The Centre of Unitary Isotopes of JB*-Algebras

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Abstract. We identify the centre of unitary isotopes of a JB*-algebra. We show that the centres of any two unitary isotopes of a JB*-algebra are isometrically Jordan *-isomorphic to each other. However, there need be no inclusion between centres of the two unitary isotopes.

1. Basics

We begin by recalling (from [3], for instance) the following concepts of homotope and isotope of Jordan algebras.

Let \( J \) be a Jordan algebra, cf. [3], and \( x \in J \). The \( x \)-homotope of \( J \), denoted by \( J[x] \), is the Jordan algebra consisting of the same elements and linear algebra structure as \( J \) but a different product, denoted by \( \cdot_x \), defined by

\[ a \cdot_x b = \{axb\} \]

for all \( a, b \) in \( J[x] \). By \( \{pqr\} \) we will always denote the Jordan triple product of \( p, q, r \) defined in the Jordan algebra \( J \) as below:

\[ \{pqr\} = (p \circ q) \circ r - (p \circ r) \circ q + (q \circ r) \circ p, \]

where \( \circ \) stands for the original Jordan product in \( J \). An element \( x \) of a Jordan algebra \( J \) with unit \( e \) is said to be invertible if there exists \( x^{-1} \in J \), called the inverse of \( x \), such that \( x \circ x^{-1} = e \) and \( x^2 \circ x^{-1} = x \). The set of all invertible elements of \( J \) will be denoted by \( J_{\text{inv}} \). In this case, \( x \) acts as the unit for the homotope \( J[x^{-1}] \) of \( J \).

If \( J \) is a unital Jordan algebra and \( x \in J_{\text{inv}} \) then by \( x \)-isotope of \( J \), denoted by \( J[x] \), we mean the \( x^{-1} \)-homotope \( J[x^{-1}] \) of \( J \). We denote the multiplication \( \cdot_{x^{-1}} \) of \( J[x] \) by \( \circ_x \).

The following lemma gives the invariance of the set of invertible elements in a unital Jordan algebra on passage to any of its isotopes.
Lemma 1.1. For any invertible element $a$ in a unital Jordan algebra $\mathcal{J}$, $\mathcal{J}_{\text{inv}} = \mathcal{J}_{\text{inv}}^{|a|}$.

Proof. See Lemma 1.5 of [8]. □

Let $\mathcal{J}$ be a Jordan algebra and let $a, b \in \mathcal{J}$. The operators $T_b$ and $U_{a,b}$ are defined on $\mathcal{J}$ by $T_b(x) = b \circ x$ and $U_{a,b}(x) = \{axb\}$. We shall denote $U_{a,a}$ simply by $U_a$. The elements $a$ and $b$ are said to operator commute if $T_b \text{ commute with } T_a$.

Let $\mathcal{J}$ be a complex unital Banach Jordan algebra and let $x \in \mathcal{J}$. As usual, the spectrum of $x$ in $\mathcal{J}$, denoted by $\sigma_{\mathcal{J}}(x)$, is defined by

$$\sigma_{\mathcal{J}}(x) = \{\lambda \in \mathbb{C} : x - \lambda e \text{ is not invertible in } \mathcal{J}\}.$$

A Jordan algebra $\mathcal{J}$ with product $\circ$ is called a Banach Jordan algebra if there is a norm $\| \|$ on $\mathcal{J}$ such that $(\mathcal{J}, \| \|)$ is a Banach space and $\|a \circ b\| \leq \|a\| \|b\|$. If, in addition, $\mathcal{J}$ has a unit $e$ with $\|e\| = 1$ then $\mathcal{J}$ is called a unital Banach Jordan algebra. In the sequel, we will only be considering unital Banach Jordan algebras; the norm closure of the Jordan subalgebra $\mathcal{J}(x_1, \ldots, x_r)$ generated by $x_1, \ldots, x_r$ of Banach Jordan algebra $\mathcal{J}$ will be denoted by $\overline{\mathcal{J}(x_1, \ldots, x_r)}$.

