Irish Math. Soc. Bulletin **64** *(2009), 79–83*

Fuchs' Problem When Torsion-Free Abelian Rank-One Groups are Slender

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Abstract. We combine Baer's classification in [Duke Math. J. **3** (1937), 68–122] of torsion-free abelian groups of rank one together with elementary properties of p-adic numbers to give a new solution to research problem 26 posed by Fuchs in his book on abelian groups in 1958.

Every group considered in this paper is abelian with its group law written additively. The Baer–Specker group $\Pi = \mathbb{Z}^{\aleph_0}$ is the direct product of countably many copies of $\mathbb{Z}$. Its extensive study in the literature has given rise to a wealth of interesting problems and a number of unexpected connections with diverse mathematical areas such as homology theories in algebraic topology, infinitary logic and various aspects of set theory. We refer the reader to [2], [3], [4], [6] and [11] for good accounts of some of these connections.

For a positive integer n, let e_n denote the element of Π whose n-th coordinate equals 1 and all its other coordinates equal 0. Following Łoś, a torsion-free group G is called slender if for every homomorphism $\phi : \Pi \to G$ we have $\phi(e_n) = 0$, for all but finitely many n. Specker ([10]) showed that $\mathbb{Z}$ is slender. Research problem 26 in Fuchs' book (page 184 in [5]) reads as follows:

Problem. *Find all slender groups of rank one.*

A partial answer to the problem was given by Łoś (see Theorem 47.3 in [5]). The problem was fully solved by Sąsiada ([9]) who in fact showed that:

Theorem 1. *A torsion-free group of cardinality less than* $2^{\aleph_0}$ *is slender if and only if it has no non-trivial divisible subgroups.*

2000 *Mathematics Subject Classification.* Primary 20K15; Secondary 20K25.
Key words and phrases. Rank-one groups, slender groups, Baer–Specker group.

Sąsiada's result is superseded by Nunke's characterization of all torsion-free groups which are slender (see [7] and [8]).

Theorem 2. *A torsion-free group is slender if and only if none of its subgroups is isomorphic to* Π, $\mathbb{Q}$ *or the p-adic integers* $\mathbb{Z}_p$ *for some prime p.*

The purpose of this short note is to give a new solution to the original problem of Fuchs. We claim no novelty for the general p-adic flavor of our arguments which is not uncommon in inquiries of this type but, as far as we can tell, a solution like ours has not appeared in the literature before. Let us first briefly recall Baer's classification ([1]) of torsion-free groups of rank one (see [5] for details).

Consider a torsion-free group G and a non-zero element $g \in G$. For a prime number p, define the height $H_p(g)$ of g at p to be the maximum integer $k \in \mathbb{N}$ such that the equation $p^k x = g$ is solvable in G (if no maximum k exists, we set $H_p(g) = \infty$). The height $H(g)$ of g is the infinite tuple

$$H(g) = (H_{p_1}(g), H_{p_2}(g), \ldots, H_{p_n}(g), \ldots),$$

where p_1, p_2, …, p_n, … is the increasing sequence of prime numbers.

Consider the set H of infinite tuples $(k_1, k_2, \ldots, k_n, \ldots)$, where $k_n \in \mathbb{N} \cup \{\infty\}$ for all n. Define a partial order $\leq$ on H as follows:

$$(k_1, k_2, \ldots, k_n, \ldots) \leq (m_1, m_2, \ldots, m_n, \ldots)$$

if $k_n \leq m_n$ for all n (where it is understood that $s \leq \infty$ for all $s \in \mathbb{N} \cup \{\infty\}$).

We also define an equivalence relation $\sim$ on H as follows:

$$(k_1, k_2, \ldots, k_n, \ldots) \sim (m_1, m_2, \ldots, m_n, \ldots)$$

provided that $k_n = m_n$ for all but finitely many n and that $k_n \neq m_n$ can happen only if neither k_n nor m_n equals ∞. An equivalence class of H under $\sim$ is called a type and the set of all types is denoted by T. It is easy to see that the partial order $\leq$ on H defined above induces a partial order (also denoted by $\leq$) on the set of types T.

Now suppose that G is of rank one and let g, g' be any two non-zero elements of G. It is not difficult to show that $H(g) \sim H(g')$. Therefore, all non-zero elements of G are of the same type, which we call the type of G and denote by $T(G)$. Every torsion-free group of rank one is isomorphic to a subgroup of $\mathbb{Q}$ and Baer showed in [1] that the set of isomorphism classes of torsion-free groups of rank one is parametrized by T via the bijective correspondence given by

$G \mapsto T(G)$. In addition, a torsion-free group G_1 of rank one is isomorphic to a subgroup of a torsion-free group G_2 of rank one if and only if $T(G_1) \leq T(G_2)$.

