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Theta Functions

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Abstract. On our analytic way to the group structure of an elliptic function we meet so called theta functions. These complex functions are entire and quasi-periodic with respect to a lattice Λ. In the proof of Abel's theorem we use their properties to characterise all meromorphic functions f from $\mathbb{C}/\Lambda$ to $\mathbb{C}$. Finally we have a closer look at a very special and interesting Λ-periodic meromorphic function, the Weierstraß $\wp$-function. This function delivers an analytic way to give a group structure to an algebraic variety.

1. Introduction

First of all, we want to analyse periodic complex functions $f : \mathbb{C} \to \mathbb{C}$ with respect to a lattice Λ. So let us fix once and for all a complex number $\tau \in \mathbb{C}$, Im $\tau > 0$ and consider the lattice $\Lambda := \mathbb{Z} \oplus \tau\mathbb{Z} \subset \mathbb{C}$.

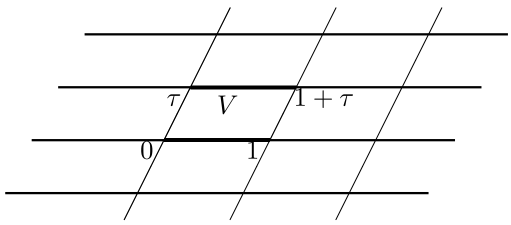

Figure 1. The lattice $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$ and its fundamental parallelogram $V = \{z = t_1 + t_2\tau \in \mathbb{C} : 0 \leq t_1, t_2 < 1\}$.

Lemma 1. *An entire doubly-periodic complex function is constant.*

To prove this lemma we need Liouville's Theorem, which we know from complex analysis. It states that each entire and bounded complex function $f : \mathbb{C} \to \mathbb{C}$ is constant.

Proof. The values of a doubly-periodic function are completely determined by the values on the closure of the fundamental parallelogram $\overline{V} = \{z \in \mathbb{C} : z = t_1 + t_2\tau \text{ for some } 0 \le t_1, t_2 \le 1\}$ which is a compact set. But a continuous function on a compact set is bounded. Hence our function is entire and bounded. Therefore it is constant by Liouville's Theorem. □

As we have seen, *entire doubly-periodic* functions are not very interesting, so in the following we will consider *entire quasi-periodic* functions and use them to prove Abel's Theorem which says what *meromorphic doubly-periodic* functions look like.

2. Theta Functions and Abel's Theorem

Definition. The *basic theta function* is defined to be the function $\theta : \mathbb{C} \to \mathbb{C}$ given by

$$\theta(z) := \theta(\tau)(z) := \sum_{n\in\mathbb{Z}} \exp(\pi i n^2 \tau) \exp(2\pi i n z)$$

Note. The function θ depends on τ. So for each $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$ we get a (not necessarily different) basic theta function. Hence there is a whole family of basic theta functions $\{\theta(\tau)\}_{\tau\in\mathbb{C},\operatorname{Im}\,\tau>0}$. But here we assume τ to be fixed, so we have only one basic theta function.

Remark. As the series in the definition above is locally uniformly unordered convergent (without proof) our basic theta function is an entire function.

Lemma 2. *The basic theta function is quasi-periodic.*

Proof. Consider $\theta(z+\lambda)$ for $\lambda \in \Lambda$, i.e. $\lambda = p\tau + q$ for p, $q \in \mathbb{Z}$.

For $\lambda = 1$, i.e. for $p = 0$ and $q = 1$ we have

$$\begin{aligned}
\theta(z+1) &\overset{def}{=} \sum_{n\in\mathbb{Z}} \exp(\pi i n^2 \tau) \exp(2\pi i n(z+1)) \\
&= \sum_{n\in\mathbb{Z}} \exp(\pi i n^2 \tau + 2\pi i n z + 2\pi i n) \\
&= \sum_{n\in\mathbb{Z}} \exp(\pi i n^2 \tau) \exp(2\pi i n z) \quad \underbrace{\exp(2\pi i n)}_{=1 \text{ for all } n\in\mathbb{Z}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n\in\mathbb{Z}} \exp(\pi i n^2\tau)\exp(2\pi i n z)\\
&\stackrel{def}{=} \theta(z)
\end{aligned}
$$

Hence the basic theta function is periodic with respect to the x-direction.

For $\lambda = \tau$, i.e., for $p = 1$ and $q = 0$ we have

$$
\begin{aligned}
\theta(z+\tau) &\stackrel{def}{=} \sum_{n\in\mathbb{Z}} \exp(\pi i n^2\tau)\exp(2\pi i n(z+\tau))\\
&= \sum_{n\in\mathbb{Z}} \exp(\pi i n^2\tau + 2\pi i n z + 2\pi i n \tau)
\end{aligned}
$$

if we complete the square and rearrange the summands then

$$
\begin{aligned}
&= \sum_{n\in\mathbb{Z}} \exp\big(\pi i n^2\tau + 2\pi i n\tau + \pi i\tau - \pi i\tau\\
&\qquad +2\pi i n z + 2\pi i z - 2\pi i z\big)\\
&= \exp(-\pi i\tau - 2\pi i z)\sum_{n\in\mathbb{Z}} \exp(\pi i(n+1)^2\tau)\exp(2\pi i(n+1)z)
\end{aligned}
$$

if we make a simple index shift $m = n+1$ then

$$
\begin{aligned}
&= \exp(-\pi i\tau - 2\pi i z)\sum_{m\in\mathbb{Z}} \exp(\pi i m^2\tau)\exp(2\pi i m z)\\
&\stackrel{def}{=} \exp(-\pi i\tau - 2\pi i z)\theta(z)
\end{aligned}
$$

Hence the basic theta function is not periodic with respect to the τ-direction as in general $\exp(-\pi i\tau - 2\pi i z) \neq 1$.

In the general case we obtain

$$
\begin{aligned}
\theta(z+\lambda) &= \theta(z + p\tau + q)\\
&\stackrel{def}{=} \sum_{n\in\mathbb{Z}} \exp(\pi i n^2\tau)\exp(2\pi i n(z + p\tau + q))\\
&= \sum_{n\in\mathbb{Z}} \exp(\pi i n^2\tau + 2\pi i n z + 2\pi i n p\tau + 2\pi i n q)
\end{aligned}
$$

if we complete the square and rearrange the summands then

$$\begin{aligned}
&= \sum_{n\in\mathbb{Z}} \exp\left(\pi i n^2\tau + 2\pi i n p\tau + \pi i p^2\tau - \pi i p^2\tau \right.\\
&\qquad \left. +2\pi i n z + 2\pi i p z - 2\pi i p z + 2\pi i n q\right)\\
&= \exp(-\pi i p^2\tau - 2\pi i p z)\\
&\qquad \cdot \sum_{n\in\mathbb{Z}} \Big[\exp(\pi i(n+p)^2\tau)\exp(2\pi i(n+p)z)\\
&\qquad\qquad \underbrace{\exp(2\pi i n q)}_{=1 \text{ for all } n\in\mathbb{Z}} \Big]\\
&= \exp(-\pi i p^2\tau - 2\pi i p z)\\
&\qquad \cdot \sum_{n\in\mathbb{Z}} \exp(\pi i(n+p)^2\tau)\exp(2\pi i(n+p)z)
\end{aligned}$$

if we make a simple index shift $m = n + p$ then

$$\begin{aligned}
&= \exp(-\pi i p^2\tau - 2\pi i p z) \sum_{m\in\mathbb{Z}} \exp(\pi i m^2\tau)\exp(2\pi i m z)\\
&\overset{def}{=} \exp(-\pi i p^2\tau - 2\pi i p z)\theta(z)
\end{aligned}$$