The following elementary properties of Banach Jordan algebras are similar to those of Banach algebras and their proofs are a fairly routine modifications of these [1, 2, 7, 9].

Lemma 1.2. Let $\mathcal{J}$ be a Banach Jordan algebra with unit $e$ and $x_1, \ldots, x_r \in \mathcal{J}$.

(i) If $\mathcal{J}(x_1, \ldots, x_r)$ is an associative subalgebra of $\mathcal{J}$, then $\overline{\mathcal{J}(x_1, \ldots, x_r)}$ is a commutative Banach algebra.

(ii) $T_{x_1}$ and $U_{x_1, x_2}$ are continuous with $\|T_{x_1}\| \leq \|x_1\|$ and $\|U_{x_1, x_2}\| \leq 3\|x_1\| \|x_2\|$.

(iii) $\mathcal{J}(x_1, \ldots, x_r)$ is a closed subalgebra of $\mathcal{J}$.

(iv) If $\mathcal{J}$ is unital then $\mathcal{J}(e, x_1)$ is a commutative Banach algebra.

(v) If $x \in \mathcal{J}$ and $\|x\| < 1$ then $e - x$ is invertible and $(e - x)^{-1} = \sum_{n=0}^{\infty} x^n \in \mathcal{J}(e, x)$.

(vi) If $K$ is a closed Jordan subalgebra of $\mathcal{J}$ containing $e$ and $x \in K$ such that $\mathbb{C} \setminus \sigma_{\mathcal{J}}(x)$ is connected then $\sigma_{\mathcal{J}}(x) = \sigma_K(x)$. 

A complex Banach Jordan algebra $J$ with isometric involution $^*$ (see [6], for instance) is called a JB$^*$-algebra if $\|\{xx^*x\}\| = \|x\|^3$ for all $x \in J$.

The class of JB$^*$-algebras was introduced by Kaplansky in 1976 (see [10]) around the same time when a related class called JB-algebras was being studied by Alfsen, Shultz and Størmer (see [1]).

A real Banach Jordan algebra $J$ is called a JB-algebra if $\|x\|^2 = \|x^2\| \leq \|x^2 + y^2\|$ for all $x, y \in J$.

These two classes of algebras are linked as follows (see [10, 13]).

**Theorem 1.3.**
(a) If $A$ is a JB$^*$-algebra then the set of self-adjoint elements of $A$ is a JB-algebra.
(b) If $B$ is a JB-algebra then under a suitable norm the complexification $\mathbb{C}B$ of $B$ is a JB$^*$-algebra.

There is an easier subclass of these algebras. Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the full algebra of bounded linear operators on $\mathcal{H}$.

(a) Any closed self-adjoint complex Jordan subalgebra of $\mathcal{B}(\mathcal{H})$ is called a JC$^*$-algebra.
(b) Any closed real Jordan subalgebra of self-adjoint operators of $\mathcal{B}(\mathcal{H})$ is called a JC-algebra.

Any JB$^*$-algebra isometrically $^*$-isomorphic to a JC$^*$-algebra is also called a JC$^*$-algebra; similarly, any JB-algebra isometrically isomorphic to a JC-algebra is also called a JC-algebra.

It is easy to verify that a JC$^*$-algebra is a JB$^*$-algebra and a JC-algebra is a JB-algebra. It might be expected, conversely, that every JB-algebra is a JC-algebra (with a corresponding statement for JB$^*$-algebras and JC$^*$-algebras) but unfortunately this is not true (for details see [1]).

2. Unitary Isotopes of a JB$^*$-algebra

In [8], we presented a study of unitary isotopes of JB$^*$-algebras. In this section, we recall some facts from [8] which are needed for the sequel.

Let $J$ be a JB$^*$-algebra. The element $u \in J$ is called unitary if $u^* = u^{-1}$, the inverse of $u$. The set of all unitary elements of $J$
will be denoted by $\mathcal{U}(\mathcal{J})$. If $u$ is a unitary element of $JB^*$-algebra $\mathcal{J}$ then the isotope $\mathcal{J}^{[u]}$ is called a unitary isotope of $\mathcal{J}$.

