(a)}$, and $w(\cdot\,:\,a) < 0$ on the other two sides. Hence, by the process outlined above, we see that for each admissible a up to and including $a = 0$ we have $w(P\,:\,a) < 0$ in each such domain. Hence, $u(P) < u(P^{\perp})$ for all P in $\Sigma(a)$.

We recall that the point where the line segment AQ crosses the x axis is the center of the largest inscribed circle that can be placed inside of the triangle ABC. Label this point $P_0 = (x_0, 0)$. We also have the following.

Proposition 2.1. *The function u is strictly increasing on the segment* $[(x,0)\,:\,0 < x < x_0]$.

Proof. Let points $P_1 = (x_1, 0)$ and $P_2 = (x_2, 0)$ be given with $0 < x_1 < x_2 < x_0$. Starting with the line L_a through P_1 and moving the line continuously toward the parallel segment AQ, we note the following. The reflection is a continuous process in the parameter a. Also, if we write $P_j(P)$ as the projection of the point P onto the real axis, it is also a continuous process. Let $P_2 = P_j(P_1^{\perp})$ where $P_1^{\perp}$ is the unique point which is the reflection of the given point P_1 and a new line $L_{a'}$. There is a unique choice of a' that will do this. We know from the information about u in the original triangle Λ that

$$u(P_1^{\perp}) < u\big(P_j(P_1^{\perp})\big) = u(P_2).$$

Thus

$$u(P_1) < u(P_1^{\perp}) < u\big(P_j(P_1^{\perp})\big) = u(P_2). \tag{7}$$

This completes the proof. $\square$

Remark. If we examine the C^2 expansion of u at P_0, we find, for $x' = x_0 + \varepsilon$ and $x'' = x_0 - \varepsilon$,

$$u(x', 0) = u(x_0, 0) + u_x(x_0, 0)(\varepsilon) + \mathcal{O}(\varepsilon^2).$$

Similarly,

$$u(x'', 0) = u(x_0, 0) + u_x(x_0, 0)(-\varepsilon) + \mathcal{O}(\varepsilon^2).$$

Subtracting these two equations and using the ideas of reflections above, we have

$$0 < u(x', 0) - u(x'', 0) = u_x(x_0, 0)(2\varepsilon) + \mathcal{O}(\varepsilon^2).$$

Thus the "sign" of the right side of this equation is determined by the first term. Since $\varepsilon > 0$ we have that $u_x(P_0) > 0$.

Note that we could have begun this process from the vertex B with appropriate similar conclusion in a "different part" of triangle ABC.

We note at this time that, in addition to the above two constructions, there is also another simple subdomain of triangle ABC to which we can apply this process. To wit, let the point $P_m = (\frac{c}{2}, 0)$ be the mid point of the line joining the origin to the point $(c, 0)$. Now we can consider the subdomain of the original triangle determined by the part of the sides AC and BC from the points $\left(\frac{c}{2}, \pm y_m\right)$ where the line $x = \frac{c}{2}$ meets these sides. The symmetry line is the line through $\frac{c}{2}$ joining the points determined by $\left(\frac{c}{2}, +y_m\right)$ and $\left(\frac{c}{2}, -y_m\right)$. Let the triangle determined by these points and the vertex $(c, 0)$ be labeled Δ. Clearly, reflection of Δ in the line $L_{\frac{c}{2}}$ remains inside ABC and, applying the technique above, we have for $P^{\perp}$ the reflection of P in the line $x = \frac{c}{2}$, for all $P \in \Delta$,

$$u(P) < u(P^{\perp}).$$

The following is true.

Proposition 2.2. *The function $u(x, 0)$ is strictly decreasing as x increases from $\frac{c}{2}$ to c.*

Proof. Let $(x', 0)$ and $(x'', 0)$ be given with $\frac{c}{2} < x' < x'' < c$. The proof is similar to that given in Proposition 2.1 and we omit the details. □

It is also true (by a discussion similar to that for the expansion of u near P_0) that $u_x\left(\frac{c}{2}\right) < 0$.

At this stage we have shown that any maximum for u can only occur between the points P_0 and P_m on the real axis.