Hence the basic theta function θ is quasi-periodic with

$$\begin{aligned}
\theta(z+\lambda) &= \theta(z + p\tau + q)\\
&= \exp(-\pi i p^2\tau - 2\pi i p z)\theta(z)
\end{aligned}$$

for all $\lambda = p\tau + q \in \Lambda$ and $z \in \mathbb{C}$. □

Definition. We define

$$e(\lambda, z) := \exp(-\pi i p^2\tau - 2\pi i p z)$$

and call this the *automorphy factor*.

Remark. We have $e(\lambda_1 + \lambda_2, z) = e(\lambda_1, z + \lambda_2)e(\lambda_2, z)$ for all λ_1, $\lambda_2 \in \Lambda$.

Let λ_1, $\lambda_2 \in \Lambda$, i.e. $\lambda_1 = p_1\tau + q_1$ and $\lambda_2 = p_2\tau + q_2$ for some p_1, p_2, q_1, $q_2 \in \mathbb{Z}$, and thus $\lambda_1 + \lambda_2 = (p_1 + p_2)\tau + (q_1 + q_2) \in \Lambda$. Then

$$\begin{aligned}
e(\lambda_1 + \lambda_2, z) &= e((p_1 + p_2)\tau + (q_1 + q_2), z) \\
&\overset{def}{=} \exp(-\pi i(p_1 + p_2)^2\tau - 2\pi i(p_1 + p_2)z)) \\
&= \exp(-\pi i p_1^2\tau - 2\pi i p_1 p_2\tau - \pi i p_2^2\tau - 2\pi i p_1 z - 2\pi i p_2 z) \\
&\overset{def}{=} \exp(-\pi i p_1^2\tau - 2\pi i p_1 p_2\tau - 2\pi i p_1 z)e(\lambda_2, z) \\
&= \exp(-\pi i p_1^2\tau - 2\pi i p_1 z - 2\pi i p_1 p_2\tau - \underbrace{2\pi i p_1 q_2}_{\exp(2\pi i p_1 q_2)=1}) \\
&\qquad \cdot e(\lambda_2, z) \\
&= \exp(-\pi i p_1^2\tau - 2\pi i p_1(z + \lambda_2))e(\lambda_2, z) \\
&\overset{def}{=} e(\lambda_1, z + \lambda_2)e(\lambda_2, z)
\end{aligned}$$

Summary. The basic theta function $\theta : \mathbb{C} \to \mathbb{C}$ is entire and quasi-periodic with automorphy factor e, i.e., we have

$$\theta(z + \lambda) = e(\lambda, z)\theta(z) = \exp(-\pi i p^2\tau - 2\pi i p z)\theta(z) \tag{1}$$

for all $\lambda = p\tau + q \in \Lambda$ and all $z \in \mathbb{C}$.

Now we want to enlarge our category of theta functions. So far we have only one (basic) theta function corresponding to the point $0 \in \mathbb{C}$ (and each point $q \in \mathbb{Z} \subset \mathbb{C}$). Now, for our fixed τ, we will define a new theta function for each point in $\mathbb{C}$. Therefore let's start with our old theta function and translate z by a fixed ξ, i.e. consider $\theta(z + \xi)$ for $\xi = a\tau + b$ for some fixed a, $b \in \mathbb{R}$:

$$\begin{aligned}
\theta(z + \xi) &= \theta(z + a\tau + b) \\
&\overset{def}{=} \sum_{n\in\mathbb{Z}} \exp(\pi i n^2\tau)\exp(2\pi i n(z + a\tau + b)) \\
&= \sum_{n\in\mathbb{Z}} \exp(\pi i n^2\tau + 2\pi i n z + 2\pi i n a\tau + 2\pi i n b)
\end{aligned}$$

If we complete the square and rearrange the summands then we obtain

$$\begin{aligned}\theta(z+\xi) = \sum_{n\in\mathbb{Z}} \exp\big(&\pi i n^2\tau + 2\pi i n a\tau + \pi i a^2\tau - \pi i a^2\tau \\ &+2\pi i n(z+b) + 2\pi i a(z+b) - 2\pi i a(z+b)\big) \\ = \exp(-\pi i a^2\tau &- 2\pi i a(z+b)) \\ &\cdot \sum_{n\in\mathbb{Z}} \exp(\pi i(n+a)^2\tau)\exp(2\pi i(n+a)(z+b))\end{aligned}$$

Note that the sum $\sum_{n\in\mathbb{Z}} \exp(\pi i(n+a)^2\tau)\exp(2\pi i(n+a)(z+b))$ looks very similar to the sum in the definition of our basic theta function above.

Definition. For $\xi = a\tau + b$ and $a,\, b \in \mathbb{R}$ the *modified theta function* is defined to be the function $\theta_\xi : \mathbb{C} \to \mathbb{C}$ given by

$$\theta_\xi(z) := \theta_\xi(\tau)(z) := \sum_{n\in\mathbb{Z}} \exp(\pi i(n+a)^2\tau)\exp(2\pi i(n+a)(z+b))$$

and ξ is called *theta characteristic*.

Note. From the calculation above we obtain a relation between the basic theta function and the modified theta function with characteristic $\xi = a\tau + b$ for some fixed $a,\, b \in \mathbb{R}$:

$$\theta_\xi(z) = \sum_{n\in\mathbb{Z}} \exp(\pi i(n+a)^2\tau)\exp(2\pi i(n+a)(z+b)) \tag{2}$$

$$= \exp(\pi i a^2\tau + 2\pi i a(z+b))\theta(z+\xi) \tag{3}$$

for all $z \in \mathbb{C}$.

Remark. As the series in the definition is locally uniformly unordered convergent (without proof) the modified theta functions are entire functions.