**Theorem 2.1.** Let $u$ be a unitary element of the $JB^*$-algebra $\mathcal{J}$. Then the isotope $\mathcal{J}^{[u]}$ is a $JB^*$-algebra having $u$ as its unit with respect to the original norm and the involution $*_{u}$ defined by $x^{*_{u}} = \{ux^{*}u\}$.

*Proof.* See Theorem 2.4 of [8].

Recall (from [3], for instance) that a Jordan algebra is said to be special if it is isomorphic to a Jordan subalgebra of some associative algebra. We require the following fact.

**Lemma 2.2.** If $\mathcal{J}$ is a special Jordan algebra and $a \in \mathcal{J}$, then $\mathcal{J}^{[a]}$ is a special Jordan algebra.

*Proof.* See Lemma 1.3 in [8].

**Theorem 2.3.** The unitary isotope of a $JC^*$-algebra is again a $JC^*$-algebra.

*Proof.* This follows from Theorem 2.1 and Lemma 2.2 (also see [8, Theorem 2.12]).

We close this section by noting following facts.

**Lemma 2.4.** Let $\mathcal{J}$ be a $JB^*$-algebra with unit $e$. Then $u \in \mathcal{U}(\mathcal{J}) \implies e \in \mathcal{U}(\mathcal{J}^{[u]})$. Moreover $\mathcal{J}^{[u]}[e] = \mathcal{J}$.

*Proof.* See Lemma 2.7 of [8].

Next theorem establishes the invariance of unitaries on passage to unitary isotopes of a $JB^*$-algebra.

**Theorem 2.5.** For any unitary element $u$ in the $JB^*$-algebra $\mathcal{J}$, $\mathcal{U}(\mathcal{J}) = \mathcal{U}(\mathcal{J}^{[u]})$.

*Proof.* See Theorem 2.8 of [8].

**Corollary 2.6.** Let $\mathcal{J}$ be a $JB^*$-algebra with unit $e$ and let $u, v \in \mathcal{U}(\mathcal{J})$. Then

(i) $\mathcal{J}^{[u]}[v] = \mathcal{J}^{[v]}$.

(ii) The relation of being unitary isotope is an equivalence relation in the class of unital $JB^*$-algebras.

*Proof.* See Corollary 2.9 of [8].
3. Centre of Unitary Isotopes

In this section, we identify the centre of unitary isotopes in terms of the centre of the original $JB^*$-algebra. We recall the following definition from [14].

**Definition 3.1.** Let $\mathcal{J}$ be a unital $JB^*$-algebra and let

$$C(\mathcal{J}) = \{ x \in J_{sa} : x \text{ operator commutes with every } y \in J_{sa} \}.$$  

Then the centre of $\mathcal{J}$, denoted by $Z(\mathcal{J})$, is defined by

$$Z(\mathcal{J}) = C(\mathcal{J}) + iC(\mathcal{J}).$$

**Remark 3.2.** It is known from [14] that $Z(\mathcal{J})$ is a $C^*$-algebra, and if $\mathcal{J}$ is a $JC^*$-algebra with $\mathcal{J} \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ then

$$Z(\mathcal{J}) = \{ x \in \mathcal{J} : xy = yx \quad \forall y \in \mathcal{J} \}.$$  

To investigate further properties of the centre we need the following lemma.

**Lemma 3.3.** Let $\mathcal{J}$ be a $JB^*$-algebra and let $x \in Z(\mathcal{J})$. Then for all $y \in \mathcal{J}$,

(i) $T_xT_y = T_yT_x$;

(ii) $T_xU_y = U_yT_x$;

(iii) $U_xU_y = U_yU_x$;

(iv) if $u \in \mathcal{J}$ is unitary then $(x \circ u^*) \circ u = x$.