3. Numerical information on the location of the maxima

Although we have given some estimates, based on the geometry of our given triangle ABC, as to where a maximum may occur, we have not been successful in proving that such an (absolute) maximum is unique or its exact location. The numerical data following implies that indeed the maximum point is unique and, moreover, its position depends only on the geometry of the triangle and not on the function $f(z)$. From our point of view this is surprising.

We constructed a pseudo-triangle ABC consisting of a lattice of square cells from -15 to 15 on the y-axis, with variable geometry allowing up to 90 cells on the x-axis. The location of vertex C is given by $(c, 0)$. When that information is given, the program initiates the computation by giving an initial value of $u = 0.1$ to all points (i, j) such that $|j| \leq \text{int}\,[15\big(1 - \big(\frac{i}{c}\big)\big)]$, $0 \leq i \leq c$ and giving a value $u = 0$ to all other points in the grid, so that the boundary grid points are zero. Observe that this pseudo-triangle is an S-C domain.

The Laplacian of u is approximated by the average of the four adjacent grid points and added to $f(u)$, giving

$$u_{\text{new}}(i,j) = \frac{1}{4}\left\{u(i{+}1,j){+}u(i{-}1,j){+}u(i,j{+}1){+}u(i,j{-}1)\right\}{+}f(u(i,j)),$$

where (i, j) is the grid point where u is evaluated. The results are stored in a new matrix of grid points u_{new} for the next iteration. All boundary grid points are kept at zero. Finally, the sum of the squares of the differences between the new and old values is obtained, and when its square root is less than a given tolerance, the iterations are stopped and the maximum grid point values are located.

We used three different $f(u)$: $f_1(u) = .03u^2$, $f_2(u) = .01(1{+}u^2)/4$, and $f_3(u) = .005u(1 + u)$. We ran the program for different values for c between $20 \leq c \leq 90$. In all cases examined, the maximum values (though different) occurred in the same place, $u(x, 0)$, where the x-value is shown in the table:

c	20	26	30	40	50	60	70	80	90
x	7	8	9	11	12	14	15	16	17

Different tolerances were used: 10^{-4} for f_1 and f_2, and 10^{-9} for f_3, but the location of the maximum did not change for all three functions. The choice of $c = 26$ is an attempt to make the pseudo-triangle ABC as close as possible to an equilateral triangle, where the center of the inscribed circle is located at $x = 5\sqrt{3} \approx 8.66$, so an x value of 8 in the table is a reasonable approximation to the maximum point's location.

References

[1] Berestyski, H. and Nirenberg, L., *On the method of moving planes and the sliding method*, Bal. Soc. Brasil. Mat. (N.S.) 22 (1991) 1–37.

[2] Berestyski, H., Nirenberg, L. and Varadhan, S.R.S., *The principal eigenvalue and maximum principle for second-order elliptic operators in general domains*, Comm. Pure and Applied Math., 47 (1994) 47–92.

[3] Du, Y., *Order structure and topological methods in nonlinear partial differential equations*, Vol. 1. World Scientific 2006.

[4] Fraenkel, L.E., *Introduction to maximum principles and symmetry in elliptic problems.* Cambridge University Press 2000.

[5] Gidas, B., Ni, Wei-Ming and Nirenberg, L., *Symmetry and related properties via the maximum principle*, Commun. Math. Phys. 66 (1979) 209–243.

Joseph Cima Dr. Cima received his M.A. and Ph.D degrees at Pennsylvania State University. He is a professor of mathematics at the University of North Carolina at Chapel Hill. He has served as Secretary of the American Mathematics Society. His interests are in Several Complex Variable Theory, Operator Theory and Harmonic Analysis.

William Derrick Dr. Derrick received his Ph.D. degree at Indiana University. His research interests are in Complex Analysis, Partial Differential Equations, Applied Mathematics and Modeling. He is currently Professor Emeritus at the University of Montana in Missoula, Montana.