We now give a new solution to the original problem of Fuchs:

Theorem 3. *A torsion-free group of rank one is slender if and only if it is not isomorphic to* $\mathbb{Q}$.

Proof. One direction is standard: Let Σ denote the direct sum of countably many copies of $\mathbb{Z}$. Since $\mathbb{Q}$ is an injective $\mathbb{Z}$-module, the homomorphism $\Sigma \to \mathbb{Q}$ defined by

$$(x_1, x_2, \ldots, x_n, \ldots) \mapsto \sum_{n=1}^{\infty} x_n$$

can be lifted to a homomorphism $\phi : \Pi \to \mathbb{Q}$ such that $\phi(e_n) \neq 0$, for all n. Therefore, $\mathbb{Q}$ is not slender.

To prove the converse, let G be a torsion-free group of rank one which is not isomorphic to $\mathbb{Q}$. From the discussion of types given before, it follows that $T(G)$ can be represented by an infinite tuple $(k_1, k_2, \ldots, k_n, \ldots)$, where $k_n \neq \infty$ for at least one $n \in \mathbb{N}$. It is easy to see that this tuple is equivalent to $(k_1, k_2, \ldots, k_{n-1}, 0, k_{n+1}, \ldots)$ and that

$$(k_1, k_2, \ldots, k_{n-1}, 0, k_{n+1}, \ldots) \leq (\infty, \infty, \ldots, \infty, 0, \infty, \ldots).$$

As the latter tuple represents the type of the group $\mathbb{Z}_{(p)}$ of rational numbers with denominators not divisible by the prime $p = p_n$, we may assume that G is a subgroup of $\mathbb{Z}_{(p)}$. Since subgroups of slender groups are obviously slender, it suffices to show that $\mathbb{Z}_{(p)}$ is slender.

Let $i : \mathbb{Z}_{(p)} \to \mathbb{Z}_p$ denote the natural embedding of $\mathbb{Z}_{(p)}$ in the ring of p-adic integers. We will use the well-known fact that for $y \in \mathbb{Z}_{(p)}$ the p-adic expansion of $i(y)$ is ultimately periodic, i.e. when we write

$$i(y) = \sum_{j=0}^{\infty} a_j p^j$$

then there exist positive integers N and r such that $a_{j+r} = a_j$ for all $j \geq N$. Now let $\phi : \Pi \to \mathbb{Z}_{(p)}$ be a homomorphism such that $\phi(e_n) \neq 0$ for infinitely many n. Without loss of generality, we may assume that $\phi(e_n) \neq 0$ for all n. We can therefore find, for each n, an integer b_n such that $\phi(b_n e_n)$ is a positive integer. Let $y_n = \phi(b_n e_n)$. Then $i(y_n)$ has a finite non-zero p-adic expansion for all n. Now consider $x = (p^{c_1}b_1, p^{c_2}b_2, \ldots) \in \Pi$, where the sequence of natural numbers c_1, c_2, … has been chosen to satisfy

$$p^{c_{n+1}} > p^n \ (p^{c_n}y_n + p^{c_{n-1}}y_{n-1} + \ldots + p^{c_1}y_1)$$

for all n. Let $y = \phi(x)$. Note that for all n we get

$$y = p^{c_1}y_1 + p^{c_2}y_2 + \ldots + p^{c_n}y_n + \phi(0, 0, \ldots, 0, p^{c_{n+1}}b_{n+1}, p^{c_{n+2}}b_{n+2}, \ldots).$$

We clearly have $c_{n+1} \leq c_{n+2} \leq \ldots$, so

$$(0, 0, \ldots, p^{c_{n+1}}b_{n+1}, p^{c_{n+2}}b_{n+2}, \ldots)$$

is divisible by $p^{c_{n+1}}$ in Π. Hence,

$$y \equiv p^{c_1}y_1 + p^{c_2}y_2 + \ldots + p^{c_n}y_n \pmod{p^{c_{n+1}}}.$$

for all n. By our assumption on c_1, c_2, …, we see that the p-adic expansion of $i(y)$ is obtained by writing down the non-zero p-adic expansion of $p^{c_1}i(y_1)$, followed by at least 1 zero digit, followed by the non-zero p-adic expansion of $p^{c_2}i(y_2)$, followed by at least 2 zero digits, etc. Hence, the p-adic expansion of $i(y)$ contains arbitrarily large blocks of zero digits followed by non-zero blocks of digits and this contradicts the fact that the p-adic expansion of $i(y)$ is ultimately periodic. □

Acknowledgement. I thank Yasuyuki Kachi for asking me a question which, unbeknownst to me at the time, was equivalent to the slenderness of $\mathbb{Z}$ and provided the motivation for this work.

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Received on 15 December 2008.