Lemma 3. *Modified theta functions are quasi-periodic functions.*

Proof. Let a, $b \in \mathbb{R}$ such that $\xi = a\tau + b$ is the characteristic of the modified theta function θ_ξ. Consider $\theta_\xi(z+\lambda)$ for $\lambda = p\tau + q \in \Lambda$.

$$\begin{aligned}
\theta_\xi(z+\lambda) &\overset{(3)}{=} \exp(\pi i a^2\tau + 2\pi i a(z+\lambda+b))\theta(z+\lambda+\xi) \\
&\overset{(1)}{=} \exp(\pi i a^2\tau + 2\pi i a(z+\lambda+b))e(\lambda, z+\xi)\theta(z+\xi) \\
&\overset{(3)}{=} \exp(\pi i a^2\tau + 2\pi i a(z+\lambda+b))e(\lambda, z+\xi) \\
&\qquad \cdot \exp(-\pi i a^2\tau - 2\pi i a(z+b))\theta_\xi(z) \\
&= \exp(2\pi i a\lambda)\exp(-\pi i p^2\tau - 2\pi i p(z+\xi))\theta_\xi(z) \\
&= \exp(2\pi i a\lambda - \pi i p^2\tau - 2\pi i p(z+\xi))\theta_\xi(z)
\end{aligned}$$

Hence the modified theta function θ_ξ is quasi-periodic with

$$\begin{aligned}
\theta_\xi(z+\lambda) &= \theta_{a\tau+b}(z+p\tau+q) \\
&= \exp(2\pi i a\lambda - \pi i p^2\tau - 2\pi i p(z+\xi))\theta_\xi(z)
\end{aligned}$$

for all $\lambda = p\tau + q \in \Lambda$ and $z \in \mathbb{C}$. □

Definition. Let a, $b \in \mathbb{R}$ be fixed and let $\xi = a\tau + b$. We define

$$e_\xi(\lambda, z) := \exp(2\pi i a\lambda - \pi i p^2\tau - 2\pi i p(z+\xi))$$

and call this the *automorphy factor*.

Remark. Let a, $b \in \mathbb{R}$ be fixed and let $\xi = a\tau + b$. We have $e_\xi(\lambda_1+\lambda_2, z) = e_\xi(\lambda_1, z+\lambda_2)e_\xi(\lambda_2, z)$ for all λ_1, $\lambda_2 \in \Lambda$.

Let λ_1, $\lambda_2 \in \Lambda$, i.e. $\lambda_1 = p_1\tau + q_1$ and $\lambda_2 = p_2\tau + q_2$ for some p_1, p_2, q_1, $q_2 \in \mathbb{Z}$, and $\lambda_1 + \lambda_2 = (p_1+p_2)\tau + (q_1+q_2) \in \Lambda$. Then

$$\begin{aligned}
e_\xi(\lambda_1+\lambda_2, z) &= e_\xi((p_1+p_2)\tau + (q_1+q_2), z) \\
&\overset{def}{=} \exp\big(2\pi i a(\lambda_1+\lambda_2) - \pi i(p_1+p_2)^2\tau \\
&\qquad -2\pi i(p_1+p_2)(z+\xi)\big) \\
&= \exp\big(2\pi i a\lambda_1 + 2\pi i a\lambda_2 - \pi i p_1^2\tau - 2\pi i p_1 p_2\tau - \pi i p_2^2\tau \\
&\qquad -2\pi i p_1(z+\xi) - 2\pi i p_2(z+\xi)\big)
\end{aligned}$$

$$
\begin{aligned}
&\overset{def}{=} \exp(2\pi i a\lambda_1 - \pi i p_1^2\tau - 2\pi i p_1 p_2 \tau - 2\pi i p_1(z+\xi)) \\
&\qquad e_\xi(\lambda_2, z) \\
&= \exp\big(2\pi i a\lambda_1 - \pi i p_1^2\tau - 2\pi i p_1 p_2\tau - \underbrace{2\pi i p_1 q_2}_{\exp(2\pi i p_1 q_2)=1} \\
&\qquad -2\pi i p_1(z+\xi)\big)\, e_\xi(\lambda_2, z) \\
&= \exp(2\pi i a\lambda_1 - \pi i p_1^2\tau - 2\pi i p_1(z+\lambda_2+\xi)) e_\xi(\lambda_2, z) \\
&\overset{def}{=} e_\xi(\lambda_1, z+\lambda_2) e_\xi(\lambda_2, z)
\end{aligned}
$$

Summary. Let $\xi = a\tau + b$ with a, $b \in \mathbb{R}$ fixed. The modified theta function with characteristic ξ is entire and quasi-periodic with automorphy factor e_ξ, i.e. we have

$$
\begin{aligned}
\theta_\xi(z+\lambda) &= e_\xi(\lambda, z)\theta_\xi(z) && (4) \\
&= \exp(2\pi i a\lambda - \pi i p^2\tau - 2\pi i p(z+\xi))\theta_\xi(z) && (5)
\end{aligned}
$$

for all $\lambda = p\tau + q \in \Lambda$ and all $z \in \mathbb{C}$.

Now we want to determine all zeros of all theta functions. Therefore we consider a special modified theta function, the theta function with characteristic $\sigma := \frac{1}{2}\tau + \frac{1}{2}$. In this case the determination of zeros is very simple because the zeros are easy to describe.

Lemma 4. *θ_σ is an odd function, i.e. $\theta_\sigma(-z) = -\theta_\sigma(z)$ for all $z \in \mathbb{C}$. In particular we have $\theta_\sigma(0) = 0$.*

Proof. We have

$$
\begin{aligned}
\theta_\sigma(-z) &= \theta_{\frac{1}{2}\tau+\frac{1}{2}}(-z) \\
&\overset{def}{=} \sum_{n\in\mathbb{Z}} \left[\exp\left(\pi i \left(n+\frac{1}{2}\right)^2 \tau\right) \right. \\
&\qquad \left. \exp\left(2\pi i \left(n+\frac{1}{2}\right)\left(-z+\frac{1}{2}\right)\right)\right]
\end{aligned}
$$

if we make a simple index shift $m = -n - 1$ then

$$= \sum_{m\in\mathbb{Z}} \left[\exp\left(\pi i \left(-m - \frac{1}{2}\right)^2 \tau\right) \right.$$

$$\left. \exp\left(2\pi i \left(-m - \frac{1}{2}\right)\left(-z + \frac{1}{2}\right)\right)\right]$$

$$= \sum_{m\in\mathbb{Z}} \left[\exp\left(\pi i \left(m + \frac{1}{2}\right)^2 \tau\right) \right.$$

$$\left. \exp\left(2\pi i \left(m + \frac{1}{2}\right)\left(z + \frac{1}{2}\right) - 2\pi i \left(m + \frac{1}{2}\right)\right)\right]$$

$$= \sum_{m\in\mathbb{Z}} \left[\exp\left(\pi i \left(m + \frac{1}{2}\right)^2 \tau\right) \right.$$

$$\left. \exp\left(2\pi i \left(m + \frac{1}{2}\right)\left(z + \frac{1}{2}\right)\right) \stackrel{def}{=} -\theta_\sigma(z). \right.$$

□

From complex analysis we know a simple way to count zeros and poles of a meromorphic function $f : \mathbb{C} \to \mathbb{C}$:

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}(z)\, dz = \text{total number of zeros - total number of poles}$$

where γ is a piecewise smooth path that runs around each zero and each pole exactly one time. We will use this integral to determine all zeros of the theta functions θ_σ with $\sigma = \frac{1}{2}\tau + \frac{1}{2}$.