**Proof.** Let $x = a + ib$ and $y = c + id$ with $a, b \in C(\mathcal{J})$ and $c, d \in J_{sa}$. Then

$$T_xT_y = (T_a + iT_b)(T_c + iT_d) = T_aT_c + iT_aT_d + iT_bT_c - T_bT_d = T_aT_c + iT_aT_b + iT_cT_b - T_cT_a = T_yT_x$$

as $a, b \in C(\mathcal{J})$ which proves (i).

(ii). Since $U_y = 2T_y^2 - T_y^2$, we have

$$T_xU_y = T_x(2T_y^2 - T_y^2) = 2T_xT_y^2 - T_xT_y^2 = (2T_y^2 - T_y^2)T_x = U_yT_x$$

by part (i) (note that the associativity of $\mathcal{B}(\mathcal{J})$ is used here).

(iii). Since $x \in Z(\mathcal{J})$, $x^2 \in Z(\mathcal{J})$ by Remark 3.2. Hence by part (ii),

$$U_xU_y = (2T_x^2 - T_x^2)U_y = 2T_x^2U_y - T_x^2U_y = 2U_yT_x^2 - U_yT_x^2 = U_yU_x.$$  

(iv). By part (i), $(x \circ u^*) \circ u = T_bT_xu^* = T_xT_{u^*} = T_xe = x$. □
Theorem 3.4. Let \( \mathcal{J} \) be a \( JB^\ast \)-algebra with unit \( e \) and let \( b \in \mathcal{Z}(\mathcal{J}) \). Then for any unitary \( u \in \mathcal{U}(\mathcal{J}) \) and for any \( x \in \mathcal{J} \) we have

(i) \((u^\ast \circ x) \circ u = u^\ast \circ (x \circ u)\);
(ii) \(\{ (b \circ u)u^\ast x \} = b \circ x\).

Proof. (i). If \( \mathcal{J} \) is special then

\[
(u^\ast \circ x) \circ u = \frac{1}{4}(u(u^\ast x + xu^\ast) + (u^\ast x + xu^\ast)u) = \frac{1}{4}(2x + xu^\ast + u^\ast xu) = \frac{1}{4}(u^\ast (ux + xu) + (ux + xu)u^\ast) = u^\ast \circ (x \circ u).
\]

Hence, by the Shirshov–Cohn theorem with inverses \([5]\), we have in the general case \((u^\ast \circ x) \circ u = u^\ast \circ (x \circ u)\).

(ii). Since \( b \in \mathcal{Z}(\mathcal{J}) \) and \( u \in \mathcal{U}(\mathcal{J}) \), we get by Lemma 3.3 (iv) that \((b \circ u) \circ u^\ast = b\).

Again by Lemma 3.3 (i),

\[
(u^\ast \circ x) \circ (b \circ u) = T_{(u^\ast \circ x)}T_bu = T_bT_{(u^\ast \circ x)}u = b \circ (u \circ (x \circ u^\ast))
\]

and

\[
u^\ast \circ ((b \circ u) \circ x) = T_{u^\ast}T_xT_bu = T_bT_{u^\ast}T_xu = b \circ (u^\ast \circ (x \circ u))
\]

so by part (i)

\[
(u^\ast \circ x) \circ (b \circ u) = u^\ast \circ ((b \circ u) \circ x).
\]

Thus by (1) and (2),

\[
\{ (b \circ u)u^\ast x \} = \{(b \circ u) \circ u^\ast \} \circ x + (u^\ast \circ x) \circ (b \circ u) - ((b \circ u) \circ x) \circ u^\ast = b \circ x.
\]

We now need a characterisation of the centre in terms of Hermitian operators. These are defined in terms of the numerical range of operators as follows (see \([14]\), for example).

