MacCool’s Proof of Morley’s Miracle

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One of the most beautiful results in plane geometry is known as Morley’s Miracle (1899). In essence it states that the triangle \( XYZ \) in the figure below is always equilateral. It features prominently on the front cover of the popular work [2] but is “still not as well-known as it deserves to be” [3]. The excellent web article [1] continues to track its development and also hosts a wide variety of proofs. None of the early proofs was easy but since 1990 elementary ones have emerged which are backward in the sense that they start from the equilateral triangle and eventually reconstruct the original. Finding a direct proof that matches them in brevity and simplicity has always been an elusive goal [3].

So I was amazed to find just such a proof in MacCool’s notebooks and indeed it was so short that I nearly missed it. At first glance
he seemed to be merely doodling, but moments later he had finished the proof and was working on something completely different.

Those readers who haven’t heard of MacCool’s notebooks may be surprised to learn that I am still less than halfway through the first one. Translation from the Ogham script is proving a long slow process and I am deeply indebted to one correspondent who reviewed and improved upon my original efforts, often spotting intricacies that I had overlooked. Although the gist of his arguments is always clear MacCool delights in recording only a minimum of information, and this particular proof was little more than a sketch decorated with jottings of line segments and angles. Like all the rest so far it is based solely on straight line geometry and similar triangles, but anyone interested in more advanced concepts may be pleased to know that diagrams containing circles begin to appear early in book two.

In his doodle the unit of measure is the perpendicular $DX$, and the lengths of $BX$ and $CX$ are $s$ and $s'$. $E$ and $F$ are points on $BC$ where $\angle BXE = \angle FXC = 60^\circ$. $P$ and $P'$ are where $BP = s$ and $CP' = s'$ and $S$ is constructed so that $BS = s$ and $\angle SBX = 120^\circ$. This makes the four marked angles $60^\circ$ (even if $\triangle ABC$ is obtuse). The rest of his construction is self-explanatory.

Now by (vi) and (vii) $2ST = 2SU + 2UT = s + 2(s - 2s^{-1}) = 3s - 4s^{-1}$ and $\triangle BQV \sim \triangle BDX$ yields $VQ = 1 - 4s^{-2}$ so $PQ = PV + VQ = 3 - 4s^{-2}$ thus

$$2ST = sPQ.$$  

Then by (iv) and (v)

$$AY = \left(\frac{AC}{s'}\right)\left(\frac{PS}{ST}\right) = \frac{2AC.PS}{ss'PQ}.$$  

If $W$ is the midpoint of $PS$ then since $\triangle BSP$ is isosceles $\triangle BWP$ and $\triangle EDX$ will have identical angles, hence $\triangle BWP \sim \triangle EDX$ giving $XE = 2s/PS$. Therefore

$$XE.AY = \frac{4AC}{s'PQ}$$  

and, by symmetry,

$$\frac{XE.AY}{XF.AZ} = \left(\frac{4AC}{s'PQ}\right)\left(\frac{sP'Q'}{4AB}\right) = \frac{saAC.P'Q'}{s'AB.PQ} = 1$$  

because $PQ(AB/s) = P'Q'(AC/s')$ is the height of $\triangle ABC$. However this means $AZ:AY = XE:XF$ and as $\angle ZAY = \alpha = \angle EXF$
Notes
(i) \( \alpha + \beta + \gamma = 60^\circ \) (so \( \angle EXF = \alpha \))
(ii) \( \angle CYA = \angle CLP' = 120^\circ + \beta \)
(iii) \( \angle TSB = \angle SPX = \angle BXE = \angle FXC = 60^\circ \)
(iv) \( \angle AY = (AC/s')LP' \) (as \( \triangle CLP' \sim \triangle CAY \))
(v) \( LP' = PS/ST \) (as \( \angle PLY = \angle SPT = 60^\circ - \beta \))
(vi) \( XR = RV = 2/s \) (as \( \triangle BDX \sim \triangle PRX \sim \triangle PVR \))
(vii) \( SU = s/2 \) (as \( \triangle BSU \) is half of an equilateral triangle)

then \( \triangle AZY \sim \triangle XEF \). Hence \( \angle YZA = \angle FEX = 60^\circ + \beta \) and
\( \angle AYZ = \angle XFE = 60^\circ + \gamma \). Analogous arguments for \( \triangle BXZ \) and
\( \triangle CYX \) show \( \angle ZXB = 60^\circ + \gamma \), \( \angle CXY = 60^\circ + \beta \) and \( \angle BZX = \angle CYX = 60^\circ + \alpha \). All the angles in the doodle may now be deduced
in terms of \( \alpha, \beta, \gamma \) and it transpires that every angle of \( \triangle XYZ \) is
\( 60^\circ \).

Here are some comments on the proof leading to a slight variation
that may help to make it more intuitive. The underlying idea is to
treat it as a series of left/right linkages. The results that \( 2ST = sPQ \)
and \( XE = 2s/PS \) are clearly “internal” to the left hand side. On
the other hand \( LP' \) has a foot in each camp since it can be expressed
both in terms of objects from the left \( PS/ST \) and objects from the
right \( s'/AY/AC \). Equating these expressions gives a “cross-linkage”
\( XE.PQ = 4AC/(s'AY) \) and its companion \( 4AB/(sAZ) = XF.P'Q' \)
which may then be combined to form the complicated looking quo-
tient above. Even MacCool seems to have been shocked by the final
devastating cross-linkage \(PQ.AB/s = P'Q'.AC/s'\) which reduces this quotient to unity. After that the rest is plain sailing.

References

Morley’s Miracle from Interactive Mathematics Miscellany and Puzzles


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