Lemma 5. *We have $\theta_\sigma(z) = 0$ precisely for all $z \in \Lambda$ and all zeros are simple zeros.*

Proof. Consider the fundamental parallelogram $V := \{z \in \mathbb{C} : z = t_1\tau + t_2 \text{ for some } 0 \le t_1, t_2 < 1\}$. Choose $w \in \mathbb{C}$ such that the border of $V_w := w + V$ contains no zeros of θ_σ and $0 \in V_w$.

Further consider the following paths along the border of V_w:

$$\alpha : [0,1] \to \mathbb{C}; t \mapsto w + t$$
$$\beta : [0,1] \to \mathbb{C}; t \mapsto w + 1 + t\tau$$
$$\gamma : [0,1] \to \mathbb{C}; t \mapsto w + (1-t) + \tau$$
$$\delta : [0,1] \to \mathbb{C}; t \mapsto w + (1-t)\tau$$

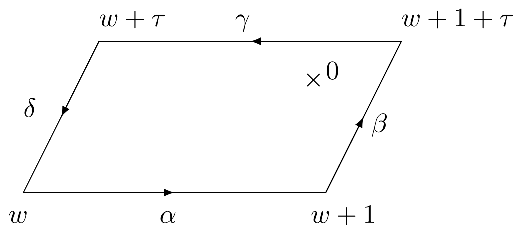

FIGURE 2

In the above figure $w \in \mathbb{C}$ is chosen such that the border of the parallelogram $V_w = w + V$ contains no zeros of f and such that $0 \in V_w$. The paths α, β, γ and δ run along the border of V_w. Note that

$$\gamma(t) = w + (1-t) + \tau = \alpha(1-t) + \tau$$

and

$$\delta(t) = w + (1-t)\tau = \beta(1-t) - 1.$$

We want to show that $\frac{1}{2\pi i}\int_{\partial V_w} \frac{\theta'_\sigma}{\theta_\sigma}(z)\, dz = 1.$

Therefore we will show that

$$\frac{1}{2\pi i}\int_\gamma \frac{\theta'_\sigma}{\theta_\sigma}(z)\, dz = 1 - \frac{1}{2\pi i}\int_\alpha \frac{\theta'_\sigma}{\theta_\sigma}(z)\, dz$$

and

$$\frac{1}{2\pi i}\int_\delta \frac{\theta'_\sigma}{\theta_\sigma}(z)\, dz = -\frac{1}{2\pi i}\int_\beta \frac{\theta'_\sigma}{\theta_\sigma}(z)\, dz.$$

$$\begin{aligned}
\frac{1}{2\pi i}\int_\gamma \frac{\theta'_\sigma}{\theta_\sigma}(z)\, dz &= \frac{1}{2\pi i}\int_0^1 \frac{\theta'_\sigma}{\theta_\sigma}(\gamma(t))\gamma'(t)\, dt \\
&= \frac{1}{2\pi i}\int_0^1 \frac{\theta'_\sigma}{\theta_\sigma}(\alpha(1-t)+\tau)(-1)\, dt \\
&= -\frac{1}{2\pi i}\int_\alpha \frac{\theta'_\sigma}{\theta_\sigma}(z+\tau)\, dz \\
&= -\frac{1}{2\pi i}\int_\alpha \frac{e'_\sigma(\tau,z)\theta_\sigma(z) + e_\sigma(\tau,z)\theta'_\sigma(z)}{e_\sigma(\tau,z)\theta_\sigma(z)}\, dz \\
&= -\frac{1}{2\pi i}\int_\alpha \frac{e'_\sigma(\tau,z)}{e_\sigma(\tau,z)}\, dz - \frac{1}{2\pi i}\int_\alpha \frac{\theta'_\sigma}{\theta_\sigma}(z)\, dz
\end{aligned}$$

when we use $e_\sigma(\tau,z)=\exp(2\pi i\frac{1}{2}\tau-\pi i\tau-2\pi i(z+\sigma))$ then the above expression becomes

$$\begin{aligned}
-\frac{1}{2\pi i}\int_\alpha \frac{\exp'(-2\pi i(z+\sigma))}{\exp(-2\pi i(z+\sigma))}\,dz-\frac{1}{2\pi i}\int_\alpha\frac{\theta'_\sigma}{\theta_\sigma}(z)\,dz \\
=-\frac{1}{2\pi i}\int_\alpha -2\pi i\,dz-\frac{1}{2\pi i}\int_\alpha\frac{\theta'_\sigma}{\theta_\sigma}(z)\,dz \\
=1-\frac{1}{2\pi i}\int_\alpha\frac{\theta'_\sigma}{\theta_\sigma}(z)\,dz
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2\pi i}\int_\delta\frac{\theta'_\sigma}{\theta_\sigma}(z)\,dz &= \frac{1}{2\pi i}\int_0^1\frac{\theta'_\sigma}{\theta_\sigma}(\delta(t))\delta'(t)\,dt \\
&= \frac{1}{2\pi i}\int_0^1\frac{\theta'_\sigma}{\theta_\sigma}(\beta(1-t)-1)(-\tau)\,dt \\
&= -\frac{1}{2\pi i}\int_\beta\frac{\theta'_\sigma}{\theta_\sigma}(z-1)\,dz \\
&= -\frac{1}{2\pi i}\int_\beta\frac{e'_\sigma(-1,z)\theta_\sigma(z)+e_\sigma(-1,z)\theta'_\sigma(z)}{e_\sigma(-1,z)\theta_\sigma(z)}\,dz \\
&= -\frac{1}{2\pi i}\int_\beta\frac{e'_\sigma(-1,z)}{e_\sigma(-1,z)}\,dz-\frac{1}{2\pi i}\int_\beta\frac{\theta'_\sigma(z)}{\theta_\sigma(z)}\,dz
\end{aligned}$$

when we use

$$e_\sigma(-1,z)=\exp(-2\pi i\frac{1}{2})$$

then

$$\begin{aligned}
&= -\frac{1}{2\pi i}\int_\beta\frac{\exp'(-\pi i)}{\exp(-\pi i)}\,dz-\frac{1}{2\pi i}\int_\beta\frac{\theta'_\sigma(z)}{\theta_\sigma(z)}\,dz \\
&= -\frac{1}{2\pi i}\int_\beta\frac{\theta'_\sigma(z)}{\theta_\sigma(z)}\,dz
\end{aligned}$$

Then we have

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\partial V_w}\frac{\theta'_\sigma}{\theta_\sigma}(z)\,dz &= \frac{1}{2\pi i}\int_\alpha\frac{\theta'_\sigma}{\theta_\sigma}(z)\,dz+\frac{1}{2\pi i}\int_\beta\frac{\theta'_\sigma}{\theta_\sigma}(z)\,dz \\
&\quad+\frac{1}{2\pi i}\int_\gamma\frac{\theta'_\sigma}{\theta_\sigma}(z)\,dz+\frac{1}{2\pi i}\int_\delta\frac{\theta'_\sigma}{\theta_\sigma}(z)\,dz \\
&= 1.
\end{aligned}$$