**Definition 3.5.** If \( \mathcal{J} \) is a complex unital Banach Jordan algebra with unit \( e \) and \( D(\mathcal{J}) = \{ f \in \mathcal{J}^\ast : ||f|| = 1 \} \) then, for \( a \in \mathcal{J} \), the numerical range of \( a \), denoted by \( W(a) \), is defined by \( W(a) = \{ f(a) : f \in D(\mathcal{J}) \} \). The element \( a \) is called Hermitian if \( W(a) \subseteq \mathbb{R} \). The set of all Hermitian elements of \( \mathcal{J} \) is denoted by \( \text{Her} \mathcal{J} \).
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The Hermitian elements in a unital JB*-algebra are exactly the self-adjoint elements (see [13]) but we shall need the following characterisation of the Hermitian operators on a JB*-algebra, given in [14].

**Theorem 3.6.** Let $\mathcal{J}$ be a JB*-algebra with unit $e$. Then $S \in \text{Her}(\mathcal{B}(\mathcal{J}))$ if and only if $S = T_a + \delta$ where $\delta$ is a *-derivation and $a = S(e)$ is self-adjoint.

We can now give a characterisation of the centre of a unitary isotope.

**Theorem 3.7.** Let $\mathcal{J}$ be a JB*-algebra with unit $e$ and let $u \in \mathcal{U}(\mathcal{J})$. Let $A$ be a JC*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ with unit $e_A$ and let $w \in \mathcal{U}(A)$.

(i) If $x \in Z(\mathcal{J})$ then $u \circ x \in Z(\mathcal{J}^u)$.
(ii) If $a \in Z(\mathcal{A}^u)$ then $(a \circ w^*) \circ w = a$.
(iii) If $z \in Z(\mathcal{J}^u)$ then $u \circ (u^* \circ z) = z$.
(iv) Define $\psi : Z(\mathcal{J}) \to Z(\mathcal{J}^u)$ by $\psi(x) = u \circ x$. Then $\psi$ is an isometric *-isomorphism of $Z(\mathcal{J})$ onto $Z(\mathcal{J}^u)$.

**Proof.** (i). Let $x = a + ib$ where $a, b \in Z(\mathcal{J})_{sa}$. Let $S = T_a \in \text{Her}(\mathcal{B}(\mathcal{J}))$. Then

$$S(e) = T_a(e) = a \circ e = a \quad \text{and} \quad S(u) = u \circ a.$$ 

As $S \in \text{Her}(\mathcal{B}(\mathcal{J}))$, $S(u) \in (\mathcal{J}^u)_{sa}$ by Theorem 3.6. By Theorem 3.4 (ii),

$$S(y) = T_a(y) = a \circ y = ((a \circ u)u^*y) = (a \circ u)u_y$$

for all $y \in \mathcal{J}$. Therefore, $S(y) = L_{S(u)}^u(y)$ for all $y \in \mathcal{J}$, where operator $L_{S(u)}^u$ stands for the multiplication by $S(u)$ in $\mathcal{J}^u$. Moreover, as $a \in Z(\mathcal{J})$ we get by [14, Theorem 14] that $S^2 \in \text{Her}(\mathcal{B}(\mathcal{J})) = \text{Her}(\mathcal{B}(\mathcal{J}^u))$ because $\mathcal{B}(\mathcal{J}^u) = \mathcal{B}(\mathcal{J})$ (see Theorem 2.1). So again by [14, Theorem 14], $S(u) \in Z(\mathcal{J}^u)$ as $S = L_{S(u)}^u$. Therefore, $u \circ a \in Z(\mathcal{J}^u)_{sa}$. Similarly, $u \circ b \in Z(\mathcal{J}^u)_{sa}$. Hence $u \circ x = u \circ a + iu \circ b \in Z(\mathcal{J}^u)$. 

(ii). By Remark 3.2,

$$Z(\mathcal{A}) = \{x \in \mathcal{A} : xy = yx\}. \quad (3)$$
By Theorem 2.3, the isotope $A^w$ is a $JC^*$-algebra and
\[ Z(A^w) = \{ x \in A : xw^*y = yw^*x \}. \quad (4) \]

Now, if $a \in Z(A^w)$ then (by (4)) $aw^*y = yw^*a$ for all $y \in A$. In particular,
\[ aw^* = w^*a. \quad (5) \]

By part (i), $a \circ w^* = e_A \circ w \ a \in Z(A^{w[e_A]}) = Z(A)$ . So we have by (4) that
\[ (a \circ w^*) \circ w = (a \circ w^*)w = \frac{1}{2}(aw^* + w^*a)w \]

hence by (5)
\[ (a \circ w^*) \circ w = (aw^*)w = a(w^*w) = a, \]
as required .