(J. Cima) Dept. of Mathematics, University of North Carolina, Chapel Hill, NC 27599-3250, USA

(W. Derrick) Department of Mathematics, University of Montana, Missoula, Montana 59812, USA

E-mail address, J. Cima: `cima@email.unc.edu`

Irish Math. Soc. Bulletin
Number 68, Winter 2011, 65–66
ISSN 0791-5578

PROBLEMS

IAN SHORT

The first problem was contributed by Finbarr Holland.

Problem 68.1. A polynomial is said to be *stable* if all its roots have negative real part. Suppose that

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \qquad a_n \neq 0,$$

is stable. Prove that

$$q(z) = a_{n-1} z^{n-1} + 2a_{n-2} z^{n-2} + \cdots + (n-1)a_1 z + n a_0$$

is also stable.

The second problem is a classic

Problem 68.2. Let A denote the set of positive integers that do not contain a 9 in their decimal expansion. Determine whether the sum

$$\sum_{n \in A} \frac{1}{n}$$

converges or diverges.

I discovered the final problem while trying to generalise the formula

$$|AB||A \cap B| = |A||B|,$$

which holds when A and B are subgroups of a finite group.

Problem 68.3. Given subsets U and V of a finite group G, define

$$UV = \{uv \,:\, u \in U, v \in V\}$$

and

$$U^{-1} = \{u^{-1} \,:\, u \in U\}.$$

Prove that

$$|AB||A^{-1}A \cap BB^{-1}| \geqslant |A||B|,$$

for any pair of subsets A and B of G.

Received 9-2-2012.

We invite readers to suggest their own problems, and to offer comments and solutions to past problems. In later issues we will publish solutions and acknowledge problem solvers. Please email submissions to `imsproblems@gmail.com`.

Department of Mathematics and Statistics, The Open University, Milton Keynes MK7 6AA, United Kingdom

Letter to the Editor

School of Applied Science and Computing,
Department of Applied Science,
ITT Dublin,
Tallaght,
Dublin.

Dear Sir,

I am writing to enquire if members would have views on the following question.

Would it be feasible or worthwhile for the IOTs and Universities to get together to consider offering appropriate courses at Masters level and beyond with the aim of advancing mathematical knowledge and providing also a means by which mathematicians and others can develop their interest in different areas perhaps outside of their own specialized areas? Attending a course can strengthen ones knowledge more that say attending a single seminar. Also due to time commitments it may not be possible to attend regular seminars outside of ones own college or institute. The possibility of attending courses would offer the general mathematical community and the mathematical community in the IOTs particularly a very valuable way of engaging in continuous professional development. Courses could be held at weekends or outside of current teaching time, for example between May and September.

The pace of change in mathematical research is increasing all the time and it seems to me that provision of courses at the appropriate level could help enormously to help keep up with the pace of this change. It seems to me eminently sensible for colleges and institutes to pool resources and get together to provide such educational opportunities for mathematicians.

Experience in hosting short one day courses in various areas at IT Tallaght has been very positive but it is felt that longer periods providing an opportunity for more in depth knowledge research and analysis could be even more beneficial.

I would be interested in reading members views on such a proposal.

Sincerely,
Dr. Cora Stack.

Instructions to Authors

The *Bulletin* is typeset with LaTeX; therefore, authors should submit articles to the *Bulletin* as LaTeX input files together with any macro files or input files for the production of graphics. Typescripts are **not** acceptable. Source files should be suitable for processing using `pdflatex`. All files should be accompanied by any non-standard style or input files which have been used. Private `METAFONT` and TEX macros, reference input files and files for diagrams must also accompany submissions. The preferred method of submission is by e-mail to

`mailto://ims.bulletin@gmail.com`

Only if this is not possible, authors may send a CD with their files to the Editor together with a printout of the typescript. If articles are submitted in electronic form, no hard copies need to be sent unless requested by the Editor. Once a paper is accepted in final form, the author(s) will be responsible to produce an output according to the *Bulletin's* standard layout. To this end, authors will be supplied with style files. Since normally no proofs are sent out before publication, it is the author's responsibility to carefully check for any misprints or typos. After publication, authors will be sent a free copy of the issue their article(s) appeared in as well as a pdf file containing their article(s). The Society grants authors free use of *this* pdf file; hence they may post it on personal web sites or electronic archives.