As θ_σ is holomorphic in $\overline{V_w}$, i.e. it doesn't have any poles, we know that θ_σ has a single zero. And by Lemma 4 this zero is in 0. Now consider $\overline{V}_w + \lambda = \overline{V}_{w+\lambda}$ for some $\lambda \in \Lambda$. As $\theta_\sigma(z+\lambda) = e_\sigma(\lambda, z)\theta_\sigma(z)$ we obtain that θ_σ has the only zero $0 + \lambda = \lambda$ in $\overline{V}_{w+\lambda}$ and this is a simple zero. But $\mathbb{C} = \cup_{\lambda\in\Lambda}\overline{V}_{w+\lambda}$. Hence θ_σ has zeros exactly in Λ and all zeros are simple. □

Corollary 6. *Let $\xi = a\tau + b$ with a, $b \in \mathbb{R}$. We have $\theta_\xi(z) = 0$ precisely for all $z \in \sigma - \xi + \Lambda$ and all its zeros are simple.*

Proof. We know $\theta_\sigma(z) = 0$ if and only if $z \in \Lambda$ and all the zeros are simple. Hence

$$\begin{aligned}
\theta_\xi(z) = 0 &\overset{(3)}{\Leftrightarrow} \exp(\pi i a^2\tau + 2\pi i a(z+b))\theta(z+\xi) = 0 \\
&\overset{(3)}{\Leftrightarrow} \exp\big(\pi i a^2\tau + 2\pi i a(z+b)\big) \\
&\quad \cdot \exp\Big(-\pi i\Big(\frac{1}{2}\Big)^2\tau - 2\pi i\frac{1}{2}\Big(z+\xi-\frac{1}{2}\tau-\frac{1}{2}+\frac{1}{2}\Big)\Big) \\
&\qquad \cdot \theta_\sigma(z+\xi-\sigma) = 0 \\
&\Leftrightarrow z + \xi - \sigma \in \Lambda \\
&\Leftrightarrow z \in \sigma - \xi + \Lambda
\end{aligned}$$

In particular we have $\theta(z) = 0$ if and only if $z \in \sigma + \Lambda$. □

So far we have considered entire quasi-periodic functions. Now we want to use our knowledge about them to see what meromorphic doubly-periodic functions with given zeros a_i and poles b_j of given order n_i resp. m_j and number n resp. m look like. Furthermore we will decide whether such a function exists or not and whether it is unique or not.

Abel's Theorem 7. *There is a meromorphic function on $\mathbb{C}/\Lambda$ with zeros $[a_i]$ of order n_i for $1 \leq i \leq n$ and poles $[b_j]$ of order m_j for $1 \leq j \leq m$ if and only if $\sum_{i=1}^n n_i = \sum_{j=1}^m m_j$ and $\sum_{i=1}^n n_i[a_i] = \sum_{j=1}^m m_j[b_j]$.*

Moreover, such a function is unique up to a constant factor.

Proof. "⇒" Let $f : \mathbb{C}/\Lambda \to \mathbb{C}$ be a meromorphic function with zeros $[a_i]$ of order n_i and poles $[b_j]$ of order m_j. Choose $w \in \mathbb{C}$ such that $V_w = \{w + z \in \mathbb{C} : z = t_1\tau + t_2 \text{ for some } 0 \leq t_1, t_2 < 1\}$ contains a representative a_i resp. b_j for every zero resp. pole of f. Further consider the paths

$$\begin{aligned}\alpha &: [0,1] \to \mathbb{C}; t \mapsto w + t \\ \beta &: [0,1] \to \mathbb{C}; t \mapsto w + 1 + t\tau \\ \gamma &: [0,1] \to \mathbb{C}; t \mapsto w + (1-t) + \tau \\ \delta &: [0,1] \to \mathbb{C}; t \mapsto w + (1-t)\tau\end{aligned}$$

along the border of V_w and the paths

$$\begin{aligned}\alpha_i &: [0,1] \to \mathbb{C}; t \mapsto a_i + r_i e^{2\pi i t} \\ \beta_j &: [0,1] \to \mathbb{C}; t \mapsto b_j + s_j e^{2\pi i t}\end{aligned}$$

around the zeros resp. poles of f where r_i resp. s_j is chosen small enough that $D_i = \{z \in \mathbb{C} : |z - a_i| < r_i\}$ resp. $D'_j = \{z \in \mathbb{C} : |z - b_j| < s_j\}$ contains no other zeros or poles of f.

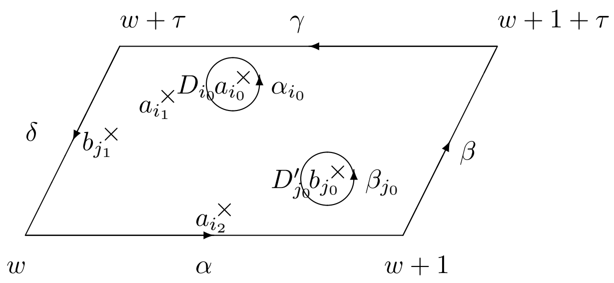

FIGURE 3

Here, $w \in \mathbb{C}$ is chosen such that the parallelogram $V_w = w + V$ contains a representative a_i resp. b_j for every zero resp. pole of f. The paths α, β, γ and δ run along the border of V_w, the paths α_{i_0} around the zero a_{i_0} of f and the path β_{j_0} around the pole b_{j_0} of f.

First we show that $\sum_{i=1}^n n_i a_i - \sum_{j=1}^m m_j b_j \in \Lambda$ as follows:

$$\begin{aligned}\sum_{i=1}^n n_i a_i - \sum_{j=1}^m m_j b_j &= \sum_{i=1}^n \frac{1}{2\pi i} \int_{\alpha_i} z \frac{f'}{f}(z)\, dz + \sum_{j=1}^m \frac{1}{2\pi i} \int_{\beta_j} z \frac{f'}{f}(z)\, dz \\ &= \frac{1}{2\pi i} \int_{\partial V_w} z \frac{f'}{f}(z)\, dz \in \Lambda\end{aligned}$$

To establish the first equality note that we can write

$$f(z) = c_i(z - a_i)^{n_i} h_i(z)$$

for a constant c_i and with $h_i(a_i) = 1$ around a_i and hence

$$f'(z) = c_i n_i (z - a_i)^{n_i - 1} \overline{h}_i(z)$$

with $\overline{h}_i(a_i) = 1$. We obtain

$$z\frac{f'}{f}(z) = z\frac{n_i}{z - a_i}\frac{\overline{h}_i}{h_i}(z)$$

with $\frac{\overline{h}_i}{h_i}(a_i) = 1$. Hence we have

$$\frac{1}{2\pi i}\int_{\alpha_i} z\frac{f'}{f}(z)\, dz = n_i a_i$$

by Cauchy's integral formula for discs. The same holds for the poles of f.