(iii) Now, let $v$ be any unitary in $Z(J^{[u]})$ (the centre of the unitary isotope $J^{[u]}$ of the $JB^*$-algebra $J$). Then $v$ is a unitary in $J$ by Theorem 2.5. By [8, Corollary 1.14], $J(e, u, u^*, v, v^*)$ is a $JC^*$-algebra and $v \in Z((J(e, u, u^*, v, v^*))^{[u]})$. Hence, by (ii),
\[ u \circ (u^* \circ v) = v. \quad (6) \]

If $z \in Z(J^{[u]})$, then by the Russo–Dye Theorem (cf. [11]) for $C^*$-algebras there exist unitaries $v_j \in Z(J^{[u]})$ and scalars $0 \leq \lambda_j \leq 1$ with $\sum_{j=1}^n \lambda_j = 1$ for some $n \in \mathbb{N}$ such that $\sum_{j=1}^n \lambda_j v_j$, because $\|\sum_{j=1}^n \lambda_j v_j\| < 1$ (recall that $Z(J^{[u]})$ is a $C^*$-algebra). Hence, by (6),
\[ u \circ (u^* \circ z) = u \circ (u^* \circ (\|z\| + 1) \sum_{j=1}^n \lambda_j v_j) \]
\[ = (\|z\| + 1) \sum_{j=1}^n \lambda_j (u \circ (u^* \circ v_j)) \]
\[ = (\|z\| + 1) \sum_{j=1}^n \lambda_j v_j = z. \]

(iv). As $\psi = T_u \mid_{Z(J)}$, $\psi$ is linear and continuous by Lemma 1.2 (i).
Let $z \in Z(J^{[u]})$. Applying part (i) to $J^{[u]}$ we get $e \circ u \ z \in Z(J^{[u]}^{[u]})$. But $J^{[u]}^{[u]} = J$ by Lemma 2.4 and $e \circ u \ z = \{e u^*z\} = u^* \circ z$. Hence $u^* \circ z \in Z(J)$. Moreover, $\psi(u^* \circ z) = u \circ (u^* \circ z) = z$ by part (iii). Thus $\psi$ maps $Z(J)$ onto $Z(J^{[u]})$. 
Further, \( \| \psi(x) \| \leq \| u \| \| x \| \) while, by Lemmas 3.3 (i) and 1.2 (ii),
\[
\| x \| = \| T_x T_u u \| = \| T_u T_x u \| \leq \| x \circ u \| = \| \psi(x) \| .
\]
Thus \( \psi \) is an isometry.

Finally, as \( \psi(e) = u \) and \( u \) is the unit of \( J[u] \) it follows from [12, Theorem 6] that \( \psi \) is an isometric \( * \)-isomorphism. \( \blacksquare \)

**Corollary 3.8.** Let \( J \) be a unital JB*-algebra. Then, for all \( u, v \in U(J) \), \( Z(J[u]) \) is isometrically Jordan \( * \)-isomorphic to \( Z(J[v]) \).

**Proof.** By Theorem 2.5, \( v \in U(J) \). Hence, by Theorem 3.7, \( Z(J[u]) \) is isometrically \( * \)-isomorphic to \( Z(J[v]) \). However, by Corollary 2.6 (i), \( J[u][v] = J[v] \). This gives the required result. \( \blacksquare \)

An alternative proof of above Corollary 3.8 can be obtained by noting that \( Z(J[u]) \) is isometrically \( * \)-isomorphic to \( Z(J[v]) \) and \( Z(J[v]) \) is isometrically \( * \)-isomorphic to \( Z(J[v]) \) by Theorem 3.7 (applied twice). As the next example shows there need be no inclusion between the centre of a unital JB*-algebra and the centre of its isotopes. In the following discussion \( M_2(\mathbb{C}) \) denotes the standard complexification of the real Jordan algebra of all \( 2 \times 2 \) symmetric matrices.