The second equality is clear since V_w contains a representative for every zero and pole of f in $\mathbb{C}/\Lambda$.

To see, that $\frac{1}{2\pi i}\int_{\partial V_w} z\frac{f'}{f}(z)\, dz$ is an element of Λ, note that

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\gamma} z\frac{f'}{f}(z)\, dz &= \frac{1}{2\pi i}\int_0^1 \gamma(t)\frac{f'}{f}(\gamma(t))\gamma'(t)\, dt \\
&= \frac{1}{2\pi i}\int_0^1 (\alpha(1-t)+\tau)\frac{f'}{f}((\alpha(1-t)+\tau))(-1)\, dt \\
&= -\frac{1}{2\pi i}\int_0^1 \alpha(1-t)\frac{f'}{f}(\alpha(1-t))\, dt \\
&\qquad - \frac{1}{2\pi i}\int_0^1 \tau\frac{f'}{f}(\alpha(1-t))\, dt \\
&= -\frac{1}{2\pi i}\int_{\alpha} z\frac{f'}{f}(z)\, dz - \tau\frac{1}{2\pi i}\int_{\alpha}\frac{f'}{f}(z)\, dz
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2\pi i}\int_\delta z\frac{f'}{f}(z)\,dz &= \frac{1}{2\pi i}\int_0^1 \delta(t)\frac{f'}{f}(\delta(t))\delta'(t)\,dt\\
&= \frac{1}{2\pi i}\int_0^1 (\beta(1-t)-1)\frac{f'}{f}((\beta(1-t)-1))(-\tau)\,dt\\
&= -\frac{1}{2\pi i}\int_0^1 \beta(1-t)\frac{f'}{f}(\beta(1-t))\tau\,dt\\
&\qquad + \frac{1}{2\pi i}\int_0^1 \frac{f'}{f}(\beta(1-t))\tau\,dt\\
&= -\frac{1}{2\pi i}\int_\beta z\frac{f'}{f}(z)\,dz + \frac{1}{2\pi i}\int_\beta \frac{f'}{f}(z)\,dz
\end{aligned}$$

hence

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\partial V_w} z\frac{f'}{f}(z)\,dz &= \frac{1}{2\pi i}\int_\alpha z\frac{f'}{f}(z)\,dz + \frac{1}{2\pi i}\int_\beta z\frac{f'}{f}(z)\,dz\\
&\qquad + \frac{1}{2\pi i}\int_\gamma z\frac{f'}{f}(z)\,dz + \frac{1}{2\pi i}\int_\delta z\frac{f'}{f}(z)\,dz\\
&= -\tau\frac{1}{2\pi i}\int_\alpha \frac{f'}{f}(z)\,dz + \frac{1}{2\pi i}\int_\beta \frac{f'}{f}(z)\,dz \in \Lambda
\end{aligned}$$

since $\frac{1}{2\pi i}\int_\beta \frac{f'}{f}(z)\,dz,\ \frac{1}{2\pi i}\int_\alpha \frac{f'}{f}(z)\,dz \in \mathbb{Z}$.

Secondly we show that

$$\begin{aligned}
\sum_{i=1}^n n_i - \sum_{j=1}^m m_j &= \frac{1}{2\pi i}\int_{\partial V_w} \frac{f'}{f}(z)\,dz\\
&= 0
\end{aligned}$$

Again the first equality is clear, since V_w contains a representative for every zero and pole of f in $\mathbb{C}/\Lambda$.

The second equality follows from:

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}(z)\,dz &= \frac{1}{2\pi i}\int_0^1 \frac{f'}{f}(\gamma(t))\gamma'(t)\,dt\\ &= \frac{1}{2\pi i}\int_0^1 \frac{f'}{f}(\alpha(1-t)+\tau)(-1)\,dt\\ &= -\frac{1}{2\pi i}\int_0^1 \frac{f'}{f}(\alpha(1-t))\,dt\\ &= -\frac{1}{2\pi i}\int_\alpha \frac{f'}{f}(z)\,dz\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2\pi i}\int_\delta \frac{f'}{f}(z)\,dz &= \frac{1}{2\pi i}\int_0^1 \frac{f'}{f}(\delta(t))\delta'(t)\,dt\\ &= \frac{1}{2\pi i}\int_0^1 \frac{f'}{f}(\beta(1-t)-1)(-\tau)\,dt\\ &= -\frac{1}{2\pi i}\int_0^1 \frac{f'}{f}(\beta(1-t))\tau\,dt\\ &= -\frac{1}{2\pi i}\int_\beta \frac{f'}{f}(z)\,dz\end{aligned}$$

hence

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial V_w} \frac{f'}{f}(z)\,dz &= \frac{1}{2\pi i}\int_\alpha \frac{f'}{f}(z)\,dz + \frac{1}{2\pi i}\int_\beta \frac{f'}{f}(z)\,dz\\ &\qquad + \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}(z)\,dz + \frac{1}{2\pi i}\int_\delta \frac{f'}{f}(z)\,dz\\ &= 0.\end{aligned}$$

"$\Leftarrow$" Now let $[a_i]$, $[b_j] \in \mathbb{C}/\Lambda$ and n_i, $m_j \in \mathbb{N}$ for $1 \le i \le n$ and $1 \le j \le m$ be such that $\sum_{i=n}^n n_i = \sum_{j=m}^m m_j$ and $\sum_{i=1}^n n_i[a_i] = \sum_{j=1}^m m_j[b_j]$. We will construct a meromorphic function $f : \mathbb{C}/\Lambda \to \mathbb{C}$ with zeros $[a_i]$ of order n_i and poles $[b_j]$ of order m_j. We choose representatives a_i, $b_j \in \mathbb{C}$ for $[a_i]$ resp. $[b_j]$ such that $\sum_{i=1}^n n_i a_i = \sum_{j=1}^m m_j b_j$ and define the function

$$g : \mathbb{C} \to \mathbb{C};\, z \mapsto \frac{\prod_{i=1}^n \theta_\sigma(z-a_i)^{n_i}}{\prod_{j=1}^m \theta_\sigma(z-b_j)^{m_j}}$$

where θ_σ is the theta function with characteristic $\frac{1}{2}\tau + \frac{1}{2}$. Obviously g is a meromorphic function with zeros in $a_i + \Lambda$ of order n_i and poles in $b_j + \Lambda$ of order m_j. We have to show that g is doubly-periodic with respect to Λ. Therefore we have to show that $g(z+\lambda) = g(z)$ for all $\lambda \in \Lambda$. It suffices to show that $g(z+1) = g(z)$ and $g(z+\tau) = g(z)$.

$$g(z+1) = \frac{\prod_{i=1}^n \theta_\sigma(z+1-a_i)^{n_i}}{\prod_{j=1}^m \theta_\sigma(z+1-b_j)^{m_j}} = \frac{\prod_{i=1}^n \theta_\sigma(z-a_i)^{n_i}}{\prod_{j=1}^m \theta_\sigma(z-b_j)^{m_j}} = g(z)$$

and

$$\begin{aligned} g(z+\tau) &= \frac{\prod_{i=1}^n \theta_\sigma(z+\tau-a_i)^{n_i}}{\prod_{j=1}^m \theta_\sigma(z+\tau-b_j)^{m_j}} \\ &= \frac{\prod_{i=1}^n (e_\sigma(\tau, z-a_i)\theta_\sigma(z-a_i))^{n_i}}{\prod_{j=1}^m (e_\sigma(\tau, z-b_j)\theta_\sigma(z-b_j))^{m_j}} \\ &= \frac{\prod_{i=1}^n e_\sigma(\tau, z-a_i)^{n_i}}{\prod_{j=1}^m e_\sigma(\tau, z-b_j)^{m_j}} \frac{\prod_{i=1}^n \theta_\sigma(z-a_i)^{n_i}}{\prod_{j=1}^m \theta_\sigma(z-b_j)^{m_j}} \\ &= \frac{\prod_{i=1}^n e_\sigma(\tau, z-a_i)^{n_i}}{\prod_{j=1}^m e_\sigma(\tau, z-b_j)^{m_j}} \cdot g(z) \end{aligned}$$

but

$$\begin{aligned} \frac{\prod_{i=1}^n e_\sigma(\tau, z-a_i)^{n_i}}{\prod_{j=1}^m e_\sigma(\tau, z-b_j)^{m_j}} &= \frac{\prod_{i=1}^n \exp(-2\pi i(z-a_i+\sigma))^{n_i}}{\prod_{j=1}^m \exp(-2\pi i(z-b_j+\sigma))^{m_j}} \\ &= \frac{\prod_{i=1}^n \exp(-2\pi i(z+\sigma))^{n_i}}{\prod_{j=1}^m \exp(-2\pi i(z+\sigma))^{m_j}} \\ &\qquad \cdot \frac{\prod_{i=1}^n \exp(2\pi i a_i)^{n_i}}{\prod_{j=1}^m \exp(2\pi i b_j)^{m_j}} \\ &= \frac{\exp(-2\pi i(z+\sigma))^{\sum_{i=1}^n n_i}}{\exp(-2\pi i(z+\sigma))^{\sum_{j=1}^m m_j}} \\ &\qquad \cdot \frac{\exp(2\pi i \sum_{i=1}^n n_i a_i)}{\exp(2\pi i \sum_{j=1}^m m_j b_j)} \\ &= 1. \end{aligned}$$

So $g(z+\tau) = g(z)$ as well. Hence g is doubly periodic w.r.t. Λ and the function $f : \mathbb{C}/\Lambda \to \mathbb{C}$ with $f([z]) = g(z)$ is well-defined and a solution.

Now suppose we are given two meromorphic functions f, g : $\mathbb{C}/\Lambda \to \mathbb{C}$ with zeros $[a_i]$ of order n_i and poles $[b_j]$ of order m_j. Then $\frac{f}{g}$ has no zeros or poles. Hence it is constant. □

3. Weierstrass ℘-function

Now we want to capitalize on our work above. Therefore we consider a very special periodic function, the Weierstraß ℘-function.

Definition. The *Weierstraß* $\wp - function$ is defined to be the function $\wp : \mathbb{C} \to \mathbb{C}$ given by

$$\wp(z) = \frac{1}{z^2} + \sum_{0 \neq \lambda \in \Lambda} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

Proposition 8. (Without proof) *$\wp$ is a Λ-periodic meromorphic function with poles of order 2 exactly in Λ.*

The following lemma gives a connection between the Weierstraß ℘-function and our well known theta function with characteristic $\sigma = \frac{1}{2} + \frac{1}{2}\tau$.

Lemma 9. *There is a constant $c \in \mathbb{C}$ such that*

$$\wp(z) = -\left(\frac{\theta'_\sigma}{\theta_\sigma} \right)'(z) + c$$

Note. The quotient $\frac{\theta'_\sigma}{\theta_\sigma}$ isn't doubly-periodic, but the derivative $\left(\frac{\theta'_\sigma}{\theta_\sigma}\right)'$ is doubly-periodic.

To see this consider $\frac{\theta'_\sigma}{\theta_\sigma}(z+\lambda)$ for some $\lambda = p\tau + q \in \Lambda$.

$$\begin{aligned}
\frac{\theta'_\sigma}{\theta_\sigma}(z+\lambda) &\overset{(5)}{=} \frac{(e_\sigma(\lambda,z)\theta_\sigma(z))'}{e_\sigma(\lambda,z)\theta_\sigma(z)} = \frac{e'_\sigma(\lambda,z)\theta_\sigma(z) + e_\sigma(\lambda,z)\theta'_\sigma(z)}{e_\sigma(\lambda,z)\theta_\sigma(z)} \\
&\overset{def}{=} \frac{\exp'(\pi i\lambda - \pi i p^2\tau - 2\pi i p(z+\sigma))\theta_\sigma(z) + e_\sigma(\lambda,z)\theta'_\sigma(z)}{e_\sigma(\lambda,z)\theta_\sigma(z)} \\
&= \frac{-2\pi i p e_\sigma(\lambda,z)\theta_\sigma(z) + e_\sigma(\lambda,z)\theta'_\sigma(z)}{e_\sigma(\lambda,z)\theta_\sigma(z)} \\
&= -2\pi i p + \frac{\theta'_\sigma}{\theta_\sigma}(z) \neq \frac{\theta'_\sigma}{\theta_\sigma}(z)
\end{aligned}$$

as in general $p \neq 0$. From the equation $\frac{\theta'_\sigma}{\theta_\sigma}(z+\lambda) = -2\pi i p + \frac{\theta'_\sigma}{\theta_\sigma}(z)$ above it follows directly that $\left(\frac{\theta'_\sigma}{\theta_\sigma}\right)'$ is doubly-periodic.

Proof. We know that θ_σ is holomorphic and has its zeros precisely in the lattice points $\lambda \in \Lambda$. That means that the expansion of $\frac{\theta'_\sigma}{\theta_\sigma}$ in a Laurent series around 0 looks like

$$\frac{\theta'_\sigma}{\theta_\sigma}(z) = a_{-1}\frac{1}{z} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \text{ terms of higher order}$$

for some constants $a_i \in \mathbb{C}$. We can choose a neighborhood U of 0 such that 0 is the only zero of θ_σ in U. As 0 is a single zero we know that

$$a_{-1} = \operatorname{Res}_0\left(\frac{\theta'_\sigma}{\theta_\sigma}\right) = \int_\alpha \frac{\theta'_\sigma}{\theta_\sigma}(z)\, dz = 1$$

where $\alpha : [0,1] \to \mathbb{C}; t \mapsto re^{2\pi i t}$ for some suitable r. We conclude

$$\frac{\theta'_\sigma}{\theta_\sigma}(z) = \frac{1}{z} + a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \text{ terms of higher order}$$

and calculate

$$\left(\frac{\theta'_\sigma}{\theta_\sigma}\right)'(z) = -\frac{1}{z^2} + a_1 + 2a_2 z + 3a_3 z^2 + \text{ terms of higher order}$$