**Example 3.9.** If \( u \in U(M_2(\mathbb{C})) \setminus Z(M_2(\mathbb{C})) \) then the unit \( e \notin Z(M_2(\mathbb{C})) \).

Indeed, \( M_2(\mathbb{C})[u] \) is a 4-dimensional \( C^* \)-algebra by Theorem 2.3 with 1-dimensional centre by the above Theorem 3.7. As \( u \) does not belong to \( Z(M_2(\mathbb{C})) \), \( u \notin Sp(e) \) where \( Sp(e) \) denotes the linear span of \( e \), and hence \( e \notin Sp(u) \). This gives that \( e \notin Z(M_2(\mathbb{C})[u]) \).

As a final point on the relationships between the centres it should be noted in the proof of Theorem 3.7 (i) that if \( a \in Z(J) \) and \( S = T_a \) then \( S \) is left multiplication in any unitary isotope. In order to study the \( * \)-derivations it might be hoped that if \( T \in Her B(J) \) then there exists a unitary isotope \( J[u] \) such that \( T \) is left multiplication operator in \( Her B(J[u]) \) since as linear spaces \( B(J) = B(J[u]) \) so \( Her B(J) = Her B(J[u]) \). Unfortunately, this fails even when \( J = M_2(\mathbb{C}) \). As all \( * \)-derivations are inner in this case, it follows that \( T \in Her B(M_2(\mathbb{C})) \) if and only if \( T = l_a + r_b \) where \( a, b \in (M_2(\mathbb{C}))_{sa} \) and \( l_a(x) = ax \) and \( r_b(x) = xb \).
Corollary 3.10. If \( a, b \in \mathcal{M}_2(\mathbb{C}) \) are given by
\[
\begin{pmatrix}
1 & 0 \\
0 & 2
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
6 & 0 \\
0 & 23
\end{pmatrix}
\]
and \( T \in \text{Her}(\mathcal{B}(\mathcal{M}_2(\mathbb{C}))) \) is defined by \( T = l_a + r_b \),
then \( T \) is not left multiplication in any unitary isotope.

Proof. It was noted in Example 3.9 that if \( u \in \mathcal{U}(\mathcal{M}_2(\mathbb{C})) \) then
\( \mathcal{M}_2(\mathbb{C})[u] \) is a four-dimensional \( C^* \)-algebra with a one-dimensional
centre so is isomorphic to \( \mathcal{M}_2(\mathbb{C}) \). By [4, Theorem 10], \( \sigma(T) = \sigma(a) + \sigma(b) = \{7, 8, 24, 25\} \).

On the other hand, if \( L_u[c] \in \text{Her}(\mathcal{B}(\mathcal{M}_2(\mathbb{C}))) \) with say \( \sigma(\mathcal{M}_2(\mathbb{C}))[c] = \{\lambda_1, \lambda_2\} \) then \( \sigma(L_u[c]) = \{\lambda_1, \lambda_1 + \lambda_2, \lambda_2\} \) again by [4, Theorem 10],
so \( \sigma(L_u[c]) \) contains only three points. Hence \( \sigma(T) \neq \sigma(L_u[c]) \) for
any unitary \( u \in \mathcal{U}(\mathcal{M}_2(\mathbb{C})) \). \( \square \)

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References

16 (1971), 115–125.
Soc. 10 (1959), 32–41.
(1967), 315–325.
[7] F. W. Shultz, On normed Jordan algebras which are Banach dual spaces,
[8] A. A. Siddiqui, Positivity of invertibles in unitary isotopes of \( JB^* \)-algebras,
submitted.
[9] H. Upmeier, Symmetric Banach manifolds and Jordan \( C^* \)-algebras,
Amsterdam, 1985.
algebras, Functional Analysis: Surveys and recent results (North Holland,
1977).

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