If we add $\wp$ and $\left(\frac{\theta'_\sigma}{\theta_\sigma}\right)'$ then we obtain

$$\wp(z) + \left(\frac{\theta'_\sigma}{\theta_\sigma}\right)'(z) = \sum_{0 \neq \lambda \in \Lambda} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right) + a_1 + 2a_2 z + 3a_3 z^2 + \ldots$$

From this sum we see directly that $\wp + \left(\frac{\theta'_\sigma}{\theta_\sigma}\right)'$ doesn't have any poles in U. Hence $\wp + \left(\frac{\theta'_\sigma}{\theta_\sigma}\right)'$ is holomorphic in a neighborhood of 0 and thus holomorphic everywhere. As it is in addition doubly-periodic (since $\wp$ is as well as $\left(\frac{\theta'_\sigma}{\theta_\sigma}\right)'$ doubly-periodic) we know from our very first lemma that it must be constant. □

The Weierstraß $\wp$-function satisfies a number of equations and differential equation. This feature makes the Weierstraß $\wp$-function to be of interest. The most important differential equation that is satisfied by the Weierstraß $\wp$-function is the following:

Theorem 10. *The Weierstraß $\wp$ -function satisfies the differential equation*

$$\wp'(z)^2 = c_3\wp(z)^3 + c_2\wp(z)^2 + c_1\wp(z) + c_0$$

where the constants

$$c_3 = 4\ ,\ c_2 = 0\ ,\ c_1 = -60 \sum_{0\neq\lambda\in\Lambda} \frac{1}{\lambda^4} \text{ and } c_0 = -140 \sum_{0\neq\lambda\in\Lambda} \frac{1}{\lambda^6}$$

depend on the lattice Λ.

Proof. Consider $\wp(z) - \frac{1}{z^2} = \sum_{0\neq\lambda\in\Lambda}\left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right)$. This function is holomorphic in a neighborhood of 0. We can expand the summands $\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}$:

$$\begin{aligned}\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} &= \frac{1}{\lambda^2}\left(\frac{1}{\left(1-\frac{z}{\lambda}\right)^2} - 1\right)\\ &= \frac{1}{\lambda^2}\left(\left(\sum_{n=0}^{\infty}\left(\frac{z}{\lambda}\right)^n\right)^2 - 1\right)\\ &= \frac{1}{\lambda^2}\left(2\frac{z}{\lambda} + 3\frac{z^2}{\lambda^2} + 4\frac{z^3}{\lambda^3} + 5\frac{z^4}{\lambda^4} + \dots\right)\\ &= 2\frac{z}{\lambda^3} + 3\frac{z^2}{\lambda^4} + 4\frac{z^3}{\lambda^5} + 5\frac{z^4}{\lambda^6} + \dots\end{aligned}$$

This sum is absolutely convergent for all $z \in \mathbb{C}$ with $|z| < |\lambda|$; in particular in a neighbourhood of 0.
To simplify the big sum from above we define $s_n := \sum_{0\neq\lambda\in\Lambda} \frac{1}{\lambda^{n+2}}$ for $n \in \mathbb{N}$. Note that $s_n = 0$ for all odd $n \in \mathbb{N}$. We obtain

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + 2s_1 z + 3s_2 z^2 + 4s_3 z^3 + 5s_4 z^4 + \dots\\ &= \frac{1}{z^2} + 3s_2 z^2 + 5s_4 z^4 + 7s_6 z^6 \dots\end{aligned}$$

which is true in a neigborhood of 0. With the constants

$$c_3 = 4\ ,\ c_2 = 0\ ,\ c_1 = -60 \sum_{0\neq\lambda\in\Lambda} \frac{1}{\lambda^4} \text{ and } c_0 = -140 \sum_{0\neq\lambda\in\Lambda} \frac{1}{\lambda^6}$$

we obtain

$$\wp(z) = \frac{1}{z^2} - \frac{c_1}{20} z^2 - \frac{c_0}{28} z^4 + \text{ terms of higher order}$$

hence

$$\wp'(z) = -\frac{2}{z^3} - \frac{c_1}{10} z - \frac{c_0}{7} z^3 + \text{ terms of higher order}$$

$$\wp'(z)^2 = \frac{4}{z^6} + \frac{2c_1}{5}\frac{1}{z^2} + \frac{4c_0}{7} + \text{ terms of higher order}$$

and

$$\wp(z)^3 = \frac{1}{z^6} - \frac{3c_1}{20}\frac{1}{z^2} - \frac{3c_0}{28} + \text{ terms of higher order}$$

Now consider

$$f(z) := \wp'(z)^2 - c_3\wp(z)^3 - c_1\wp(z) - c_0$$

The series of f has only positive powers of z. Hence f is holomorphic around 0. Hence it is holomorphic everywhere. And as it is doubly-periodic, it is constant. But the constant part of the series is $\frac{4}{7}c_0 + 4 \cdot \frac{3}{28}c_0 - c_0 = 0$. Hence $f = 0$. □

We will mention one more equation that is satisfied by the Weierstraß $\wp$-function:

Remark. Remember that our lattice Λ is generated by 1 and τ. Hence the set of zeros of $\wp'$ is given by $\left(\frac{1}{2} + \Lambda\right) \cup \left(\frac{\tau}{2} + \Lambda\right) \cup \left(\frac{1+\tau}{2} + \Lambda\right)$. Set $e_1 := \wp\left(\frac{1}{2}\right)$, $e_2 := \wp\left(\frac{\tau}{2}\right)$, $e_3 := \wp\left(\frac{1+\tau}{2}\right) \in \mathbb{C}$. Then we have

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$$

and

$$\begin{aligned} e_1 + e_2 + e_3 &= 0 \\ e_1e_2 + e_1e_3 + e_2e_3 &= \frac{1}{4}c_1 \\ e_1e_2e_3 &= -\frac{1}{4}c_0 \end{aligned}$$

where c_0 and c_1 are the constants from above.

Finally we will see how to use the Weierstraß $\wp$-function to give a group structure to an elliptic curve.

Remark. If we consider the elliptic curve

$$C := \{(x, y) \in \mathbb{C}^2 \text{ such that } y^2 = c_3x^3 + c_2x^2 + c_1x + c_0\}$$

for the constants

$$c_3 = 4 \text{ , } c_2 = 0 \text{ , } c_1 = -60 \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^4} \text{ and } c_0 = -140 \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^6}$$

from the theorem above then we have a bijection

$$\mathbb{C}/\Lambda \setminus \{0\} \to C \text{ given by } z \mapsto (\wp(z), \wp'(z))$$

In particular we can give the variety C the group structure of $\mathbb{C}/\Lambda$.

This can be extended to an embedding of $\mathbb{C}/\Lambda$ into the projective plane. For more details see the article of M. Khalid [2].

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